On the eigenvalues of the lattice spin-boson model with at most one photon

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Abstract. In the present paper we consider a lattice spin-boson model with at most one photon A, which has a 2×2 block operator matrix representation. The essential spectrum of A is analyzed. We prove that the operator matrix A has four eigenvalues. We consider the case where the special integral is an infinite. The existence condition of the eigenvalues lying in and out of the essential spectrum are found. The results presented in this paper plays an important role when we study the location of the two-particle and three-particle branches of the essential spectrum of the lattice spin-boson Hamiltonian with at most two photons, and also to showing the finiteness of the number of its eigenvalues.

1 Introduction

Block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1, 2]. One special class of block operator matrices are Hamiltonians associated with systems of non-conserved number of quasi-particles on a lattice. Their number can be unbounded as in the case of spin-boson models or bounded as in the case of "truncated" spin-boson models. They arise, for example, in the theory of solid-state physics [3], quantum field theory [4] and statistical physics [5, 6].

In [7], the generalized Friedrichs model H_1 acting in the two-particle subspace of the Fock space is considered. The well-known Faddeev operator corresponding to H_1 is constructed and the most important its properties, which are related with the number of discrete eigenvalues are strongly proved. In the paper [8] the Faddeev equation for the eigenvectors of a 2×2 operator matrix H_2 is constructed and some important properties of this equation, related with the number of eigenvalues, are studied. In [9] an operator matrix A_μ, $\mu > 0$ of order 2 is discussed. It is acting in the direct sum of one-particle and two-particle subspaces of the bosonic Fock space and here shown that for some value of the coupling constant the operator matrix A_μ has infinitely many eigenvalues on the both sides of its essential spectrum.

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In the paper [10] for a lattice spin-boson model A_2 with fixed atom and at most two photons the first Schur complement $S_1(\lambda)$ with spectral parameter λ corresponding to A_2 is constructed. Some properties of $S_1(\lambda)$ are studied. In [11], [12] for a 2×2 operator matrix the existence of the critical value of the coupling constant for which operator matrix has an infinite number of eigenvalues is shown. In [13] the two-particle and three-particle branches of the essential spectrum of a 2×2 operator matrix is described. In [14], [15] for a family of 2×2 operator matrices $\mathrm{A}_\mu(k)$, $k\in T^3:=(-\pi,\pi]^3$, $\mu>0$, it is proven that there is a value μ_0 such that only for $\mu=\mu_0$ the operator $\mathrm{A}_\mu(\overline{0})$ and $\mathrm{A}_\mu(\overline{\pi})$ has virtual level at the point $z=0=\min\sigma_{\mathrm{ess}}(\mathrm{A}_\mu(\overline{0}))$ and $z=18=\max\sigma_{\mathrm{ess}}(\mathrm{A}_\mu(\overline{\pi}))$, respectively, where $\overline{0}:=(0,0,0),\overline{\pi}:=(\pi,\pi,\pi)\in T^3$.

In [16] the position and structure of two-particle as well three-particle branches (subsets) of essential spectrum of a family of operator matrices $H(K)$, $K\in T^3:=(-\pi,\pi]$ are investigated. In [17] for some finite set $\Lambda\subset T^3$ the existence of infinitely many eigenvalues of $H(K)$ for all $K\in\Lambda$ is proved, if the associated Friedrichs model has a zero energy resonance. In [18] an analytic description of the essential spectrum of $H(K)$ is established.

In [19] for a matrix model A the relations for the spectrum, essential spectrum, and point spectrum are established. The paper [20] for a 4×4 block operator matrix A_3 the position of the $\sigma_{\mathrm{ess}}(\mathrm{A}_3)$ is described and two-particle, three-particle and four-particle branches are analyzed. In [21] for a operator matrix A the location of the essential spectrum of the operator A is described. It is established that the essential spectrum of the operator matrix A consists of the union of the no more than 14 closed intervals and there are no more than 16 simple eigenvalues.

We remark that the above mentioned properties for the two-particle and three-particle lattice model operators are discussed by many authors, see for example [22], [23], [24], [25], [26], [27]. In particular, spectral properties of these model operators related with the threshold energy resonances, threshold eigenvalues, existence of the eigenvalues located in and outside of the essential spectrum, and also relation between spectrum and numerical range are studied.

