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Faddeev equation and its symmetric version for a three-particle lattice hamiltonian

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Abstract. In the present paper we consider the three-particle lattice Hamiltonian associated to a system of three particles on the d-dimensional lattice, where the role of two-particle discrete Schroedinger operators is played by a family of Friedrichs models. We define two bounded and self-adjoint so-called channel operators and prove that the essential spectrum of considered Hamiltonian is the union of spectra of the channel operators. Since the channel operators have a more simple structure than considered Hamiltonian, this fact plays an important role in the subsequent investigations of the essential spectrum. The spectrum of the constructed channel operators are described by the spectrum of the corresponding Friedrichs model. The Faddeev equation and its symmetric version for the eigenfunctions of the considered Hamiltonian are constructed.

1 Introduction

In models of solid state physics [1,2,3] and also in lattice quantum field theory [4], one considers so-called three-particle discrete Schroedinger operators, which are lattice analogs of the three-particle Schroedinger operator in the continuous (Eucledian) space. Although the energy operator of a system of three-particles on lattice is bounded and the perturbation operator in the pair problem is a compact operator, the study of spectral properties of energy operators of systems of two and three particles on a lattice is more complex than in the continuous case. The well-known methods for an investigation of the location of essential spectra of Schroedinger operators are the Weyl creation for the one particle problem and the HWZ theorem for multi-particle problems, a modern proof of which is based on the Ruelle-Simon partition of unity. Usually to study the essential spectrum of the discrete Schroedinger operators we use the Weyl creation, Fredholm theorem and Faddeev equation [5-31].

In the present paper, we study model Hamiltonian associated to a three particles on a lattice, where the role of two-particle discrete Schroedinger operators is played by a family of Friedrichs models. This model Hamiltonian is considered as a linear, bounded and self-adjoint operator in the Hilbert space. For the study of location of the essential spectrum of the considered model Hamiltonian we introduce two channel operators and prove that the

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essential spectrum of considered Hamiltonian is the union of spectra of the channel operators. Since the channel operators have a more simple structure than considered Hamiltonian, this fact plays an important role in the subsequent investigations of the essential spectrum. The spectrum of the two channel operators are described using the spectrum of the Friedrichs models. We constract the Faddeev equation and its symmetric version for the eigenvectors of the three particle lattice Hamiltonian.

2 Literature review

In this section we discuss some spectral properties of the model Hamiltonians related with the two-particle and three-particle systems on a lattice [5-10]. In addition, we also give an information about the block operator matrices in Fock space [11-25], which has the similar spectral properties. For the convenience of readers, we will discuss the studied model and obtained results separately in each paper.

In the paper [5] the model Hamiltonian $H_{\mu,\lambda}$, $\mu, \lambda > 0$, which is related to the three-particle system on the one dimensional lattice, interacting via non-local potentials is considered. The new branches of the essential spectrum of operator $H_{\mu,\lambda}$ are studied.

In the paper [6] a Friedrichs model $\mathcal{A}(\mu_1, \mu_2)$, $\mu_1, \mu_2 > 0$ with rank two perturbation is considered. We recall that this model is related with the two quantum particle system on the three-dimensional integer lattice. The number and location of the discrete eigenvalues of $\mathcal{A}(\mu_1, \mu_2)$ are investigated with respectr two parameters μ_1, μ_2. The sufficient and necessary conditions which guarantees the equality of the spectrum of $\mathcal{A}(\mu_1, \mu_2)$ and its field of values (or numerical range) are given. The relation of the threshold eigenvalues and virtual levels with the numerical range of $\mathcal{A}(\mu_1, \mu_2)$ are established.

In the paper [7] the three-particle lattice model Hamiltonian $H_{\mu,\lambda}$, $\mu, \lambda > 0$, by making use nonlocal potential is presented. This Hamiltonian defined as a tensor sum of two Friedrichs models $h_{\mu,\lambda}$ with rank two perturbation. It is associated with the system of three quantum particles on a d-dimensional lattice. The number of eigenvalues of $h_{\mu,\lambda}$ is investigated dependently on the parameters μ, λ. The suitable conditions on the existence of eigenvalues localized inside, in the gap and below the bottom of the essential spectrum of $H_{\mu,\lambda}$ is provided.

