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Structure of the numerical range of a Friedrichs model: 1D case with rank two perturbation

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Frisrixs modeli sonli tasvirining tuzilishi: rangi ikkiga teng qo'zg'alishli 1 o'lchamli hol

Ushbu maqolada chegaralangan va o'z-o'ziga qo'shma $\mathcal{A}(\mu_1, \mu_2)$, $\mu_1, \mu_2 > 0$ Fridrixs modeli qaraladi va rangi ikkiga teng qo'zg'alishli 1 o'lchamli hol tahlil qilinadi. Odatda bunday modellar 1 o'lchamli panjaradagi ikkita kvant zarrachalar sistemasiga mos keladi. $\mathcal{A}(\mu_1, \mu_2)$ operatorning sonli tasviri μ_1 va μ_2 parametrlarda nisbatan tadqiq qilinadi. μ_α, $\alpha = 1, 2$ parametrlarning $\mathcal{A}(\mu_1, \mu_2)$ operator spektri va sonli tasviri ustma-ust tushishini ta'minlaydigan kritik qiymati topiladi.

Kalit so'zlar: Fridrixs modeli; qo'zg'alish; kvant zarrachalar; lokal bo'lmagan ta'sirlashish operatori; sonli tasvir; spektral munosabat; xos qiymat.

Структура числовой области значений модели Фридрихса: одномерный случай с двумерным возмущением

В настоящей статьи рассматривается ограниченная и самосопряженная модель Фридрихса $\mathcal{A}(\mu_1, \mu_2)$, $\mu_1, \mu_2 > 0$ и обсуждается одномерный случай с двумерным возмущением. Обычно такие модели ассоциированы с системой двух квантовых частиц в одномерной решетке. Исследуется числовая область значения оператора $\mathcal{A}(\mu_1, \mu_2)$ относительно параметров μ_1 и μ_2. Найдем критическое значение параметра μ_α, $\alpha = 1, 2$ гарантирующий совпадаемость спектра и числовой области значений оператора $\mathcal{A}(\mu_1, \mu_2)$.

Ключевые слова: модель Фридрихса; возмущения; квантовые частицы; нелокальный оператор взаимодействия; числовой область значения; спектральное включение; собственное значение.

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The numerical range and its useful properties

The numerical range is an important tool in the spectral analysis of bounded and unbounded linear operators in Hilbert spaces. For the reader's convenience, we begin by collecting some of its useful properties (see e.g. [1, 2]). Let $\mathcal{H}$ be a complex Hilbert space and let T be a bounded linear operator in $\mathcal{H}$. Then the numerical range of T is the set

$$W(T) := \{(Tx, x) : x \in \mathcal{H}, \|x\| = 1\},$$

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where $(\cdot,\cdot)$ and $\|\cdot\|$ are the inner product and norm on Hilbert space $\mathcal{H}$, respectively. Thus the numerical range of an T, like the spectrum, is a subset of the complex plane whose geometrical properties should say something about that operator. This notion was first studied by O.Toeplitz in [3]; he proved that the numerical range of a matrix contains all its eigenvalues and that its boundary is a convex curve. In [4] F.Hausdorff showed that indeed the set $W(T)$ is convex. In fact, it turned out that this continues to hold for general bounded linear operators and that the spectrum is contained in the closure $\overline{W(T)}$ (see [5]).

For $\alpha \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$ we set

$$\alpha\Omega := \{\alpha z :\ z \in \Omega\}, \quad \alpha + \Omega := \{\alpha + z :\ z \in \Omega\}.$$

In the following we formulate some properties of $W(T)$ which are immediate. For a bounded linear operator T on a Hilbert space $\mathcal{H}$:

(i) $W(T) \subset \{\lambda \in \mathbb{C} :\ |\lambda| \leq \|T\|\}$;

(ii) $W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$;

(iii) $W(I) = \{1\}$, where I is an identity operator on $\mathcal{H}$. More generally, if α and β are complex numbers, then $W(\alpha T + \beta) = \alpha W(T) + \beta$;

(iv) If $T = T^*$, then $W(T) \subset \mathbb{R}$;

