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On the Numerical Range of a Friedrichs Model with Rank Two Perturbation: Threshold Analysis Technique

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Abstract. In the paper we consider a Friedrichs model $\mathscr{A}(\mu_1,\mu_2)$, $\mu_1,\mu_2>0$ with rank two perturbation. It is related with a two quantum particle system on 3D integer lattice. The number and location of the discrete eigenvalues of $\mathscr{A}(\mu_1,\mu_2)$ are investigated. We give the sufficient and necessary conditions which guarantees the equality of the spectrum of $\mathscr{A}(\mu_1,\mu_2)$ and its field of values (or numerical range). The relation of the threshold eigenvalues and virtual levels with the numerical range of $\mathscr{A}(\mu_1,\mu_2)$ are established.

INTRODUCTION

The field of values (or numerical range) is a key tool in the investigation of the spectra of the bounded or unbounded linear operators acting in complex Hilbert spaces. Let us state its definition. Assume A is a linear bounded operator on a Hilbert space Y. Denote by $(\cdot,\cdot)$ the scalar product on the complex Hilbert space Y. For the operator A we determine the image of the unit sphere of Y under the quadratic form $y\to(Ay,y)$. We say this set the field of values (or numerical range) of the operator A and use the notation $\mathscr{W}(A)$. Precisely speaking,

$$\mathscr{W}(A):=\{(Ay,y):\,y\in Y,\,\|y\|=1\}.$$

Therefore for the field of values $\mathscr{W}(A)$, like the spectrum, we have $\mathscr{W}(A)\subset\mathbf{C}$. The notion field of values was first discussed in [1]; Toeplitz showed that the field of values of a usual matrix A is contain its spectrum, in addition the boundary of $\mathscr{W}(A)$ is a convex curve. Convexity of the set $\mathscr{W}(A)$ was shown in [2]. Further, these properties are true for general bounded linear operators, that is, the closure $\overline{\mathscr{W}(A)}$ is contain the spectrum (see [3]).

The notion of field of values is generalized by the different ways, see for example [4, 5, 6, 7, 8, 9]. In particular, the notions higher rank numerical range and tracial numerical range are among them. There are also concepts of quadratic numerical range for second order operator matrices, cubic numerical range for third order operator matrices and block numerical range for n-th order operator matrices. We remark that the numerical range of a linear operator allows to determine the location of its smallest and largest eigenvalues.

In the present paper we discuss a Hamiltonian (Friedrichs model) $\mathscr{A}(\mu_1,\mu_2)$ with rank two perturbation. This Hamiltonian is related with a two quantum particle system on 3D lattice. In the Section 2 we collect some useful properties of the numerical range. In the Section 3 we investigate the position and number of the eigenvalues of $\mathscr{A}(\mu_1,\mu_2)$. In our case the corresponding Fredholm's determinant as a function is not monotonic. Therefore we use the method, which is based on the number of discrete eigenvalues of Friedrichs model with rank one perturbation to define the position of its discrete eigenvalues. In Section 4 we find sufficient and necessary conditions which guaranties that the spectrum of $\mathscr{A}(\mu_1,\mu_2)$ is equal to the field of values.

Remark that the structure of the closure of the field of values of a 2×2 operator matrix, related with a system of at most two quantum particles on dD lattice, is investigated in detail in [10], with respect of matrix elements in any dimensions d of $\mathbf{T}^{\mathrm{d}}$.

SOME USEFUL PROPERTIES OF THE NUMERICAL RANGE

Now we list some useful properties of $\mathscr{W}(A)$ (for detailed information we refer [11, 12, 13]).

The set of all complex numbers will be denoted by $\mathbf{C}$. Next we consider a bounded linear operator A on a complex Hilbert space X.

(a) for the field of values $\mathscr{W}(A)$ of A we have the inclusion

$$\mathscr{W}(A)\subset\{a\in\mathbf{C}:|a|\le||A||;$$

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(b) the assertion $\mathscr{W}(A^*) = \{\overline{a} : \lambda \in \mathscr{W}(A)\}$ is true for the adjoint operator A^*;

(c) For the field of values of an identity operator E on X the equality $\mathscr{W}(E) = \{1\}$ holds. Additionally, the equality

$$\mathscr{W}(\alpha A + \beta) = \alpha \mathscr{W}(A) + \beta$$

is valid for an arbitrary complex quantities α and β;

(d) From self-adjoint ness of A it follows that $\mathscr{W}(A) \subset \mathbf{R}$;

(e) If $\dim(X) < \infty$, then $\mathscr{W}(A)$ is a compact (closed and bounded) set;

(f) Under unitary similarity the field of values $\mathscr{W}(A)$ of A is an invariant;

(g) Let $\sigma_{\mathrm{p}}(A)$ be the point spectrum (or set of eigenvalues) and $\sigma(A)$ be the spectrum of A. Then for the field of values $\mathscr{W}(A)$ of A the *spectral inclusion property*:

$$\mathscr{W}(A) \subset \sigma_{\mathrm{p}}(A) \quad \text{and} \quad \overline{\mathscr{W}(A)} \subset \sigma(A)$$

are valid.