2 Truncated spin-boson models on a lattice

Let T^{d} be the d-dimensional torus, $\mathrm{H}_0:=C$ be the set of all complex numbers, $\mathrm{H}_1:=L_2(T^{\mathrm{d}})$ be the Hilbert space of square integrable (complex) functions defined on T^{d}, $\mathrm{H}_2:=L_2^{\mathrm{sym}}((T^{\mathrm{d}})^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(T^{\mathrm{d}})^2$ and

$$\mathrm{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}})):=\mathrm{H}_0\oplus\mathrm{H}_1;$$
$$\mathrm{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}})):=\mathrm{H}_0\oplus\mathrm{H}_1\oplus\mathrm{H}_2;$$
$$L_m:=C^2\otimes\mathrm{F}_{\mathrm{b}}^{(n)}(L_2(T^{\mathrm{d}})),\quad n=1,2.$$

For $n=1,2$ the Hilbert space $\mathrm{F}_{\mathrm{b}}^{(n)}(L_2(T^{\mathrm{d}}))$ is called $n+1$-particle cut subspace of the symmetric Fock space $\mathrm{F}_{\mathrm{b}}(L_2(T^{\mathrm{d}}))$ for bosons.

For $n=1,2$ we write elements F_n of the space L_n in the form

$$F_1 = \{f_0^{(s)}, f_1^{(s)}(k_1); s=\pm\}; \quad F_2 = \{f_0^{(s)}, f_1^{(s)}(k_1), f_2^{(s)}(k_1,k_2); s=\pm\}.$$

The norm in L_n, $n=1,2$ is given by

$$\| F_1 \|^2 := \sum_{s=\pm}\left(| f_0^{(s)} |^2 + \int_{T^d} | f_1^{(s)}(k_1) |^2 \, dk_1 \right);$$

$$\| F_2 \|^2 := \sum_{s=\pm}\left(| f_0^{(s)} |^2 + \int_{T^d} | f_1^{(s)}(k_1) |^2 \, dk_1 + \int_{(T^d)^2} | f_2^{(s)}(k_1,k_2) |^2 \, dk_1 dk_2 \right).$$

We recall that the lattice spin-boson model with at most n ($n=1,2$) photons (truncated spin-boson model) A_n is acting in L_n as the $(n+1)\times(n+1)$ block operator matrix of the form

$$A_1 := \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^* & A_{11} \end{pmatrix}, \quad A_2 := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22} \end{pmatrix},$$

where matrix elements A_{ij} are defined by

$$A_{00} f_0^{(s)} = s\varepsilon f_0^{(s)}, \quad A_{01} f_1^{(s)} = \alpha \int_{T^d} v(t) f_1^{(-s)}(t) dt,$$

$$(A_{11} f_1^{(s)})(k_1) = (s\varepsilon + w(k_1)) f_1^{(s)}(k_1), \quad (A_{12} f_2^{(s)})(k_1) = \alpha \int_{T^d} v(t) f_2^{(-s)}(k_1,t) dt,$$

$$(A_{22} f_2^{(s)})(k_1,k_2) = (s\varepsilon + w(k_1) + w(k_2)) f_2^{(s)}(k_1,k_2), \quad \{f_0^{(s)}, f_1^{(s)}, f_2^{(s)}; s=\pm\} \in L_2.$$

Here A_{ij}^* denotes the adjoint operator to A_{ij} for $i<j$ with $i,j=0,1,2$; $w(k)$ is the dispersion of the free field, $\alpha v(k)$ is the coupling between the atoms and the field modes, $\alpha > 0$ is a real number, so-called the coupling constant, real number. We assume that $v(\cdot)$ and $w(\cdot)$ are the real-valued continuous functions on T^d.

Under these assumptions for $n=1,2$ the lattice spin-boson model with at most n photons A_n is bounded and self-adjoint in the complex Hilbert space L_n.

The standard spin-boson model with at most n, $n=1,2$ photons was completely studied in [6] for small values of the parameter (so-called coupling constant) α.

In [28] the essential spectrum of the operator matrix A_2 is described (without proof) and it is shown that the operator matrix A_2 has finitely many eigenvalues below the bottom of its essential spectrum for any values of the coupling constant α.

In [29] the number and location of the eigenvalues is determined and the corresponding eigenvectors of the operator matrix A_1 is studied. It is shown that there are no eigenvalues in a gap (i.e., between the branches of the essential spectrum) in the essential spectrum of the operator matrix A_1. The existence of the eigenvalues in the essential spectrum (i.e., embedded eigenvalues) of the operator matrix A_1 is proved. The location of the essential of the operator matrix A_2 is described. It is proven that the essential spectrum of the operator matrix A_2 is a union of at most six intervals. The structure of the essential spectrum of the operator matrix A_2 is studied and estimated its lower bound.