In the paper [8] the tensor sum $H_{\mu,\lambda}$, $\mu, \lambda > 0$ of the two bounded and self-adjoint Friedrichs models $h_{\mu,\lambda}$ with rank two perturbation is considered. The Hamiltonian $H_{\mu,\lambda}$ is associated with the system of three quantum particles on the one-dimensional lattice. The number and location of the eigenvalues of $H_{\mu,\lambda}$ is investigated with respect to the parameters μ, λ. The existence of eigenvalues located respectively inside, in the gap, and below of the bottom of the essential spectrum of $H_{\mu,\lambda}$ is proved.

In the paper [9] the model Hamiltonian H associated with the system of three particles on a d -dimensional lattice that interact by the non-local potentials is considered. The corresponding linear, bounded and self-adjoint channel operators are identified. An analogue of the Faddeev equation for the eigenfunctions of the model Hamiltonian H is constructed and the spectrum of the model Hamiltonian H is described. The location of the two-particle and three-particle branches of the essential spectrum of the model Hamiltonian H is described by the spectrum of the constructed channel operators. It is shown that the essential spectrum of the model Hamiltonian H consists the union of at most $2n + 1$ bounded closed intervals, where n is the rank of the kernel of the non-local interaction operators. The upper bound of the spectrum of the model Hamiltonian H is found. The lower bound of the essential spectrum of the model Hamiltonian H for the case $d = 1$ is estimated.

The authors of the paper [10] are consider the Schroedinger operator $H(k)$, associated with the system of two particles on a two-dimensional lattice. They show that the subspaces of even and odd functions are invariant under the operator $H(k)$.

The same properties for the block operator matrices, corresponding to the Hamiltonian of a system with non conserved number (at most two, three, four) of particles on a lattice are discussed by many authors, see for example [11-25].

By the authors of the paper [11] the special type of soluble model, corresponding to a coupled molecular and nuclear Hamiltonians H_1 is studied. This model sometimes is called generalized Friedrichs model. First they construct the well-known Faddeev operator corresponding to the Hamiltonian H_1 and then the most important properties of this operator, which are related with the number of discrete eigenvalues are proved. In addition, the formula for the counting the multiplicity of the discrete eigenvalues of the Hamiltonian H_1 is derived.

In the paper [12] the 2×2 operator matrix H_2 acting in the direct sum of the two Hilbert spaces is considered. An analog of the well-known Faddeev equation for the eigenvectors of the operator matrix H_2 is constructed and some important properties of this equation, related with the number of eigenvalues, are studied. In particular, the Birman-Schwinger principle for the operator matrix H_2 is proved.

In [13] the block operator matrix A_μ of order 2, which is depending with the coupling constant $\mu > 0$ and acting in the direct sum of one-particle and two-particle subspaces of the bosonic Fock space is investigated. The authors of this paper are show that there exist the critical values of the coupling constant that the operator matrix A_μ has infinitely many eigenvalues on the left hand side (right hand side) of the its essential spectrum.

In the paper [14] the well-known lattice spin-boson model A_2 with fixed atom and at most two photons is considered. The first Schur complement $S_1(\lambda)$ with spectral parameter λ corresponding to the lattice spin-boson model A_2 is constructed. The Birman-Schwinger principle for the lattice spin-boson model A_2 with respect to $S_1(\lambda)$ is proved. An important properties of $S_1(\lambda)$ related with the number of eigenvalues of the lattice spin-boson model A_2 is studied.

In the papers [15] and [16] the 2×2 operator matrix, acting in the cut subspace of the bosonic Fock space are considered. Here the critical value of the coupling constant for which the considered operator matrix has an infinite number of eigenvalues is find. It is shown that the part of these eigenvalues accumulate at the lower bound and another part accumulate at the upper bound of the essential spectrum. An asymptotic formula for the number of such eigenvalues in both the left and right parts of the essential spectrum are obtained.

In the paper [17] the 2×2 operator matrix, associated with the lattice systems describing three particles in interaction, without conservation of the number of particles on a d -dimensional lattice is considered. First, the two-particle and three-particle branches of the essential spectrum of the considered operator matrix are described and then it is shown that the essential spectrum consist of the union of at most three bounded closed intervals.