One special usage of $\mathscr{W}(A)$ is the study of the boundary of the spectrum $\sigma(A)$. We determine the *approximate point spectrum* $\sigma_{\mathrm{app}}(A)$ by the equality (see [13])

$$\sigma_{\mathrm{app}}(A) := \{a \in \mathbf{C} : \exists \{y_n\}_1^\infty \subset D(A),\, (A - aE)y_n \to 0, \text{as}\, n \to \infty,\, \|y_n\| = 1\}$$

and it is well known that the spectrum's boundary is contained in $\sigma_{\mathrm{app}}(A)$.

The following example shows that even for the bounded linear operator $C = C^*$ in complex Hilbert space X we can not state $\mathscr{W}(C) \subset \sigma(C)$ or $\sigma(C) \subset \mathscr{W}(C)$. Let

$$l_2(\mathbf{N}) := \{y = (y_1, y_2, \ldots, y_n, \ldots) : \sum_{k=1}^{\infty} |y_k|^2 < \infty\};$$

$$C : l_2(\mathbf{N}) \to l_2(\mathbf{N}), \quad Cy = (y_1, \frac{1}{2}y_2, \ldots, \frac{1}{n}y_n, \ldots), \quad y = (y_1, y_2, \ldots, y_n, \ldots) \in l_2(\mathbf{N}).$$

According to the simple calculations we get

$$\sigma(C) = \overline{\left\{\frac{1}{n} : n \in \mathbf{N}\right\}} = \left\{\frac{1}{n} : n \in \mathbf{N}\right\} \cup \{0\}, \quad \mathscr{W}(C) = (0, 1].$$

Here $0 \notin \mathscr{W}(C)$, since the equality

$$(Cy, y) = \sum_{k=1}^{\infty} \frac{1}{n} |y_n|^2 = 0$$

implies $y = (0, 0, \ldots) \in l_2(\mathbf{N})$. Then a natural question arises: Is there the differently from scalar a bounded self-adjoint operator, that its spectrum equals to the field of values? In this paper, we will try to answer to this question and give an example for such type operators.

FRIEDRICHS MODEL AND ITS SPECTRUM

By $\mathbf{T}^3$ we mean the 3D torus and by $L_2(\mathbf{T}^3)$ we mean the complex Hilbert space of square integrable (in general complex valued) functions on 3D torus $\mathbf{T}^3$.

Let us consider the Hamiltonian (so-called the Friedrichs model) acting on $L_2(\mathbf{T}^3)$:

$$\mathscr{A}(\mu_1, \mu_2) : L_2(\mathbf{T}^3) \to L_2(\mathbf{T}^3), \quad \mathscr{A}(\mu_1, \mu_2) = \mathscr{A}_0 - \mu_1 V_1 + \mu_2 V_2, \tag{1}$$

where the operators $\mathscr{A}_0$ and V_α, $\alpha = 1, 2$ are defined by

$$(\mathscr{A}_0 g)(x) = u(x) g(x), \quad g \in L_2(\mathbf{T}^3);$$

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$$(V_\alpha g)(x) = v_\alpha(x)\int_{\mathbf{T}^3} v_\alpha(s)g(s)ds, \quad \alpha = 1,2, \quad g \in L_2(\mathbf{T}^3).$$

Here $\mu_\alpha > 0$, $\alpha = 1,2$ are positive reals, the functions $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1,2$ are continuous on $\mathbf{T}^3$ with real-values. Using simple discussions one can show the boundedness and self-adjointness of the operator $\mathscr{A}(\mu_1,\mu_2)$ in $L_2(\mathbf{T}^3)$ defined by (1).

The Friedrichs model $\mathscr{A}(\mu_1,\mu_2)$ is related with a two particle system on a 3D lattice and interacting by the non-local potentials. Usually such operators are arising in Hubbard model [14, 15]. In this paper we discuss the case where the kernel of non-local interaction operators (integral operators) V_α, $\alpha = 1,2$ has rank two. A key problem of the spectral theory of such operators is to determine the field of values and to investigate the position and number of discrete eigenvalues lying on the left and right hand side of the essential spectrum.

Let A be a bounded self-adjoint linear operator in a Hilbert space X. The essential spectrum of the operator A will be denoted by $\sigma_{\rm ess}(A)$, the discrete spectrum $\sigma_{\rm disc}(A)$, the spectrum by $\sigma(A)$ and the resolvent set by $\rho(A)$.

We start to determine the sets $\sigma_{\rm ess}(\mathscr{A}(\mu_1,\mu_2))$ and $\sigma_{\rm disc}(\mathscr{A}(\mu_1,\mu_2))$.