(v) If $\dim \mathcal{H} < \infty$, then $W(T)$ is compact;

(vi) If $S, T : \mathcal{H} \to \mathcal{H}$ are unitarily equivalent, then $W(S) = W(T)$;

(vii) The numerical range $W(T)$ of T satisfies the so-called *spectral inclusion property*

$$W(T) \subset \sigma_{\mathrm{p}}(T), \quad \overline{W(T)} \subset \sigma(T)$$

for the point spectrum $\sigma_{\mathrm{p}}(T)$ (or set of eigenvalues) and the spectrum $\sigma(T)$ of T.

The notion of numerical range is generalized by the different ways, see for example [6, 7, 8, 9, 10]. One important use of $W(T)$ is to bound the spectrum $\sigma(T)$. The spectrum of an operator T consists of those complex numbers λ such that $T - \lambda I$ is not invertible. For our purpose the spectrum of an operator is equal to its numerical range for some case, it is enough to look at the boundary of the spectrum.

It is well known that the boundary of the spectrum is contained in the *approximate point spectrum* $\sigma_{\mathrm{app}}(T)$ (see [2]), which consists of complex numbers λ for which there exists a sequence of unit vectors $\{f_n\}$ with

$$\|(T - \lambda I)f_n\| \to 0$$

as $n \to \infty$.

The following example shows that even for the bounded self-adjoint operator B in Hilbert space $\mathcal{H}$ we can not state $\sigma(B) \subset W(B)$ or $W(B) \subset \sigma(B)$. Let

$$B : l^2 \to l^2, \quad Bx = (x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, \ldots), \quad x = (x_1, x_2, \ldots, x_n, \ldots) \in l^2.$$

It is easy to see that

$$\sigma(B) = \overline{\Big\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\Big\}} = \Big\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\Big\}, \quad W(B) = (0, 1].$$

Here $0 \notin W(B)$, since the equality

$$(Bx, x) = \sum_{k=1}^{\infty} \frac{1}{n}|x_n|^2 = 0$$

implies $x = (0, 0, ...) \in l^2$. Then a natural question arises: Is there a bounded self-adjoint operator, which is differently from scalar, that its spectrum coincide with the numerical range? In this paper, we will try to answer to this question and give an example for such type operators.

In the present paper we consider a Hamiltonian (Friedrichs model) $\mathcal{A}(\mu_1, \mu_2)$ with rank two perturbation. This Hamiltonian is associated with a system of two quantum particles on a one-dimensional lattice. We study the numerical range of $\mathcal{A}(\mu_1, \mu_2)$ dependently on μ_α, $\alpha = 1, 2$. In particular, we find the critical value of μ_α, $\alpha = 1, 2$ under which the spectrum of $\mathcal{A}(\mu_1, \mu_2)$ coincides with its numerical range $W(\mathcal{A}(\mu_1, \mu_2))$. In [11] the structure of the closure of numerical range of a 2×2 operator matrix, associated with a system of at most two quantum particles on d− dimensional lattice, was investigated in detail by terms of matrix entries for all dimensions of the torus $\mathbb{T}^{\mathrm{d}}$. Some properties of the generalized Friedrichs model related with the numerical range was studied in [12]. Formula for the quadratic numerical range of the generalized Friedrichs model was obtained in [13].

Essential and discrete spectrum of a Friedrichs model

In this section we introduce a Friedrichs model $\mathcal{A}(\mu_1, \mu_2)$ and we state the assumptions which will be needed throughout the paper. Then we analyze the essential spectrum and discrete spectrum of $\mathcal{A}(\mu_1, \mu_2)$.