In the papers [18] and [19] the family of bounded and self-adjoint 2×2 operator matrices $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^3 := (-\pi, \pi]^3$, $\mu > 0$, associated with the Hamiltonian of a system consisting of at most two particles on a three-dimensional lattice $\mathbb{Z}^3$ are investigated. It is proven that there is a value μ_0 of the parameter μ such that only for $\mu = \mu_0$ the operator matrices $\mathcal{A}_\mu(\bar{0})$ and $\mathcal{A}_\mu(\bar{\pi})$ has the virtual level at the point $z = 0 = \min \sigma_{\mathrm{ess}}\left(\mathcal{A}_\mu(\bar{0})\right)$ and $z = 18 = \max \sigma_{\mathrm{ess}}\left(\mathcal{A}_\mu(\bar{\pi})\right)$, respectively, where $\bar{0} := (0,0,0)$, $\bar{\pi} := (\pi, \pi, \pi) \in \mathbb{T}^3$. The absence of the eigenvalues of the operator matrix $\mathcal{A}_\mu(k)$ for all values of k under the assumption that $\mu = \mu_0$ is shown. The threshold energy expansions for the Fredholm determinant associated to the operator matrix $\mathcal{A}_\mu(k)$ are obtained.

In the papers [20], [21] and [22] the family of operator matrices $H(K),\ K \in \mathbb{T}^3 := (-\pi,\pi]^3$ of order three are considered. In particular, the position and structure of the two-particle as well the three-particle branches (subsets) of the essential spectrum of the operator matrix $H(K)$ are investigated in [20]. The authors of the paper [21] are find the finite set $\Lambda \subset \mathbb{T}^3$ to prove the existence of infinitely many eigenvalues of the operator matrix $H(K)$ for all $K \in \Lambda$ when the associated Friedrichs model has the zero energy resonance. It is also shown that for every $K \in \Lambda$, the number $N(K,z)$ of eigenvalues of the operator matrix $H(K)$ lying on the left of z, $z < 0$, satisfies the asymptotic relation

$$\lim_{z \to -0} \frac{N(K,z)}{|\log|z||} = U_0$$

with $0 < U_0 < \infty$, independently on the cardinality of Λ. In the paper [22], first an analogue of the Faddeev equation for the eigenfunctions of the operator matrix $H(K)$ is obtained. Then, an analytic description of the essential spectrum of $H(K)$ is established. Further, it is shown that the essential spectrum of the operator matrix $H(K)$ consists the union of at most three bounded closed intervals.

In the paper [23] the block operator matrix $\mathcal{A}$ related to a system describing two identical fermions and one particle another nature on a lattice is considered. Here the authors are reduce the problem of the study of the spectrum of the block operator matrix $\mathcal{A}$ to the investigation of the spectrum of block operator matrices of order three with a discrete variable. Then the relations for the spectrum, essential spectrum, and point spectrum are established. Two-particle and three-particle branches of the essential spectrum of the block operator matrix $\mathcal{A}$ are also determined.

The paper [24] is devoted to the study of the spectrum of the 4×4 block operator matrix $\mathcal{A}_3$, which is the non-symmetric version of the Hamiltonian related with the lattice spin boson Hamiltonian with at most 3 photons. The position of the essential spectrum $\sigma_{\text{ess}}(\mathcal{A}_3)$ is described. Its two-particle, three-particle and four-particle branches are analyzed. Moreover, the formula for the point spectrum $\sigma_{\text{p}}(\mathcal{A}_3)$ is derived. The connections for the discrete spectrum $\sigma_{\text{disc}}(\mathcal{A}_3)$ are obtained.

In the paper [25] the operator matrix $\mathcal{A}$ of order four, corresponding to the Hamiltonian of a system with non conserved number and at most four particles on a lattice is considered. First, it is shown that the operator matrix $\mathcal{A}$ is unitarily equivalent to the diagonal matrix. The diagonal elements of the obtained operator matrix are again operator matrices of order four. Then the location of the essential spectrum of the operator $\mathcal{A}$ is described. Two-particle, three-particle and four-particle branches of the essential spectrum of the operator matrix $\mathcal{A}$ are separated. It is established that the essential spectrum of the operator matrix $\mathcal{A}$ consists of the union of at most 14 closed intervals. The Fredholm determinant, whose the set of zeros is coinside with the discrete spectrum of the operator matrix $\mathcal{A}$ is constructed. Moreover, it was shown that the operator matrix $\mathcal{A}$ has at most 16 simple eigenvalues, lying outside of the essential spectrum.