By the definition of the perturbation $-\mu_1V_1+\mu_2V_2$ of $\mathscr{A}_0$ we have $(-\mu_1V_1+\mu_2V_2)^* = -\mu_1V_1+\mu_2V_2$ and $\mathrm{rank}(-\mu_1V_1+\mu_2V_2) = 2$. Hence,

$$\sigma_{\rm ess}(\mathscr{A}(\mu_1,\mu_2)) = \sigma(\mathscr{A}_0) = [m_1;m_2],$$

where the numbers m_1 and m_2 are defined by

$$m_1 := \min_{x\in\mathbf{T}^3} u(x), \quad m_2 := \max_{x\in\mathbf{T}^3} u(x).$$

For any fixed $\mu_\alpha > 0$, $\alpha = 1,2$ we define so called the Fredholm determinant

$$\Delta(\mu_1,\mu_2;\cdot) := \Delta_1(\mu_1;\cdot)\Delta_2(\mu_2;\cdot)+\mu_1\mu_2\Delta_3^2(\cdot)$$

related with the operator $\mathscr{A}(\mu_1,\mu_2))$ as an analytic function in $\mathbf{C}\setminus[m_1;m_2]$ as

$$\Delta_\alpha(\mu_\alpha;z) := 1+(-1)^\alpha\mu_\alpha\int_{\mathbf{T}^3}\frac{v_\alpha^2(t)dt}{u(t)-z},\ \alpha = 1,2,\ \Delta_3(z) := \int_{\mathbf{T}^3}\frac{v_1(t)v_2(t)dt}{u(t)-z}.$$

Using Fredholm's theorem and the Birman-Schwinger principle we obtain the following simple result.

Lemma 1. *The quantity $z \in \mathbf{C}\setminus[m_1;m_2]$ is an discrete eigenvalue of the operator $\mathscr{A}(\mu_1,\mu_2)$ iff $\Delta(\mu_1,\mu_2;z) = 0$.*

By Lemma 1 for $\sigma_{\rm disc}(\mathscr{A}(\mu_1,\mu_2)$ we obtain

$$\sigma_{\rm disc}(\mathscr{A}(\mu_1,\mu_2)) = \{z\in\mathbf{C}\setminus[m_1;m_2] : \Delta(\mu_1,\mu_2;z) = 0\}.$$

Therefore, for the spectrum $\sigma(\mathscr{A}(\mu_1,\mu_2))$ of $\mathscr{A}(\mu_1,\mu_2)$ we have

$$\sigma(\mathscr{A}(\mu_1,\mu_2)) = [m_1;m_2]\cup\{z\in\mathbf{C}\setminus[m_1;m_2] : \Delta(\mu_1,\mu_2;z) = 0\}.$$

Now we define the following Friedrichs models $\mathscr{A}_\alpha(\mu_\alpha)$, acting on $L^2(\mathbf{T}^3)$ by the rule

$$\mathscr{A}_\alpha(\mu_\alpha) := \mathscr{A}_0+(-1)^\alpha\mu_\alpha V_\alpha, \quad \alpha = 1,2. \tag{2}$$

For a bounded linear operator $A = A^*$ acting in a complex Hilbert space L we determine $L_A(\lambda)$, $\lambda\in\mathbf{R}$ as a subspace so that $(Af,f) < \lambda\|f\|^2$ for all $f\in L_A(\lambda)$ and put

$$N(\lambda,A) := \sup_{L_A(\lambda)} \dim L_A(\lambda).$$

Notice $N(\lambda,A) = +\infty$ if $\lambda > \min\sigma_{\rm ess}(A)$; if $N(\lambda,A) < +\infty$, then it is coincide with the number of discrete eigenvalues smaller than λ of A.

Assume $\mathrm{supp}\{v_\alpha(\cdot)\}$ be the support of the function $v_\alpha(\cdot)$. Lebesgue's measure of the measurable set $\Omega\subset\mathbf{T}^3$ is denoted by $\mathrm{mes}(\Omega)$.

Lemma 2. (a) *For any fixed $\mu_\alpha > 0$, $\alpha = 1,2$ the operator $\mathscr{A}(\mu_1,\mu_2)$ hasn't more than 1 simple discrete eigenvalue smaller than m_1 respectively bigger than m_2.*

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(b) *Suppose that*

$$\mathrm{mes}(\mathrm{supp}\{v_1(\cdot)\} \cap \mathrm{supp}\{v_2(\cdot)\}) = 0. \tag{3}$$

For this case the quantity $z \in \mathbf{C} \setminus [m_1; m_2]$ is a discrete eigenvalue of $\mathscr{A}(\mu_1, \mu_2)$ iff the quantity z is a discrete eigenvalue one of the operators $\mathscr{A}_\alpha(\mu_\alpha)$, $\alpha = 1, 2$.

Proof. (a) Since the operator V_α, $\alpha = 1, 2$ is a positive definite, it is clear that $\mathscr{A}(\mu_1, \mu_2) \geq \mathscr{A}_1(\mu_1)$. Therefore, $L_{\mathscr{A}(\mu_1,\mu_2)}(\lambda) \subset L_{\mathscr{A}_1(\mu_1)}(\lambda)$ for $\lambda \leq m_1$. It means that

$$N(\lambda, \mathscr{A}(\mu_1, \mu_2)) \leq N(\lambda, \mathscr{A}_1(\mu_1)), \quad \lambda \leq m_1. \tag{4}$$

Since for any fixed $\mu_1 > 0$ the function $\Delta_1(\mu_1; \cdot)$ is decreasing in the interval $(-\infty; m_1)$ we obtain $N(m_1, \mathscr{A}_1(\mu_1)) \leq 1$. Hence, by (4) it follows $N(m_1, \mathscr{A}(\mu_1, \mu_2)) \leq 1$.