Let $\mathbb{T}^1$ be the one-dimensional torus and $L^2(\mathbb{T}^1)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^1$. We consider the bounded and self-adjoint Friedrichs model $\mathcal{A}(\mu_1, \mu_2)$ acting on the Hilbert space $L^2(\mathbb{T}^1)$ as

$$\mathcal{A}(\mu_1, \mu_2) := \mathcal{A}_0 - \mu_1 V_1 + \mu_2 V_2,$$

where $\mathcal{A}_0$ is the multiplication operator by the function $u(\cdot)$:

$$(\mathcal{A}_0 f)(x) = u(x) f(x),$$

and V_α, $\alpha = 1, 2$ are non-local interaction operators:

$$(V_\alpha f)(x) = v_\alpha(x) \int_{\mathbb{T}^1} v_\alpha(t) f(t) dt, \quad \alpha = 1, 2.$$

Here $f \in L^2(\mathbb{T}^1)$; $\mu_\alpha > 0$, $\alpha = 1, 2$ are positive reals, $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1, 2$ are real-valued continuous functions on $\mathbb{T}^1$.

Throughout this paper, we assume that the function $u(\cdot)$ has an unique minimum at the point $x_1 \in \mathbb{T}^1$ and has an unique maximum at the point $x_2 \in \mathbb{T}^1$, and for $\alpha = 1, 2$ the function $v_\alpha(\cdot)$ has the continuous partial derivatives up to the third-order inclusive at some neighborhood of $x_\alpha \in \mathbb{T}^1$. From now on we suppose that

$$\operatorname{mes}(\operatorname{supp}\{v_1(\cdot)\} \cap \operatorname{supp}\{v_2(\cdot)\}) = 0, \tag{1}$$

where $\operatorname{mes}(\cdot)$ is the Lebesgue measure on $\mathbb{R}$ and $\operatorname{supp}\{v_\alpha(\cdot)\})$ is the support of the function $v_\alpha(\cdot)$.

The following example shows that the class of functions $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1, 2$ satisfying above mentioned conditions is nonempty. To prove this fact we introduce the functions of the form:

$$\begin{aligned} & u(x) = 1 - \cos x, \\ & v_1(x) = \begin{cases} \sin(2x), & x \in (-\pi/2, \pi/2) \\ 0, & \text{othervice} \end{cases}; \\ & v_2(x) = \sin(2x) - v_1(x), \quad x \in \mathbb{T}^1. \end{aligned} \tag{2}$$

Then it is easy to check that for the function $u(\cdot)$ the points $x_1 = 0$ and $x_2 = \pi$ are extremal points. For $\alpha = 1, 2$ the function $v_\alpha(\cdot)$ is an analytic in the $\delta < \pi/2$-neighborhood

$$U_\delta(x_\alpha) := \{x \in \mathbb{T}^1 : |x - x_\alpha| < \delta\}$$

of the point x_α. Validness of the condition (1) follows from the construction of $v_\alpha(\cdot)$.

By the definition the perturbation $-\mu_1 V_1 + \mu_2 V_2$ of the operator $\mathcal{A}_0$ is a self-adjoint operator of rank two. Therefore, in accordance with the Weyl theorem about the invariance of the essential spectrum under the finite rank perturbations, the essential spectrum of the operator $\mathcal{A}(\mu_1, \mu_2)$ coincides with the spectrum of $\mathcal{A}_0$:

$$\sigma_{\text{ess}}(\mathcal{A}(\mu_1, \mu_2)) = \sigma(\mathcal{A}_0) = [m_1; m_2],$$

where the numbers m_1 and m_2 are defined by

$$m_1 := \min_{x \in \mathbb{T}^1} u(x), \quad m_2 := \max_{x \in \mathbb{T}^1} u(x).$$

In order to study the spectral properties of the operator $\mathcal{A}(\mu_1, \mu_2)$, we introduce the following two bounded self-adjoint operators (Friedrichs model with rank one perturbation) $\mathcal{A}_\alpha(\mu_\alpha)$, acting on $L^2(\mathbb{T}^1)$ by the rule

$$\mathcal{A}_\alpha(\mu_\alpha) := \mathcal{A}_0 + (-1)^\alpha \mu_\alpha V_\alpha, \quad \alpha = 1, 2. \tag{3}$$