3 Three-particle lattice Hamiltonian

Let $T^1 :\equiv (-\pi;\pi]$. The operations addition and multiplication by the real numbers of elements of T^1 should be regarded as operations on R modulo $2\pi Z$. For example,

$$\frac{\pi}{4} + \pi = \frac{5\pi}{4} = -\frac{3\pi}{4} (\mathrm{mod}\ 2\,\pi),$$
$$7 \cdot \frac{\pi}{5} = 2\pi - \frac{3\pi}{5} = -\frac{3\pi}{5} (\mathrm{mod}\ 2\,\pi).$$

Then the obtained set T^1 is called one dimensional torus. Let $L_2^{\mathrm{s}}(T^2)$ be the Hilbert space of the square integrable (complex) symmetric functions defined on T^2.

In the Hilbert space $L_2^{\mathrm{s}}(T^2)$ we consider the following Hamiltonian

$$H_{\mu,\lambda}^{(\gamma)} := H_0^{(\gamma)} - \mu(V_1 + V_2) - \lambda V_3, \tag{1}$$

where the numbers $\mu, \lambda > 0$ are coupling parameters, the operator $H_0^{(\gamma)}$ is the non-perturbed operator, i.e. multiplication operator by the function $E_\gamma(\cdot,\cdot)$:

$$(H_0^{(\gamma)} f)(x,y) = E_\gamma(x,y) f(x,y),$$
$$E_\gamma(x,y) := \varepsilon(x) + \varepsilon(y) + \gamma\varepsilon(x+y),\ \varepsilon(x) := 1 - \cos(2x).$$

The operators V_α, $\alpha = 1,2,3$ are the non-local potential operators of the form:

$$(V_1 f)(x,y) = \sin y \int_{T^1} \sin t\, f(x,t) dt,$$
$$(V_2 f)(x,y) = \sin x \int_{T^1} \sin t\, f(t,y) dt,$$
$$(V_3 f)(x,y) = \int_{T^1} f(t, x+y-t) dt.$$

From the definitions one can see that they are partial integral operators.

Using the elements of the Functional Analysis course one can show that the operator $H_{\mu,\lambda}^{(\gamma)}$ acting in the Hilbert space $L_2^{\mathrm{s}}(T^2)$ by (1) is linear, bounded and self-adjoint. Readers can independently verify this assertion.

To define the essential spectrum of the operator $H_{\mu,\lambda}^{(\gamma)}$ we introduce two so-called channel operators. These operators are acting in the Hilbert space $L_2(T^2)$ as

$$H_\mu^{(\gamma,1)} := H_0^{(\gamma)} - \mu V_1,$$
$$H_\lambda^{(\gamma,2)} := H_0^{(\gamma)} - \lambda V_3.$$

It is easy to see that the operators $H_\mu^{(\gamma,1)}$ and $H_\lambda^{(\gamma,2)}$ are also linear, bounded and self-adjoint operators in the Hilbert space $L_2(T^2)$.

The operators $H_\mu^{(\gamma,1)}$ and $H_\lambda^{(\gamma,2)}$ can be decomposed into the direct integrals

$$H_\mu^{(\gamma,1)} = \int_{T^1} \oplus \left(h_\mu^{(\gamma,1)}(k) + \varepsilon(k) I\right) dk,$$
$$H_\lambda^{(\gamma,2)} = \int_{T^1} \oplus \left(h_\lambda^{(2)}(k) + \gamma\varepsilon(k) I\right) dk.$$

Here I is the identity operator in the Hilbert space $L_2(T^1)$ and the operators $h_\mu^{(\gamma,1)}(k)$ and $h_\lambda^{(2)}(k)$ are so-called family of Friedrichs models, acting in the Hilbert space $L_2(T^1)$ as

$$h_\mu^{(\gamma,1)}(k) := h_0^{(\gamma,1)}(k) - \mu v_1,\ k \in T^1,$$
$$h_\lambda^{(2)}(k) = h_0^{(2)}(k) - \lambda v_2,\ k \in T^1,$$

with

$$(h_0^{(\gamma,1)}(k) f)(x) = \big(\varepsilon(x) + \gamma\varepsilon(k+x)\big) f(x),\ (v_1 f)(x) = \sin x \int_{T^1} \sin t\, f(t) dt,$$
$$(h_0^{(2)}(k) f)(x) = \big(\varepsilon(x) + \varepsilon(k-x)\big) f(x),\ (v_2 f)(x) = \int_{T^1} f(t) dt.$$

It is clear that the operators $h_\mu^{(\gamma,1)}(k)$ and $h_\lambda^{(2)}(k)$ are linear, bounded and self-adjoint operators in the Hilbert space $L_2(T^1)$.