The assertion $N(-m_2, -\mathscr{A}(\mu_1, \mu_2)) \leq 1$ can be proven similarly.

(b) Suppose the quantity $z \in \mathbf{C} \setminus [m_1; m_2]$ is a discrete eigenvalue of $\mathscr{A}(\mu_1, \mu_2)$. We denote the corresponding eigen function by $g \in L_2(\mathbf{T}^3)$. Then the eigen function g satisfy the equation

$$(u(x) - z)g(x) - \mu_1 v_1(x) \int_{\mathbf{T}^3} v_1(s)g(s)ds + \mu_2 v_2(x) \int_{\mathbf{T}^3} v_2(s)g(s)ds = 0. \tag{5}$$

By the determination, for al values of $z \in \mathbf{C} \setminus [m_1; m_2]$ the relation $u(x) - z \neq 0$ is valid for any $x \in \mathbf{T}^3$. By this reason from (5) we get

$$g(x) = \frac{\mu_1 v_1(x) k_1 - \mu_2 v_2(x) k_2}{u(x) - z}, \tag{6}$$

where

$$k_\alpha := \int_{\mathbf{T}^3} v_\alpha(s) g(s) ds, \quad \alpha = 1, 2. \tag{7}$$

Putting the formula (6) for f to (7) and using the condition (3) we conclude that the equation (5) has a nonzero solution iff the homogeneous system of linear equations

$$\begin{gathered}\Delta_1(\mu_1; z)k_1 = 0;\\ \Delta_2(\mu_2; z)k_2 = 0\end{gathered}$$

has a non zero solution $(k_1, k_2) \in \mathbf{C}^2$, that is, if the assertion $\Delta_1(\mu_1; z)\Delta_2(\mu_2; z) = 0$ is valid. For the case $v_\alpha(p) \equiv 0$ from the definitions of $\mathscr{A}(\mu_1, \mu_2)$ and $\mathscr{A}_\alpha(\mu_\alpha)$ we obtain that $\mathscr{A}(\mu_1, \mu_2) = \mathscr{A}_\beta(\mu_\beta)$ for $\alpha \neq \beta$. Then Lemma 1 complete the proof of part (b) of the proving Lemma 2.

From part (b) of Lemma 2 it follows that under the assumption (3) there is a relation

$$\sigma_{\mathrm{disc}}(\mathscr{A}(\mu_1, \mu_2)) = \sigma_{\mathrm{disc}}(\mathscr{A}_1(\mu_1)) \cup \sigma_{\mathrm{disc}}(\mathscr{A}_2(\mu_2))$$

between discrete spectra of $\mathscr{A}(\mu_1, \mu_2)$ and $\mathscr{A}_\alpha(\mu_\alpha)$, $\alpha = 1, 2$.

Using the Fredholm determinant $\Delta_\alpha(\mu_\alpha; \cdot)$ of $\mathscr{A}_\alpha(\mu_\alpha)$ we conclude that

$$\sigma_{\mathrm{disc}}(\mathscr{A}_\alpha(\mu_\alpha)) = \{z \in \mathbf{C} \setminus [m_1; m_2] : \Delta_\alpha(\mu_\alpha; z) = 0\}; \tag{8}$$

$$\sigma(\mathscr{A}_\alpha(\mu_\alpha) = [m_1; m_2] \cup \{z \in \mathbf{C} \setminus [m_1; m_2] : \Delta(\mu_\alpha; z) = 0\}.$$

We note that the operators $\mathscr{A}_\alpha(\mu_\alpha)$, $\alpha = 1, 2$ have a simple structure than $\mathscr{A}(\mu_1, \mu_2)$. By this matter the latter equality plays an key role in next investigation of the field of values and the spectrum of $\mathscr{A}(\mu_1, \mu_2)$.

Detrmine

$$I_\alpha(z) := \int_{\mathbf{T}^3} \frac{v_\alpha^2(t)dt}{u(t) - z}, \ z \in \mathbf{R} \setminus [m_1; m_2].$$

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Since the function $I_\alpha(\cdot)$ is an increasing in the intervals $(-\infty;m_1)$ and $(m_2;+\infty)$, by the Lebesgue dominated convergence theorem there exist the following finite or infinite limits

$$I_1(m_1) = \lim_{z\to m_1-0} I_1(z),\ \ I_2(m_2) = \lim_{z\to m_2+0} I_2(z).$$

The finiteness or infiniteness of the last limits are important in the investigation the presence of the discrete eigenvalues of $\mathscr{A}(\mu_1,\mu_2)$ and the conditions under which the field of values of $\mathscr{A}(\mu_1,\mu_2)$ is closed as a set. In the following we give the examples where these limits can be finite or infinite.