Let $\mathbb{C}$ be the field of complex numbers. We define an analytic function $\Delta_\alpha(\mu_\alpha;\, \cdot)$ (the Fredholm determinant associated with the operator $\mathcal{A}_\alpha(\mu_\alpha)$) in $\mathbb{C} \setminus [m_1; m_2]$ by

$$\Delta_\alpha(\mu_\alpha;\, z) := 1 + (-1)^\alpha \mu_\alpha \int_{\mathbb{T}^1} \frac{v_\alpha^2(t) dt}{u(t) - z}.$$

Then the Birman-Schwinger principle and the Fredholm theorem imply that [14, 15] the operator $\mathcal{A}_\alpha(\mu_\alpha)$ has an eigenvalue $z_\alpha \in \mathbb{C}\setminus[m_1;m_2]$ if and only if $\Delta_\alpha(\mu_\alpha;\, z_\alpha)=0$. From here it follows that for the discrete spectrum of $\mathcal{A}_\alpha(\mu_\alpha)$ the equality

$$\sigma_{\rm disc}(\mathcal{A}_\alpha(\mu_\alpha) = \{z\in\mathbb{C}\setminus[m_1;m_2]:\, \Delta_\alpha(\mu_\alpha;\, z)=0\} \tag{4}$$

holds.

The following lemma establishes a connection between of eigenvalues of $\mathcal{A}(\mu_1,\mu_2)$ and $\mathcal{A}_\alpha(\mu_\alpha)$, $\alpha=1,2$.

Lemma. *The number $z\in\mathbb{C}\setminus[m_1;m_2]$ is an eigenvalue of $\mathcal{A}(\mu_1,\mu_2)$ if and only if z is an eigenvalue one of the operators $\mathcal{A}_\alpha(\mu_\alpha)$, $\alpha=1,2$.*

Proof. Let the number $z\in\mathbb{C}\setminus[m_1;m_2]$ be an eigenvalue of $\mathcal{A}(\mu_1,\mu_2)$ and $f\in L^2(\mathbb{T}^1)$ be the corresponding eigenfunction. Then f satisfy the equation

$$(u(x)-z)f(x)-\mu_1 v_1(x)\int_{\mathbb{T}^1} v_1(t)f(t)dt+\mu_2 v_2(x)\int_{\mathbb{T}^1} v_2(t)f(t)dt=0. \tag{5}$$

It is easy to see that for any $z\in\mathbb{C}\setminus[m_1;m_2]$ the relation $u(x)-z\neq 0$ holds for all $x\in\mathbb{T}^1$. Then the equation (5) implies

$$f(x)=\frac{\mu_1 v_1(x)C_1-\mu_2 v_2(x)C_2}{u(x)-z}, \tag{6}$$

where

$$C_\alpha := \int_{\mathbb{T}^1} v_\alpha(t)f(t)dt, \quad \alpha=1,2. \tag{7}$$

Substituting the expression (6) for f into (7) and using the condition (1) we conclude that the equation (5) has a nonzero solution if and only if the system of equations

$$\Delta_\alpha(\mu_\alpha;\, z)C_\alpha=0, \quad \alpha=1,2$$

has a nonzero solution, i.e., if the condition $\Delta_1(\mu_1;\, z)\Delta_2(\mu_2;\, z)=0$ holds. If we set $v_\alpha(x)\equiv 0$, then by the definitions of $\mathcal{A}(\mu_1,\mu_2)$ and $\mathcal{A}_\alpha(\mu_\alpha)$ we obtain that $\mathcal{A}(\mu_1,\mu_2)=\mathcal{A}_\beta(\mu_\beta)$ for $\alpha\neq\beta$. Now the equality (1) completes the proof. □

By Lemma, the discrete spectrum of $\mathcal{A}(\mu_1,\mu_2)$ and $\mathcal{A}_\alpha(\mu_\alpha)$, $\alpha=1,2$ are connected by the equality

$$\sigma_{\rm disc}(\mathcal{A}(\mu_1,\mu_2))=\sigma_{\rm disc}(\mathcal{A}_1(\mu_1))\cup\sigma_{\rm disc}(\mathcal{A}_2(\mu_2)).$$

We note that the operators $\mathcal{A}_\alpha(\mu_\alpha)$, $\alpha=1,2$ have a structure simpler than that of $\mathcal{A}(\mu_1,\mu_2)$, and therefore, the latter equality plays an important role in futher investigating the spectrum and numerical range of $\mathcal{A}(\mu_1,\mu_2)$.