In what follows $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, and $\sigma_{\mathrm{disc}}(\cdot)$ are the respective spectrum, essential spectrum and discrete spectrum of a bounded self-adjoint operator.

In accordance with the invariance of the essential spectrum under finite rank perturbations (Weyl's theorem) the essential spectrum of $h_\mu^{(\gamma,1)}(k)$ coincides with the spectrum of $h_0^{(\gamma,1)}(k)$, analogously the essential spectrum of $h_\lambda^{(2)}(k)$ coincides with the spectrum of $h_0^{(2)}(k)$ and the equalities

$$\sigma_{\text{ess}}\left(h_\mu^{(\gamma,1)}(k)\right) = \left[m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)\right],$$

$$\sigma_{\text{ess}}\left(h_\lambda^{(2)}(k)\right) = [m_2(k); M_2(k)],$$

hold, where

$$m_1^{(\gamma)}(k) := \min_{k\in T^1}\big(\varepsilon(x) + \gamma\varepsilon(k+x)\big), M_1^{(\gamma)}(k) := \max_{k\in T^1}\big(\varepsilon(x) + \gamma\varepsilon(k+x)\big),$$

$$m_2(k) := \min_{k\in T^1}\big(\varepsilon(x) + \varepsilon(k-x)\big), M_2(k) := \max_{k\in T^1}\big(\varepsilon(x) + \varepsilon(k-x)\big).$$

For any fixed numbers $\mu, \lambda, \gamma > 0$ and $k \in T$ we define the regular functions defined in $C\backslash\left[m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)\right]$ and $C\backslash[m_2(k); M_2(k)]$ respectively as

$$\Delta_\mu^{(\gamma,1)}(k,z) := 1 - \mu\int_{T^1}\frac{\sin^2(t)dt}{\varepsilon(t) + \gamma\varepsilon(k+t) - z},$$

$$\Delta_\lambda^{(2)}(k,z) := 1 - \lambda\int_{T^1}\frac{dt}{\varepsilon(t) + \varepsilon(k-t) - z}.$$

Usually the functions $\Delta_\mu^{(\gamma,1)}(k,\cdot)$ and $\Delta_\lambda^{(2)}(k,\cdot)$ are called Fredholm determinants associated with the operators $h_\mu^{(\gamma,1)}(k)$ and $h_\lambda^{(2)}(k)$ respectively.

The following Lemma describes the connection between the eigenvalues of the operator $h_\mu^{(\gamma,1)}(k)$ and zeros of the function $\Delta_\mu^{(\gamma,1)}(k,\cdot)$.

Lemma 1. For any fixed $k \in T$ and $\mu, \gamma > 0$ the number $z \in C\backslash \left[m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)\right]$ is an eigenvalue of $h_\mu^{(\gamma,1)}(k)$ if and only if $\Delta_\mu^{(\gamma,1)}(k,z) = 0$.

From Lemma 1 we obtain the following result about the discrete spectrum of the operator $h_\mu^{(\gamma,1)}(k)$.

Corollary 1. For the discrete spectrum of the family of Friedrichs models $h_\mu^{(\gamma,1)}(k)$ the equality

$$\sigma_{\text{disc}}\left(h_\mu^{(\gamma,1)}(k)\right) = \left\{z \in C\backslash\left[m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)\right]: \Delta_\mu^{(\gamma,1)}(k,z) = 0\right\}$$

holds.

The following Lemma describes the connection between the eigenvalues of the operator $h_\lambda^{(2)}(k)$ and zeros of the function $\Delta_\lambda^{(2)}(k,\cdot)$.