Example. *Let*

$$u(p) = (3-\cos x_1-\cos x_2-\cos x_3)^2,\ x=(x_1,x_2,x_3)\in \mathbf{T}^3.$$

In this case $m_1 = 0$ and $m_2 = 36$. At the point $\overline{0} := (0,0,0) \in \mathbf{T}^3$ The function $u(\cdot)$ has an unique non-degenerate global min as well at the point $\overline{\pi} := (\pi,\pi,\pi) \in \mathbf{T}^3$ has a non-degenerate global max.

(a) *If $v_\alpha(x)\equiv 1$, then $|I_\alpha(m_\alpha)| = +\infty$ for $\alpha = 1,2$.*

(b) *If $v_\alpha(x) = \sin x_1+\sin x_2+\sin x_3$, then $|I_\alpha(m_\alpha)| < +\infty$ for $\alpha = 1,2$.*

(c) *If*

$$v_1(x) = \begin{cases} \sin(2x_1)\sin(2x_2)\sin(2x_3), & x_i\in(-\pi/2,\pi/2),\, i=1,2,3;\\ 0, & \textit{otherwise;}\end{cases}$$

$$v_2(p) = \begin{cases} \cos x_1\cos x_2\cos x_3, & x_i\in \mathbf{T}^1\setminus(-\pi/2,\pi/2),=1,2,3;\\ 0, & \textit{otherwise;}\end{cases}$$

then $|I_1(m_1)| < +\infty$ and $|I_2(m_2)| = +\infty$.

(d) *If*

$$v_1(x) = \begin{cases} \cos x_1\cos x_2\cos x_3, & x_i\in(-\pi/2,\pi/2),\, i=1,2,3;\\ 0, & \textit{otherwise;}\end{cases}$$

$$v_2(x) = \begin{cases} \sin(2x_1)\sin(2x_2)\sin(2x_3), & x_i\in \mathbf{T}^1\setminus(-\pi/2,\pi/2),\, i=1,2,3;\\ 0, & \textit{otherwise;}\end{cases}$$

then $|I_1(m_1)| = +\infty$ and $|I_2(m_2)| < +\infty$.

Let us discuss the case (c). For $\delta > 0$ and $p_0 \in \mathbf{T}^3$ we set

$$U_\delta(x_0) := \{x\in \mathbf{T}^3 : |x-x_0| < \delta\}.$$

By the properties of sine and cosine functions, there are the quantities $C_1,C_2,C_3,\delta > 0$ so that

$$\begin{aligned}
&C_1|x|^4 \le |u(x)| \le C_2|x|^4, \quad x\in U_\delta(\overline{0});\\
&C_1|x-\overline{\pi}|^4 \le |u(x)-36| \le C_2|x-\overline{\pi}|^4, \quad x\in U_\delta(\overline{\pi});\\
&C_1|x_1x_2x_3| \le |v_1(x)| \le C_2|x_1x_2x_3|, \quad x\in U_\delta(\overline{0});\\
&|v_2(x)| \ge C_3, \quad x\in U_\delta(\overline{\pi}).
\end{aligned}$$

Using the last estimates one can easily show that

$$\begin{aligned}
\big|I_1(\overline{0})\big| &= \left|\int_{\mathbf{T}^3}\frac{v_1^2(s)ds}{u(s)}\right| \le C_1\int_{U_\delta(\overline{0})}\frac{|s_1||s_2||s_3|}{(|s_1|^2+|s_2|^2+|s_3|^2)^2}ds_1ds_2ds_3 + C_2 < \infty;\\
|I_2(\overline{\pi})| &= \left|\int_{\mathbf{T}^3}\frac{v_2^2(s)ds}{u(s)-36}\right| \ge C_2\int_{U_\delta(\overline{\pi})}\frac{ds_1ds_2ds_3}{(|s_1-\pi|^2+|s_2-\pi|^2+|s_3-\pi|^2)^2} = +\infty.
\end{aligned}$$

For the case $|I_\alpha(m_\alpha)| < +\infty$, $\alpha = 1,2$ we set

$$\mu_1^0 := (I_1(m_1))^{-1},\ \ \mu_2^0 := -(I_2(m_2))^{-1}.$$

The set of discrete eigenvalues of $\mathscr{A}_2(\mu_2)$ is described in the following result (theorem).

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Theorem 1. (a) *If $I_2(m_2) = -\infty$, then for all values of $\mu_2 > 0$ the Friedrichs model $\mathscr{A}_2(\mu_2)$ has an unique discrete eigenvalue lying smaller than m_2.*

(b) *Suppose that $|I_2(m_2)| < +\infty$.*

(b1) *If $0 < \mu_2 \leq \mu_2^0$, then the Friedrichs model $\mathscr{A}_2(\mu_2)$ hasn't discrete eigenvalues bigger than m_2.*

(b2) *For $\mu_2 > \mu_2^0$ the Friedrichs model $\mathscr{A}_2(\mu_2)$ has an unique discrete eigenvalue bigger than m_2.*

(c) *For all values of $\mu_2 > 0$ the Friedrichs model $\mathscr{A}_2(\mu_2)$ hasn't discrete eigenvalues smaller than m_1.*

Proof. (a) Let $I_2(m_2) = -\infty$. Then for all values of $\mu_2 > 0$ we have $\lim\limits_{z \to m_2+0} \Delta_2(\mu_2\,; z) = -\infty$.