Analysis of the numerical range of $\mathcal{A}(\mu_1,\mu_2)$

In this section we investigate the numerical range of $\mathcal{A}(\mu_1,\mu_2)$ with respect to the parameters μ_1 and μ_2. Our approach is based on the so-called threshold analysis method [14, 15].

Henceforth, we shall denote by $C_1, C_2, C_3>0$ and $\delta>0$ different positive numbers.

We note that if $v_\alpha(x_\alpha)=0$, then from the condition on $v_\alpha(\cdot)$ it follows that there exist positive numbers C_1, C_2 and δ such that the inequalities

$$C_1|x-x_\alpha|^{n_\alpha}\le|v_\alpha(x)|\le C_2|x-x_\alpha|^{n_\alpha}, \quad x\in U_\delta(x_\alpha) \tag{8}$$

hold for some $n_\alpha\in\mathbb{N}$. For example, if the function $v_\alpha(\cdot)$ is defined by (2), then $n_\alpha=1$ for $\alpha=1$. Since the function $u(\cdot)$ has an unique minimum at the point x_1 and an unique maximum at the point x_2 there exist $C_1, C_2>0$ and $\delta>0$ such that

$$C_1|x-x_\alpha|^2\le|u(x)-m_\alpha|\le C_2|x-x_\alpha|^2,\ x\in U_\delta(x_\alpha); \tag{9}$$

$$|u(x)-m_\alpha|\ge C_1, \quad x\notin U_\delta(x_\alpha). \tag{10}$$

Hence, if $v_\alpha(x_\alpha)=0$, then using the inequalities (8), (9) and (10) one can easily seen that the integral

$$\int_{\mathbb{T}^1}\frac{v_\alpha^2(t)dt}{|u(t)-m_\alpha|}$$

is positive and finite. In this case we set

$$\mu_\alpha^0 := (-1)^{\alpha+1} \left(\int_{\mathbb{T}^1} \frac{v_\alpha^2(t)dt}{u(t)-m_\alpha} \right)^{-1}, \quad \alpha=1,2.$$

For $\alpha=1,2$ we denote by $E_{\mu_\alpha}^{(\alpha)}$ an eigenvalue of $\mathcal{A}_\alpha(\mu_\alpha)$ (if exists).

Main result of the paper is the following theorem.

Theorem. (A) *Let $v_1(x_1)=0$ and $v_2(x_2)=0$. Then*
(A1) $\overline{W(\mathcal{A}(\mu_1,\mu_2))}=[m_1,m_2]$, *if* $\mu_\alpha \in (0,\mu_\alpha^0]$, $\alpha=1,2$; *moreover,* $m_\alpha \in W(\mathcal{A}(\mu_1,\mu_2))$, *if* $\mu_\alpha=\mu_\alpha^0$ *and* $n_\alpha \geq 2$ *in* (7);
(A2) $W(\mathcal{A}(\mu_1,\mu_2))=[E_{\mu_1}^{(1)},E_{\mu_2}^{(2)}]$ *with* $E_{\mu_1}^{(1)}<m_1$ *and* $E_{\mu_2}^{(2)}>m_2$, *if* $\mu_\alpha>\mu_\alpha^0$, $\alpha=1,2$;
(B) *Suppose $v_1(x_1)\neq 0$ and $v_2(x_2)\neq 0$. Then for any $\mu_\alpha>0$, $\alpha=1,2$ we have*

$$W(\mathcal{A}(\mu_1,\mu_2))=[E_{\mu_1}^{(1)},E_{\mu_2}^{(2)}],$$

where $E_{\mu_1}^{(1)}<m_1$ and $E_{\mu_2}^{(2)}>m_2$.