3 Spectrum of the block operator matrix A_1

To study the spectral properties of the block operator matrix A_1, we also consider the bounded self-adjoint operators $\mathrm{A}_1^{(\mathrm{s})}$, $\mathrm{s}=\pm$, acting in the Hilbert space $\mathrm{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}}))$ as the 2×2 block operator matrices

$$\mathrm{A}_1^{(\mathrm{s})} := \begin{pmatrix} \hat{\mathrm{A}}_{00}^{(\mathrm{s})} & \hat{\mathrm{A}}_{01} \\ \hat{\mathrm{A}}_{01}^{*} & \hat{\mathrm{A}}_{11}^{(\mathrm{s})} \end{pmatrix}$$

with the elements

$$\hat{\mathrm{A}}_{00}^{(\mathrm{s})} f_0 = \mathrm{s}\varepsilon f_0, \quad \hat{\mathrm{A}}_{01} f_1 = \alpha \int_{T^{\mathrm{d}}} v(t) f_1(t)dt,$$

$$(\hat{\mathrm{A}}_{11}^{(\mathrm{s})} f_1)(k_1) = (-\mathrm{s}\varepsilon + w(k_1)) f_1(k_1), \quad (f_0, f_1) \in \mathrm{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}})).$$

Using the elements of the Functional Analysis course it is easy to see that

$$(\hat{\mathrm{A}}_{01}^{*} f_0)(k_1) = \alpha v(k_1) f_0, \quad f_0 \in \mathrm{H}_0.$$

The operator $\hat{\mathrm{A}}_{01}$ is called annihilation operator, and the operator $\hat{\mathrm{A}}_{01}^{*}$ is called creation operator [4]. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

In what follows, $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, $\sigma_{\mathrm{pp}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$ are the respective spectrum, essential spectrum, point spectrum, and discrete spectrum of a bounded self-adjoint operator.

We establish the connection between the spectra of the block operator matrices A_1 and $\mathrm{A}_1^{(\mathrm{s})}$, $\mathrm{s}=\pm$. For the spectrum of the block operator matrix A_1 the equality $\sigma(\mathrm{A}_1) = \sigma(\mathrm{A}_1^{(+)}) \cup \sigma(\mathrm{A}_1^{(-)})$ holds. Moreover,

$$\sigma_{\mathrm{ess}}(\mathrm{A}_1) = \sigma_{\mathrm{ess}}(\mathrm{A}_1^{(+)}) \cup \sigma_{\mathrm{ess}}(\mathrm{A}_1^{(-)}),$$

$$\sigma_{\mathrm{p}}(\mathrm{A}_1) = \sigma_{\mathrm{p}}(\mathrm{A}_1^{(+)}) \cup \sigma_{\mathrm{p}}(\mathrm{A}_1^{(-)}).$$

Depending on the value of the coupling constant α the part of the discrete spectrum $\sigma_{\mathrm{disc}}(\mathrm{A}_1^{(\mathrm{s})})$ of the block operator matrix $\mathrm{A}_1^{(\mathrm{s})}$ can be located in the essential spectrum $\sigma_{\mathrm{ess}}(\mathrm{A}_1^{(-\mathrm{s})})$ of the block operator matrix $\mathrm{A}_1^{(\mathrm{s})}$. Hence, we have the relations

$$\sigma_{\mathrm{disc}}(\mathrm{A}_1) \subseteq \sigma_{\mathrm{disc}}(\mathrm{A}_1^{(+)}) \cup \sigma_{\mathrm{disc}}(\mathrm{A}_1^{(-)}),$$

$$\sigma_{\mathrm{disc}}(\mathrm{A}_1) = \{\sigma_{\mathrm{disc}}(\mathrm{A}_1^{(+)}) \cup \sigma_{\mathrm{disc}}(\mathrm{A}_1^{(-)})\} \setminus \sigma_{\mathrm{ess}}(\mathrm{A}_1).$$

More precisely,

$$\sigma_{\mathrm{disc}}(\mathrm{A}_1) = \bigcup_{\mathrm{s}=\pm} \{\sigma_{\mathrm{disc}}(\mathrm{A}_1^{(\mathrm{s})}) \setminus \sigma_{\mathrm{ess}}(\mathrm{A}_1^{(-\mathrm{s})})\}.$$

We remark that for $\mathrm{s}=\pm$ the block operator matrix $\mathrm{A}_1^{(\mathrm{s})}$ has a simpler structure than A_1, and hence, the latter facts are play important role in further investigations of the spectrum of the block operator matrix A_1.