Lemma 2. For any fixed $k \in T$ and $\gamma > 0$ the number $z \in C\backslash[m_2(k); M_2(k)]$ is an eigenvalue of $h_\lambda^{(2)}(k)$ if and only if $\Delta_\lambda^{(2)}(k,z) = 0$.

From Lemma 2 we obtain the following result about the discrete spectrum of the operator $h_\lambda^{(2)}(k)$.

Corollary 2. For the discrete spectrum of the family of Friedrichs models $h_\lambda^{(2)}(k)$ the equality

$$\sigma_{\text{disc}}\left(h_\lambda^{(2)}(k)\right) = \left\{z \in C\backslash[m_2(k); M_2(k)]: \Delta_\lambda^{(2)}(k,z) = 0\right\}$$

holds.

It can be easily seen that the Fredholm determinant as a function of the variable z is a monotonically decreasing function outside of the essential spectrum on the set of real numbers. Therefore, it is easy to determine the case where its zeros exist. The eigenvalue of the Friedrichs model $h_\mu^{(\gamma,1)}(k)$ ($h_\lambda^{(2)}(k)$) is a simple. There is no eigenvalues of the Friedrichs model $h_\mu^{(\gamma,1)}(k)$ ($h_\lambda^{(2)}(k)$) located on the right hand side of the essential spectrum.

In the following Theorem, the spectrum of the channel operators $H_{\mu}^{(\gamma,1)}$ and $H_{\lambda}^{(\gamma,2)}$ are described by the spectrum of the families of the Friedrichs models $h_{\mu}^{(\gamma,1)}(k)$ and $h_{\lambda}^{(2)}(k)$.

Theorem 1. We have the following equalities

$$\sigma\left(H_{\mu}^{(\gamma,1)}\right)=\sigma_{\text{two}}\left(H_{\mu}^{(\gamma,1)}\right)\cup\left[0;3+\frac{3\gamma}{2}\right],$$
$$\sigma\left(H_{\lambda}^{(\gamma,2)}\right)=\sigma_{\text{two}}\left(H_{\lambda}^{(\gamma,2)}\right)\cup\left[0;3+\frac{3\gamma}{2}\right],$$

where

$$\sigma_{\text{two}}\left(H_{\mu}^{(\gamma,1)}\right):=\bigcup_{k\in T}\left\{\sigma_{\text{disc}}\left(h_{\mu}^{(\gamma,1)}(k)\right)+\varepsilon(k)\right\},$$
$$\sigma_{\text{two}}\left(H_{\lambda}^{(\gamma,2)}\right):=\bigcup_{k\in T}\left\{\sigma_{\text{disc}}\left(h_{\lambda}^{(2)}(k)\right)+\gamma\varepsilon(k)\right\}.$$

In proving the stated theorem, the structure of the channel operators and the theorem about the spectrum of operators of the decomposable to the direct integral plays an important role. The number and location of eigenvalues of the Friedrichs model are important in the determining the number of closed intervals defining the spectrum of the channel operators.

The following theorem describes the location of the essential spectrum of the Hamiltonian $H_{\mu,\lambda}^{(\gamma)}$.

Theorem 2. For the essential spectrum of the Hamiltonian $H_{\mu,\lambda}^{(\gamma)}$ the equality

$$\sigma_{\text{ess}}\left(H_{\mu,\lambda}^{(\gamma)}\right)=\sigma\left(H_{\mu}^{(\gamma,1)}\right)\cup\sigma\left(H_{\lambda}^{(\gamma,2)}\right)$$

holds.

By definition, the channel operators have a simpler structure than considered operator. Therefore, studying their spectrum is a relatively easy problem. Because the spectrum of the channel operators are found using the spectrum of the Friedrichs model and the Friedrichs model has a simpler structure than channel operators. So, Theorem 1 and 2 are very important results.