Using the monotonicity property of the continuous function $\Delta_2(\mu_2\,;\cdot)$ in $(m_2; +\infty)$ and the equality

$$\lim_{z \to +\infty} \Delta_2(\mu_2\,; z) = 1, \tag{9}$$

we obtain that there exist $E_2(\mu_2) \in (m_2; +\infty)$ such that $\Delta_2(\mu_2\,; E_2(\mu_2)) = 0$. Therefore, by equality (8) the number $E_2(\mu_2) > m_2$ is an eigenvalue of $\mathscr{A}_2(\mu_2)$.

(b1) Assume that $|I_2(m_2)| < +\infty$ and $0 < \mu_2 \leq \mu_2^0$. Since the function $\Delta_2(\mu_2\,;\cdot)$ is an increasing in the interval $(m_2; +\infty)$, for any $z > m_2$ we have

$$\Delta_2(\mu_2\,; z) > \Delta_2(\mu_2\,; m_2) \geq \Delta_2(\mu_2^0; m_2) = 1 + \mu_2^0 \cdot I_2(m_2) = 0.$$

Hence, by (8) the operator $\mathscr{A}_2(\mu_2)$ has no eigenvalues in $(m_2; +\infty)$.

(b2) Let now $\mu_2 > \mu_2^0$. In this case

$$\Delta_2(\mu_2\,; m_2) < \Delta_2(\mu_2^0; m_2) = 1 + \mu_2^0 \cdot I_2(m_2) = 0,$$

that is, $\Delta_2(\mu_2\,; m_2) < 0$. Since the function $\Delta_2(\mu_2\,;\cdot)$ is an increasing in the interval $(m_2; +\infty)$, by equality (9) there exist $E_2(\mu_2) \in (m_2; +\infty)$ such that $\Delta_2(\mu_2\,; E_2(\mu_2)) = 0$. From the equality (8) we get that the quantity $E_2(\mu_2) > m_2$ is a discrete eigenvalue of $\mathscr{A}_2(\mu_2)$.

(c) It is easy to see that $\Delta_2(\mu_2\,; z) \geq 1$ for any $\mu_2 > 0$ and $z < m_1$. From here using the equality (8) we obtain that the Friedrichs model $\mathscr{A}_2(\mu_2)$ hasn't discrete eigenvalues in $(-\infty; m_1)$.

Theorem 1 is completely proved.

Now we describe the set of eigenvalues of $\mathscr{A}_1(\mu_1)$.

Theorem 2. (a) *If $I_1(m_1) = +\infty$, then for all values of $\mu_1 > 0$ the Friedrichs model $\mathscr{A}_1(\mu_1)$ has an unique discrete eigenvalue smaller than m_1.*

(b) *Assume that $I_1(m_1) < +\infty$.*

(b1) *If $0 < \mu_1 \leq \mu_1^0$, then the Friedrichs model $\mathscr{A}_1(\mu_1)$ hasn't discrete eigenvalues smaller than m_1.*

(b2) *For all values of $\mu_1 > \mu_1^0$ the Friedrichs model $\mathscr{A}_1(\mu_1)$ has an unique discrete eigenvalue smaller than m_1.*

(c) *For all values of $\mu_1 > 0$ the Friedrichs model $\mathscr{A}_1(\mu_1)$ hasn't discrete eigenvalues bigger than m_2.*

Theorem 2 can be proven by the same way as the proof of Theorem 1.

From Lemma 2, Theorems 1 and 2 it follows

Corollary 1. *Let the condition* (3) *be fulfilled.*

(a) *If either $|I_\alpha(m_\alpha)| = +\infty$ and $\mu_\alpha > 0$ for $\alpha = 1,2$ or $|I_\alpha(m_\alpha)| < +\infty$ and $\mu_\alpha > \mu_\alpha^0$ for $\alpha = 1,2$, then the operator $\mathscr{A}(\mu_1, \mu_2)$ has two simple eigenvalues E_α, $\alpha = 1,2$ such that $E_1 < m_1$ and $E_2 > m_2$.*

(b) *If $|I_\alpha(m_\alpha)| < +\infty$ and $0 < \mu_\alpha \leq \mu_\alpha^0$ for $\alpha = 1,2$, then the discrete spectrum of the Friedrichs model $\mathscr{A}(\mu_1, \mu_2)$ is an empty set.*

From Corollary 1 one can see that the operator $\mathscr{A}(\mu_1^0, \mu_2^0)$ has no eigenvalues lying outside of its essential spectrum, and hence the relation $\sigma(\mathscr{A}(\mu_1^0, \mu_2^0)) = [m_1; m_2]$ is valid for the spectrum of $\mathscr{A}(\mu_1^0, \mu_2^0)$. .

THE FIELD OF VALUES OF THE FRIEDRICHS MODEL

The section is devoted to the description the field of values $\mathscr{W}(\mathscr{A}(\mu_1^0, \mu_2^0))$ of $\mathscr{A}(\mu_1^0, \mu_2^0)$ depending on the value of the function $v_\alpha(\cdot)$, $\alpha = 1,2$. Our investigations are based on the threshold analysis techniques.