In the following we study the essential and discrete spectra of the block operator matrix $\mathrm{A}_1^{(\mathrm{s})}$, $\mathrm{s}=\pm$, in detail. For each $\mathrm{s}=\pm$, we consider the block operator matrix $\mathrm{A}_{1,0}^{(\mathrm{s})}$ acting in the Hilbert space $\mathrm{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}}))$ as

$$\mathrm{A}_{1,0}^{(\mathrm{s})} := \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{A}_1^{(\mathrm{s})} \end{pmatrix}.$$

It is easy to show that the perturbation operator $A_1^{(s)} - A_{1,0}^{(s)}$ of the block operator matrix $A_{1,0}^{(s)}$ is a rank two self-adjoint block operator matrix. By the well-known Weyl theorem on the invariance of the essential spectrum under finite rank perturbations, the essential spectrum of the block operator matrix $A_1^{(s)}$ coincides with essential spectrum of the block operator matrix $A_{1,0}^{(s)}$. Since the block operator matrix $A_{1,0}^{(s)}$ is the diagonal block operator matrix for its essential and discrete spectrum we obtain the following equalities

$$\sigma_{\text{disc}}(A_{1,0}^{(s)}) = \{0\}, \quad \sigma_{\text{ess}}(A_{1,0}^{(s)}) = [-s\varepsilon + m, -s\varepsilon + M],$$

where the numbers m and M are defined by

$$m := \min_{k_1 \in T^d} w(k_1), \quad M := \min_{k_1 \in T^d} w(k_1).$$

It follows that

$$\sigma_{\text{ess}}(A_1^{(s)}) = [-s\varepsilon + m, -s\varepsilon + M].$$

Hence, we have the following equality for the essential spectrum of the block operator matrix A_1:

$$\sigma_{\text{ess}}(A_1) = [-\varepsilon + m, -\varepsilon + M] \cup [\varepsilon + m, \varepsilon + M].$$

One can see that if $\varepsilon > (M - m)/2$, then the essential spectrum of the block operator matrix A_1 has a gap $(-\varepsilon + M, \varepsilon + m)$.

To determine and study the eigenvalues of the block operator matrix $A_1^{(s)}$ as well A_1, we define a regular function in the domain $C \setminus [-s\varepsilon + m, -s\varepsilon + M]$ as

$$\Delta^{(s)}(z) := s\varepsilon - z - \alpha^2 \int_{T^d} \frac{v^2(t)dt}{-s\varepsilon + w(t) - z}.$$

Usually the function $\Delta^{(s)}(\cdot)$ is called the Fredholm determinant associated with the block operator matrix $A_1^{(s)}$.

The following lemma establishes a relation between the eigenvalues of the block operator matrix $A_1^{(s)}$ and the zeroes of the Fredholm determinant $\Delta^{(s)}(\cdot)$.

Lemma 3.1. *The number* $z^{(s)} \in C \setminus \sigma_{\text{ess}}(A_1^{(s)})$ *is an eigenvalue of the block operator matrix* $A_1^{(s)}$ *if and only if* $\Delta^{(s)}(z^{(s)}) = 0$.

From Lemma 3.1 for the discrete spectrum of the block operator matrix $A_1^{(s)}$ we have the equality

$$\sigma_{\text{disc}}(A_1^{(s)}) = \left\{ z \in C \setminus \sigma_{\text{ess}}(A_1^{(s)}) : \Delta^{(s)}(z) = 0 \right\}.$$

Then using the fact that the operator A_1 has a diagonal block operator matrix structure one can conclude that

$$\sigma_{\text{disc}}(A_1) = \left\{ z \in C \setminus \sigma_{\text{ess}}(A_1) : \Delta^{(+)}(z)\Delta^{(-)}(z) = 0 \right\}.$$

To represent some results about existence of the eigenvalues of the block operator matrix A_1 within this section we assume that $m = 0$ and

$$\int_{T^d} \frac{v^2(t)dt}{w(t)} < \infty, \quad \int_{T^d} \frac{v^2(t)dt}{M - w(t)} < \infty \tag{1}$$

and we set

$$\alpha_1 := \sqrt{M + 2\varepsilon} \left(\int_{T^d} \frac{v^2(t)dt}{M - w(t)} \right)^{-1/2}, \quad \alpha_2 := \sqrt{2\varepsilon} \left(\int_{T^d} \frac{v^2(t)dt}{w(t)} \right)^{-1/2},$$

$$\alpha_3 := \sqrt{M}\left(\int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{2\varepsilon + M - w(t)}\right)^{-1/2}.$$