We introduce the following Hilbert space

$$\boldsymbol{L_2^{(2)}(T)\coloneqq\{\varphi=(\varphi_1,\varphi_2)\colon\ \varphi_i\in L_2(T),\qquad i=1,2\}.}$$

For each $\boldsymbol{z\in C\backslash\sigma_{ess}\left(H_{\mu,\lambda}^{(\gamma)}\right)}$ we introduce block operator matrix acting in the Hilbert space $\boldsymbol{L_2^{(2)}(T)}$ as

$$\boldsymbol{T_{\mu,\lambda}^{(\gamma)}(z)}:=\begin{pmatrix}\boldsymbol{T_{11}(\mu,\lambda,\gamma;z)} & \boldsymbol{T_{12}(\mu,\lambda,\gamma;z)}\\ \boldsymbol{T_{21}(\mu,\lambda,\gamma;z)} & \boldsymbol{0}\end{pmatrix}.$$

Here $\boldsymbol{T_{ij}(\mu,\lambda,\gamma;z)\colon L_2(T^1)\to L_2(T^1),i,j=1,2}$ are the integral operators given by

$$\boldsymbol{(T_{11}(\mu,\lambda,\gamma;z)\varphi_1)(x)=\frac{\mu\sin x}{\Delta_{\mu}^{(\gamma,1)}(x,z-\varepsilon(x))}\int_{T^1}\frac{\sin t\varphi_1(t)}{E_{\gamma}(x,t)-z}dt},$$
$$\boldsymbol{(T_{12}(\mu,\lambda,\gamma;z)\varphi_2)(x)=\frac{\lambda}{\Delta_{\mu}^{(\gamma,1)}(x,z-\varepsilon(x))}\int_{T^1}\frac{\sin(t-x)\varphi_2(t)}{E_{\gamma}(x,t-x)-z}dt},$$
$$\boldsymbol{(T_{21}(\mu,\lambda,\gamma;z)\varphi_1)(x)=\frac{2\mu}{\Delta_{\gamma}^{(2)}(x,z-\gamma\varepsilon(x))}\int_{T^1}\frac{\sin(x-t)\varphi_1(t)}{E_{\gamma}(t,x-t)-z}dt}.$$

The following Theorem establishes a relation between eigenvalues of the operators $H_{\mu,\lambda}^{(\gamma)}$ and $T_{\mu,\lambda}^{(\gamma)}(z)$.

Theorem 3. The number $z\in C\backslash\sigma_{\text{ess}}\left(H_{\mu,\lambda}^{(\gamma)}\right)$ is an eigenvalue of the operator $H_{\mu,\lambda}^{(\gamma)}$ if and only if the number 1 is an eigenvalue of the block operator matrix $T_{\mu,\lambda}^{(\gamma)}(z)$.

Remark 3. We note that the equation $\varphi=T_{\mu,\lambda}^{(\gamma)}(z)\varphi$ is usually called the analogue of the Faddeev equation for eigenfunctions of $H_{\mu,\lambda}^{(\gamma)}$.

Here

$$\varphi = T_{\mu,\lambda}^{(\gamma)}(z)\varphi,$$
$$\varphi = \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} = \begin{pmatrix}(T_{11}(\mu,\lambda,\gamma;z)\varphi_1)(x) + (T_{12}(\mu,\lambda,\gamma;z)\varphi_2)(x)\\ (T_{21}(\mu,\lambda,\gamma;z)\varphi_1)(x)\end{pmatrix}.$$

Usually the Faddeev equation is constructed by analyzing the equation with respect to the eigenvalue. In this case, the methods of solving linear integral equations are very useful. Applying this equation, we easily determine the location of the essential spectrum.

Remark. If $z < \min \sigma_{\text{ess}}\left(H_{\mu,\lambda}^{(\gamma)}\right)$, then for all $x \in T^1$ we have $\Delta_\mu^{(\gamma,1)}(x;z) > 0$ and $\Delta_\lambda^{(2)}(x;z) > 0$.

For each $z < \min \sigma_{\text{ess}}\left(H_{\mu,\lambda}^{(\gamma)}\right)$ in the Hilbert space $L_2^{(2)}(T)$ we define the following block operator matrix

$$\hat{T}_{\mu,\lambda}^{(\gamma)}(z) := \begin{pmatrix}\hat{\mathrm{T}}_{11}(\mu,\lambda,\gamma;z) & \hat{\mathrm{T}}_{12}(\mu,\lambda,\gamma;z)\\ \hat{\mathrm{T}}_{21}(\mu,\lambda,\gamma;z) & 0\end{pmatrix}$$

where the operators $\hat{\mathrm{T}}_{11}(\mu,\lambda,\gamma;z)$, $\hat{\mathrm{T}}_{12}(\mu,\lambda,\gamma;z)$, $\hat{\mathrm{T}}_{12}(\mu,\lambda,\gamma;z)$ are defined in the Hilbert space $L_2(T)$ as integral operators:

$$\left(\hat{\mathrm{T}}_{11}(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\gamma};\boldsymbol{z})\boldsymbol{\psi}_1\right)(\boldsymbol{x}) = \frac{\mu \sin x}{\sqrt{\Delta_\mu^{(\gamma,1)}(x,z-\varepsilon(x))}}\int_{T^1}\frac{\sin t\,\psi_1(t)}{(E_\gamma(x,t)-z)\sqrt{\Delta_\mu^{(\gamma,1)}(t,z-\varepsilon(t))}}\boldsymbol{dt},$$
$$\left(\hat{\mathrm{T}}_{12}(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\gamma};\boldsymbol{z})\boldsymbol{\psi}_2\right)(\boldsymbol{x}) = \frac{\lambda}{\sqrt{\Delta_\mu^{(\gamma,1)}(x,z-\varepsilon(x))}}\int_{T^1}\frac{\sin(t-x)\psi_2(t)}{(E_\gamma(t,t-x)-z)\sqrt{\Delta_\gamma^{(2)}(t,z-\gamma\varepsilon(t))}}\boldsymbol{dt},$$
$$\left(\hat{\mathrm{T}}_{21}(\boldsymbol{\mu},\boldsymbol{\lambda},\boldsymbol{\gamma};\boldsymbol{z})\boldsymbol{\psi}_1\right)(\boldsymbol{x}) = \frac{2\mu}{\sqrt{\Delta_\gamma^{(2)}(x,z-\gamma\varepsilon(x))}}\int_{T^1}\frac{\sin(x-t)\psi_1(t)}{(E_\gamma(t,x-t)-z)\sqrt{\Delta_\mu^{(\gamma,1)}(t,z-\varepsilon(t))}}\boldsymbol{dt}.$$

The following Theorem can be proved similarly to Theorem 3 and establishes a relation between eigenvalues of $H_{\mu,\lambda}^{(\gamma)}$ and $\hat{T}_{\mu,\lambda}^{(\gamma)}(z)$.

Theorem 4. The number $z < \min \sigma_{\text{ess}}\left(H_{\mu,\lambda}^{(\gamma)}\right)$ is an eigenvalue of $H_{\mu,\lambda}^{(\gamma)}$ if and only if the number $\lambda = 1$ is an eigenvalue of $\hat{T}_{\mu,\lambda}^{(\gamma)}(z)$ and their multiplicities coincide.

Remark. The operator equation $\psi = \hat{T}_{\mu,\lambda}^{(\gamma)}(z)\psi$ is a symmetrized version of the Faddeev equation for the eigenfunctions of $H_{\mu,\lambda}^{(\gamma)}$.

Here

$$\psi = \hat{T}_{\mu,\lambda}^{(\gamma)}(z)\psi,$$
$$\psi = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}(\hat{\mathrm{T}}_{11}(\mu,\lambda,\gamma;z)\psi_1)(x) + (\hat{\mathrm{T}}_{12}(\mu,\lambda,\gamma;z)\psi_2)(x)\\ (\hat{\mathrm{T}}_{21}(\mu,\lambda,\gamma;z)\psi_1)(x)\end{pmatrix}.$$

As in the usual Faddeev equation, the methods of solving the system of linear integral equations are used to construct the symmetric Faddeev equation. Only in this case, the fact that the Fredholm determinant is a positive function for a given number z is used.

We notice that the Faddeev equation is very useful for the determine of the location of the essential spectrum of the operator $H_{\mu,\lambda}^{(\gamma)}$. Symmetric version of the Faddeev equation is an important for studying the discrete spectrum of the operator $H_{\mu,\lambda}^{(\gamma)}$.

4 Conclusion

In the present paper the three-particle lattice Hamiltonian associated to a system of three particles on the d-dimensional lattice is studied. Two bounded and self-adjoint so-called channel operators are defined. The spectrum of the constructed channel operators are described by the spectrum of the corresponding Friedrichs model. It is established that the essential spectrum of the considered Hamiltonian is equal to the union of the spectrum of the two channel operators. The Faddeev equation and its symmetric version for the eigenfunctions of the considered Hamiltonian are constructed. The latter equations are play key role when we investigate the essential and discrete spectrum.

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