During the section, we suppose that at the point $x_1 \in \mathbf{T}^3$ the function $u(\cdot)$ has an unique non degenerate global min and at the point $x_2 \in \mathbf{T}^3$ has an unique non degenerate max, and for $\alpha = 1,2$ at some neighborhood of $x_\alpha \in \mathbf{T}^3$ the continuous partial derivatives up to the third-order inclusive of the function $v_\alpha(\cdot)$ are exist.

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Now we establish that the class of the parameter functions $v_\alpha(\cdot)$, $\alpha=1,2$ satisfying above mentioned conditions is nonempty. Let these functions have the forms:

$$
\begin{aligned}
&u(x)=3-\cos x_1-\cos x_2-\cos x_3;\\
&v_1(x)=\begin{cases}\cos x_1\cos x_2\cos x_3, & x_i\in(-\pi/2,\pi/2),\, i=1,2,3;\\ 0, & \text{otherwise;}\end{cases}\\
&v_2(x)=\cos x_1\cos x_2\cos x_3-v_1(x),\quad x=(x_1,x_2,x_3)\in\mathbf{T}^3.
\end{aligned}
$$

For this special case the point $x_1=(0,0,0)$ the function $u(\cdot)$ has an unique non-degenerate global min and at the point $x_2=(\pi,\pi,\pi)$ has an unique non-degenerate max. For $\alpha=1,2$, the function $v_\alpha(\cdot)$ is an analytic in the $\delta<\pi/2$-neighborhood $U_\delta(x_\alpha)$ of the point p_α. Validness of the condition (3) follows from the definition of $v_\alpha(\cdot)$.

Theorem 3. *Let the condition* (3) *be fulfilled. If* $v_\alpha(x_\alpha)=0$ *for* $\alpha=1,2$, *then*

$$\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))=[m_1;m_2](=\sigma(\mathscr{A}(\mu_1^0,\mu_2^0))).$$

Proof. First of all we recall that by Corollary 1 we have

$$\sigma(\mathscr{A}(\mu_1^0,\mu_2^0))=\sigma_{\mathrm{ess}}(\mathscr{A}(\mu_1^0,\mu_2^0))=[m_1;m_2].$$

From the boundedness and self-adjointness of $\mathscr{A}(\mu_1^0,\mu_2^0)$ we get that its spectrum and the field of values are related with the equality $\overline{\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))}=[m_1;m_2]$. If in addition $v_\alpha(x_\alpha)=0$, $\alpha=1,2$, then we show that $\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))=[m_1;m_2]$. It remains to prove that $m_\alpha\in\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))$.

We determine the function

$$g_\alpha(x)=(-1)^{\alpha+1}\frac{\mu_\alpha^0 v_\alpha(x)}{u(x)-m_\alpha}. \tag{10}$$

We proceed to show that this function satisfies the equation $\mathscr{A}(\mu_1^0,\mu_2^0)f_\alpha=m_\alpha f_\alpha$. Indeed,

$$
\begin{aligned}
(\mathscr{A}(\mu_1^0,\mu_2^0)-m_\alpha)g_\alpha(x)\quad &=(u(x)-m_\alpha)(-1)^{\alpha+1}\frac{\mu_\alpha^0 v_\alpha(x)}{u(x)-m_\alpha}\\
&-\mu_1^0 v_1(x)(-1)^{\alpha+1}\mu_\alpha^0\int_{\mathbf{T}^3}\frac{v_1(s)v_\alpha(s)ds}{u(s)-m_\alpha}\\
&+\mu_2^0 v_2(x)(-1)^{\alpha+1}\mu_\alpha^0\int_{\mathbf{T}^3}\frac{v_2(s)v_\alpha(s)ds}{u(s)-m_\alpha}.
\end{aligned}
$$

Using the assumption (3) and definition of μ_α^0 we obtain

$$(\mathscr{A}(\mu_1^0,\mu_2^0)-m_\alpha)g_\alpha(x)=(-1)^{\alpha+1}\mu_\alpha^0 v_\alpha(x)\left[1+(-1)^\alpha\mu_\alpha^0\int_{\mathbf{T}^3}\frac{v_\alpha^2(s)ds}{u(s)-m_\alpha}\right]=0,\quad \alpha=1,2.$$

The task is now to prove that $g_\alpha\in L_2(\mathbf{T}^3)$ if $v_\alpha(x_\alpha)=0$. Indeed. Let $v(x_\alpha)=0$. Using the fact that continuous partial derivatives up to the third-order inclusive at some neighborhood of $x_\alpha\in\mathbf{T}^3$ of the function $v_\alpha(\cdot)$ are exist, we obtain that there exist the quantities $C_1,C_2,C_3>0$, $n_\alpha\in\mathbf{N}$ and $\delta>0$ so that

$$C_1|x-x_\alpha|^{n_\alpha}\le|v_\alpha(x)|\le C_2|x-x_\alpha|^{n_\alpha},\quad x\in U_\delta(x_\alpha). \tag{11}$$