Lemma 3.2. *The following statements hold:*

(a) *For all $\alpha > 0$, the block operator matrix $\mathrm{A}_1^{(-)}$ has a unique simple eigenvalue to the left of $-\varepsilon$. If $\alpha \in (0, \alpha_1]$, then the block operator matrix $\mathrm{A}_1^{(-)}$ does not have any eigenvalues to the right of $M+\varepsilon$. for $\alpha > \alpha_1$ the block operator matrix $\mathrm{A}_1^{(-)}$ has a unique simple eigenvalue to the right of $M+\varepsilon$.*

(b) *If $\alpha \in (0, \min\{\alpha_2, \alpha_3\}]$, then the block operator matrix $\mathrm{A}_1^{(+)}$ does not have any eigenvalues to the left of $-\varepsilon$ and to the right of $M+\varepsilon$. For $\alpha > \max\{\alpha_2, \alpha_3\}$, the block operator matrix $\mathrm{A}_1^{(+)}$ has one simple eigenvalue to the left of $-\varepsilon$ and and one to the right of $M+\varepsilon$.*

Let us introduce the following quantities

$$\alpha_{\min} := \min\{\alpha_1, \alpha_2, \alpha_3\}, \quad \alpha_{\max} := \max\{\alpha_1, \alpha_2, \alpha_3\}.$$

The following statement is a consequence of Lemma 3.2.

Corollary 3.3. *For all $\alpha > 0$ the block operator matrix A_1 has at least one and at most four eigenvalues. Moreover, if $\alpha \in (0, \alpha_{\min}]$, then the block operator matrix A_1 has a unique simple (isolated) eigenvalue to the left of $-\varepsilon$ and if $\alpha \in (\alpha_{\max}, +\infty)$, then the block operator matrix A_1 has two eigenvalues to the left of $-\varepsilon$ and two to the right of $M+\varepsilon$.*

Remark 3.4. *In Corollary 3.3, the eigenvalue E_0 of the block operator matrix A_1 that exists for all $\alpha > 0$ is usually called the ground state. It is easy to see that the components of the corresponding vector eigenfunction have the forms*

$$f_0^{(+)} = 0, f_0^{(-)} = \mathrm{const} \neq 0, f_1^{(+)}(p) = -\frac{\alpha v(p) f_0^{(-)}}{\varepsilon + w(p) - E_0}, f_1^{(-)}(p) = 0.$$

The following lemma holds.

Lemma 3.5. *Let $\varepsilon > M/2$. Then for all $\alpha > 0$ the block operator matrix A_1 does not have any eigenvalues in the gap $(M-\varepsilon, \varepsilon)$. If $\alpha \in (0, \alpha_3)$, then the block operator matrix A_1 has an eigenvalue in the essential spectrum, in particular, in the interval $(\varepsilon, M+\varepsilon)$.*

From the discussions presented in this section, we can conclude that the existence of isolated or embedded eigenvalues of the block operator matrix A_1 is tightly connected with the properties of the block operator matrix $\mathrm{A}_1^{(\mathrm{s})}$, $\mathrm{s} = \pm$, and that $\sigma_{\mathrm{disc}}(\mathrm{A}_1) \neq \varnothing$.

4 Main result

In this section we assume that

$$\int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{w(t)-m} = \infty, \quad \int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{M - w(t)} = \infty. \tag{2}$$

The case where the latter integrals are finite with $m = 0$ was discussed in [29].

Let us consider some examples:

$$\mathrm{d} = 1, \quad v(k_1) = 1, \quad w(k_1) = 2 - \cos k_1.$$

Then $m = 1$, $M = 3$ and there exist the numbers $C_1, C_2, C_3, C_4 > 0$ and $\delta_1, \delta_2 > 0$ such that

$$C_1k_1^2 \le 1-\cos k_1 \le C_2k_1^2, \quad k_1 \in (-\delta_1;\delta_1);$$
$$C_3\,|\,k_1-\pi\,|^2 \le 1+\cos k_1 \le C_4\,|\,k_1-\pi\,|^2, \quad k_1 \in (\pi-\delta;\pi+\delta);$$
$$1-\cos k_1 \ge C_1, \quad k_1 \in T^1 \setminus (-\delta_1;\delta_1);$$
$$1+\cos k_1 \le C_3, \quad k_1 \in T^1 \setminus (\pi-\delta;\pi+\delta).$$