Proof. (A) Assume $v_1(x_1)=0$ and $v_2(x_2)=0$. First we discuss the case $\mu_\alpha \in (0,\mu_\alpha^0]$ for $\alpha=1,2$. Then from monotonicity property of $\Delta_\alpha(\mu_\alpha;\, z)$ by μ_α and z, and also from the definition of μ_α^0 we get

$$\Delta_1(\mu_1;\, z) \geq \Delta_1(\mu_1^0;\, z) > \Delta_1(\mu_1^0;\, m_1)=0 \text{ for } z<m_1;$$
$$\Delta_2(\mu_2;\, z) \leq \Delta_2(\mu_2^0;\, z) < \Delta_2(\mu_2^0;\, m_2)=0 \text{ for } z>m_2.$$

The latter two assertions means that the operator $\mathcal{A}_1(\mu_1)$ (resp. $\mathcal{A}_2(\mu_2)$) has no eigenvalues lying in $(-\infty,m_1)$ (resp. $(m_2,+\infty)$).

From the positivity of the operator V_1 it follows easily that the assertions

$$((\mathcal{A}_1(\mu_1)-z)g,g)=\int_{\mathbb{T}^1}(u(t)-z)|g(t)|^2dt-\mu_1(V_1g,g)<0$$

hold for any $\mu_1>0$, $z>m_2$ and $g\in L^2(\mathbb{T}^1)$. In the same manner one can see that the assertions

$$((\mathcal{A}_2(\mu_2)-z)g,g)=\int_{\mathbb{T}^1}(u(t)-z)|g(t)|^2dt+\mu_2(V_2g,g)>0$$

hold for any $\mu_2>0$, $z<m_1$ and $g\in L^2(\mathbb{T}^1)$. Then it is obvious that for all $\mu_2>0$ the operator $\mathcal{A}_2(\mu_2)$ has no eigenvalues lying on the l.h.s. of m_1 and for all $\mu_1>0$ the operator $\mathcal{A}_1(\mu_1)$ has no eigenvalues lying on the r.h.s. of m_2. Hence, the equality (4) and Lemma imply that the operator $\mathcal{A}(\mu_1,\mu_2)$ has no eigenvalues outside of $[m_1,m_2]$. Therefore,

$$\sigma(\mathcal{A}(\mu_1,\mu_2))=\sigma_{\rm ess}(\mathcal{A}(\mu_1,\mu_2))=[m_1;m_2]$$

for all $\mu_\alpha\in(0,\mu_\alpha^0]$, $\alpha=1,2$. Since $\mathcal{A}(\mu_1,\mu_2)$ is a bounded and self-adjoint, by the convexity of the numerical range we have $\overline{W(\mathcal{A}(\mu_1,\mu_2))}=[m_1,m_2]$.

Now let us consider the case $\mu_\alpha=\mu_\alpha^0$ and $n_\alpha\geq 2$ in (8) for $\alpha\in\{1,2\}$. We introduce the function of the form

$$f_\alpha(x)=(-1)^{\alpha+1}\frac{\mu_\alpha^0 v_\alpha(x)}{u(x)-m_\alpha} \tag{11}$$

and show that this function satisfies the equation $\mathcal{A}(\mu_1^0,\mu_2^0)f_\alpha=m_\alpha f_\alpha$:

$$\begin{aligned}(\mathcal{A}(\mu_1^0,\mu_2^0)-m_\alpha)f_\alpha(x)&=(u(x)-m_\alpha)(-1)^{\alpha+1}\frac{\mu_\alpha^0 v_\alpha(x)}{u(x)-m_\alpha}\\&-\mu_1^0 v_1(x)(-1)^{\alpha+1}\mu_\alpha^0\int_{\mathbb{T}^1}\frac{v_1(t)v_\alpha(t)dt}{u(t)-m_\alpha}\\&+\mu_2^0 v_2(x)(-1)^{\alpha+1}\mu_\alpha^0\int_{\mathbb{T}^1}\frac{v_2(t)v_\alpha(t)dt}{u(t)-m_\alpha}.\end{aligned}$$