From here it follows that

$$\int_{T^1} \frac{v^2(t)dt}{w(t)} = \int_{T^1} \frac{dt}{1-\cos t} \ge \int_{-\delta}^{\delta} \frac{dt}{1-\cos t} \ge \frac{1}{C_1}\int_{-\delta}^{\delta} \frac{dt}{t^2} = +\infty;$$
$$\int_{T^1} \frac{v^2(t)dt}{2-w(t)} = \int_{T^1} \frac{dt}{1+\cos t} \ge \int_{\pi-\delta}^{\pi+\delta} \frac{dt}{1+\cos t} \ge \frac{1}{C_3}\int_{\pi-\delta}^{\pi+\delta} \frac{dt}{|\,t-\pi\,|^2} = +\infty.$$

Therefore, the conditions (2) are satisfied.
In the case

$$\mathrm{d}=1, \quad v(k_1)=\sin k_1, \quad w(k_1)=2-\cos k_1$$

we have

$$\int_{T^1} \frac{v^2(t)dt}{w(t)} = \int_{T^1} \frac{\sin^2 t dt}{1-\cos t} + C_2 \le \int_{-\delta}^{\delta} \frac{\sin^2 t dt}{1-\cos t} \le C_1\int_{-\delta}^{\delta} \frac{t^2 dt}{t^2} + C_2 < +\infty;$$
$$\int_{T^1} \frac{v^2(t)dt}{2-w(t)} = \int_{T^1} \frac{\sin^2 t dt}{1+\cos t} \le \int_{\pi-\delta}^{\pi+\delta} \frac{dt}{1+\cos t} + C_2 \le \frac{1}{C_3}\int_{\pi-\delta}^{\pi+\delta} \frac{|\,t-\pi\,|^2\, dt}{|\,t-\pi\,|^2} < +\infty.$$

Here we also use the fact that $\sin k_1 : \ k_1$ as $k_1 \in (-\delta_1;\delta_1)$.

For the special case $m<0<M$ we set $\overline{\alpha}_{\min} := \min\{\overline{\alpha}_1, \overline{\alpha}_2\}$ and $\overline{\alpha}_{\max} := \max\{\overline{\alpha}_1, \overline{\alpha}_2\}$, where

$$\overline{\alpha}_1 := \sqrt{-m}\left(\int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{2\varepsilon+w(t)-m}\right)^{-1/2}, \overline{\alpha}_2 := \sqrt{M}\left(\int_{T^{\mathrm{d}}} \frac{v^2(t)dt}{2\varepsilon+M-w(t)}\right)^{-1/2}.$$

Main result of this paper is the following theorem.

Theorem 4.1. (a) *For all values of the coupling constant* $\alpha>0$ *the block operator matrix* A *has four simple eigenvalues* E_k, $k=1,2,3,4$ *satisfying the condition* $E_1 \le E_2 \le E_3 \le E_4$. *Moreover* $E_1 < -\varepsilon+m$ *and* $E_4 > \varepsilon+M$.

(b) *If* $m<0<M$ *and* $\alpha > \overline{\alpha}_{\max}$, *then for the eigenvalues of the block operator matrix* A *we have* $E_1 \le E_2 < -\varepsilon+m$ *and* $E_4 \ge E_3 > M+\varepsilon$.

(c) *Let* $m<0<M$, $\varepsilon \le (M-m)/2$ *and* $0<\alpha<\overline{\alpha}_{\min}$. *then for the eigenvalues of the block operator matrix* A *we have*

$$E_1 < -\varepsilon+m < E_3 \le E_4 \le M+\varepsilon < E_4,$$

that is, the eigenvalues E_2 *and* E_3 *of the block operator matrix* A *are located in the essential spectrum, in particular, in the interval* $(-\varepsilon+m;M+\varepsilon)$.

This Theorem plays an important role when we study the location of the two-particle and three-particle branches of the essential spectrum of the lattice spin-boson Hamiltonian with at most two photons, and also to showing the finiteness of the number of its eigenvalues.

5 Conclusion

In the present paper we introduce the lattice spin-boson model with at most one photon A, which has a 2×2 block operator matrix representation. We analyze the essential spectrum of the block operator matrix A. We prove that the block operator matrix A has four eigenvalues. During our investigations we discuss the case where the special integral is an infinite. We find the existence condition of the eigenvalues lying in and out of the essential spectrum.

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