Using the assumption (1) and definition of μ_α^0 we obtain

$$(\mathcal{A}(\mu_1^0,\mu_2^0)-m_\alpha)f_\alpha(x)=(-1)^{\alpha+1}\mu_\alpha^0 v_\alpha(x)\left[1+(-1)^\alpha\mu_\alpha^0\int_{\mathbb{T}^1}\frac{v_\alpha^2(t)dt}{u(t)-m_\alpha}\right]=0,\quad \alpha=1,2.$$

Next we prove that $f_\alpha \in L^2(\mathbb{T}^1)$. By the additivity property of the integral we have

$$\int_{\mathbb{T}^1} |f_\alpha(t)|^2 dt = (\mu_\alpha^0)^2 \int_{U_\delta(x_\alpha)} \frac{v_\alpha^2(t)dt}{(u(t)-m_\alpha)^2} + (\mu_\alpha^0)^2 \int_{\mathbb{T}^1\setminus U_\delta(x_\alpha)} \frac{v_\alpha^2(t)dt}{(u(t)-m_\alpha)^2}. \tag{12}$$

Then by (8) and (9) for the first summand on the right-hand side of (12) we obtain

$$\int_{U_\delta(x_\alpha)} \frac{v_\alpha^2(t)dt}{(u(t)-m_\alpha)^2} \le C_1 \int_{U_\delta(x_\alpha)} \frac{|t-x_\alpha|^{2n_\alpha}dt}{|t-x_\alpha|^4}.$$

The finiteness of the latter integral follows from the condition $n_\alpha \ge 2$.

It follows from the continuity of the function $v_\alpha(\cdot)$ on the compact set $\mathbb{T}^1$ and (10) that

$$\int_{\mathbb{T}^1\setminus U_\delta(x_\alpha)} \frac{v_\alpha^2(t)dt}{(u(t)-m_\alpha)^2} \le C_1 \int_{\mathbb{T}^1\setminus U_\delta(x_\alpha)} dt < +\infty.$$

So,$f_\alpha \in L^2(\mathbb{T}^1)$. Therefore, $m_\alpha \in W(\mathcal{A}(\mu_1^0,\mu_2^0))$ for $\alpha=1,2$, that is,

$$W(\mathcal{A}(\mu_1^0,\mu_2^0)) = [m_1; m_2]$$

under the assumption $n_\alpha \ge 2$ in (8).

If $\mu_\alpha > \mu_\alpha^0$ for $\alpha=1,2$, then it is evident that

$$\Delta_1(\mu_1;\, m_1) < \Delta_1(\mu_1^0;\, m_1) = 0, \quad \Delta_2(\mu_2;\, m_2) > \Delta_2(\mu_2^0;\, m_2) = 0.$$

Taking into account the last two facts and the equalities

$$\lim_{z\to-\infty} \Delta_1(\mu_1;\, z) = +\infty, \quad \lim_{z\to+\infty} \Delta_2(\mu_2;\, z) = -\infty,$$

we conclude that there exist the points $E_{\mu_1}^{(1)} \in (-\infty, m_1)$ and $E_{\mu_2}^{(2)} \in (m_1, +\infty)$ such that $\Delta_1(\mu_1;\, E_{\mu_1}^{(1)}) = 0$ and $\Delta_2(\mu_2;\, E_{\mu_2}^{(2)}) = 0$. By the equality (4) it means that the numbers $E_{\mu_1}^{(1)}$ and $E_{\mu_2}^{(2)}$ are the eigenvalues of $\mathcal{A}_1(\mu_1)$ and $\mathcal{A}_2(\mu_2)$, respectively, and hence, by Lemma they are eigenvalues of $\mathcal{A}(\mu_1,\mu_2)$. We denote by $f_\alpha \in L^2(\mathbb{T}^1)$ the corresponding eigenfunction with $\|f_\alpha\| = 1$. Then

$$E_{\mu_\alpha}^{(\alpha)} = (\mathcal{A}(\mu_1,\mu_2)f_\alpha, f_\alpha),$$

that is, $E_{\mu_\alpha}^{(\alpha)} \in W(\mathcal{A}(\mu_1,\mu_2))$ for $\alpha=1,2$. Now the equalities

$$E_{\mu_1}^{(1)} = \min_{\|f\|=1} (\mathcal{A}(\mu_1,\mu_2)f,f), \quad E_{\mu_2}^{(2)} = \max_{\|f\|=1} (\mathcal{A}(\mu_1,\mu_2)f,f)$$

completes the proof of part (A) of Theorem.

(B) Let $v_\alpha(x_\alpha) \ne 0$, $\alpha=1,2$. Since the function $v_\alpha(\cdot)$ is a continuous on a closed set $\mathbb{T}^1$, there exist positive numbers C_1 and δ such that

$$|v_\alpha(x)| \ge C_1, \quad x \in U_\delta(x_\alpha).$$

Then using the inequality (9) one can easily seen that the integral

$$\int_{\mathbb{T}^1} \frac{v_\alpha^2(t)dt}{|u(t)-m_\alpha|} \ge C_1 \int_{U_\delta(x_\alpha)} \frac{dt}{|t-x_\alpha|^4} = +\infty.$$

The Lebesgue dominated convergence theorem yields

$$\lim_{z\to m_1-0} \Delta_1(\mu_1\,; z) = -\infty, \quad \lim_{z\to m_2+0} \Delta_2(\mu_2\,; z) = +\infty.$$

Now analysis similar to that in the proof of part (A2) of Theorem completes the proof of part (B). Theorem 1 is completely proved. □

From Theorem it follows the following assertion.

Corollary. (A) *If $v_1(x_1)=0$ and $v_2(x_2)\ne 0$, then for any $\mu_2>0$ we have*
(A1) $\overline{W(\mathcal{A}(\mu_1,\mu_2))} = [m_1, E_{\mu_2}^{(2)}]$ *with* $E_{\mu_2}^{(2)} > m_2$, *if* $\mu_1 \in (0,\mu_1^0]$; *moreover, the claim* $m_1 \in W(\mathcal{A}(\mu_1,\mu_2))$ *holds,*

if $\mu_1 = \mu_1^0$ *and* $n_1 \geq 2$ *in* (7)*;*
(A2) $W(\mathcal{A}(\mu_1, \mu_2)) = [E_{\mu_1}^{(1)}, E_{\mu_2}^{(2)}]$ *with* $E_{\mu_1}^{(1)} < m_1$ *and* $E_{\mu_2}^{(2)} > m_2$*, if* $\mu_1 > \mu_1^0$*;*
(B) *If* $v_1(x_1) \neq 0$ *and* $v_2(x_2) = 0$*, then for any* $\mu_1 > 0$ *we have*
(B1) $\overline{W(\mathcal{A}(\mu_1, \mu_2))} = [E_{\mu_1}^{(1)}, m_2]$ *with* $E_{\mu_1}^{(1)} < m_1$*, if* $\mu_2 \in (0, \mu_2^0)$*; moreover the claim* $m_2 \in W(\mathcal{A}(\mu_1, \mu_2))$ *holds, if* $\mu_2 = \mu_2^0$ *and* $n_2 \geq 2$ *in (7);*
(B2) $W(\mathcal{A}(\mu_1, \mu_2)) = [E_{\mu_1}^{(1)}, E_{\mu_2}^{(2)}]$ *with* $E_{\mu_1}^{(1)} < m_1$ *and* $E_{\mu_2}^{(2)} > m_2$*, if* $\mu_2 > \mu_2^0$*.*

We remark that at the first sight, the convexity of the numerical range seems to be useful property, e.g. to show that the spectrum of an operator lies in a half plane. However, the numerical range often gives a poor localization of the spectrum and it cannot capture finer structures such as the separation of the spectrum in two or three parts. The spectrum of the Friedrichs model $\mathcal{A}(\mu_1, \mu_2)$ may consist of a union of one, two, or three sets. The present paper is devoted to the study of numerical range in the case where the spectrum of $\mathcal{A}(\mu_1, \mu_2)$ is a purely essential. Mainly we analyzed the problem of whether boundary points of the essential spectrum of $\mathcal{A}(\mu_1, \mu_2)$ belong to its numerical range.

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