By the assumption at the point $x_1\in\mathbf{T}^3$ the function $u(\cdot)$ has an unique global non-degenerate min and at the point $x_2\in\mathbf{T}^3$ has an unique global non-degenerate max, and hence there exist the quantities $C_1,C_2,C_3,\delta>0$ so that

$$C_1|x-x_\alpha|^2\le|u(x)-m_\alpha|\le C_2|x-x_\alpha|^2,\quad x\in U_\delta(x_\alpha), \tag{12}$$

$$|u(x)-m_\alpha|\ge C_3,\quad x\in\mathbf{T}^3\setminus U_\delta(x_\alpha). \tag{13}$$

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By the additivity property of the integral we obtain the conclusion

$$\int_{\mathbf{T}^3} |g_\alpha(s)|^2 dt = (\mu_\alpha^0)^2 \int_{U_\delta(x_\alpha)} \frac{v_\alpha^2(s)ds}{(u(s)-m_\alpha)^2} + (\mu_\alpha^0)^2 \int_{\mathbf{T}^3\setminus U_\delta(s_\alpha)} \frac{v_\alpha^2(s)ds}{(u(s)-m_\alpha)^2}. \tag{14}$$

Then by (11) and (12) we obtain for the first summand of (14) the estimation

$$\int_{U_\delta(x_\alpha)} \frac{v_\alpha^2(s)ds}{(u(s)-m_\alpha)^2} \leq C_1 \int_{U_\delta(x_\alpha)} \frac{|s-x_\alpha|^{2n_\alpha}ds}{|s-x_\alpha|^4}.$$

Now, changing the coordinate system

$$\begin{aligned}
&s_1 = x_\alpha^{(1)} + r\sin\psi\cos\varphi,\\
&s_2 = x_\alpha^{(2)} + r\sin\psi\sin\varphi,\\
&s_3 = x_\alpha^{(3)} + r\cos\psi, \quad 0 \leq r \leq \delta, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \psi \leq \pi,
\end{aligned}$$

we obtain the finiteness of the last integral, where $s = (s_1, s_2, s_3)$ and $x_\alpha = (x_\alpha^{(1)}, x_\alpha^{(2)}, x_\alpha^{(3)})$.
The function $v_\alpha(\cdot)$ is continuous on $\mathbf{T}^3$, the estimation (13) implies that

$$\int_{\mathbf{T}^3\setminus U_\delta(x_\alpha)} \frac{v_\alpha^2(s)ds}{(u(s)-m_\alpha)^2} \leq C_1 \int_{\mathbf{T}^3\setminus U_\delta(x_\alpha)} ds < +\infty.$$

Therefore, if $v(x_\alpha) = 0$, then $g_\alpha \in L_2(\mathbf{T}^3)$. In this case we obtain

$$(\mathscr{A}(\mu_1^0,\mu_2^0)g_\alpha/\|g_\alpha\|, g_\alpha/\|g_\alpha\|) = m_\alpha(g_\alpha/\|g_\alpha\|, g_\alpha/\|g_\alpha\|) = m_\alpha.$$

Therefore, $m_\alpha \in \mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))$ for $\alpha = 1,2$, that is, $\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0)) = [m_1; m_2]$.

Remark 1. *When we prove the Theorem* 3 *we show that if* $v_\alpha(x_\alpha) = 0$ *for* $\alpha = 1,2$, *then the point* $z = m_\alpha$ *is not only bound of the essential spectrum of* $\mathscr{A}(\mu_1^0,\mu_2^0)$, *but it is also an eigenvalue of* $\mathscr{A}(\mu_1^0,\mu_2^0)$. *By this reason it is called the threshold eigenvalue of* $\mathscr{A}(\mu_1^0,\mu_2^0)$.

Theorem 4. *Let the condition* (3) *be fulfilled. If* $v_\alpha(p_\alpha) \neq 0$ *for* $\alpha = 1,2$, *then* $\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0)) = (m_1; m_2)$.

Proof. Once again we recall that $\sigma(\mathscr{A}(\mu_1^0,\mu_2^0)) = [m_1; m_2]$. We show that if $v_\alpha(x_\alpha) \neq 0$ for $\alpha = 1,2$, then $m_\alpha \notin \mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))$, $\alpha = 1,2$. Suppose, this assertion is false, that is, $m_\alpha \in \mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))$, $\alpha = 1,2$. Then there exists the function $g_\alpha \in L_2(\mathbf{T}^3)$ with $\|g_\alpha\| = 1$ such that $(\mathscr{A}(\mu_1^0,\mu_2^0)f_\alpha, f_\alpha) = m_\alpha$. A trivial verification shows that the function g_α has the form (10).

From the continuity of the function $v_\alpha(\cdot)$ on $\mathbf{T}^3$ and $v(p_\alpha) \neq 0$ we get that there are the quantities $C_1, \delta > 0$ so that

$$|v_\alpha(x)| \geq C_1, \quad x \in U_\delta(x_\alpha). \tag{15}$$

Applying (12) and (15) we receive

$$\int_{\mathbf{T}^3} |g_\alpha(s)|^2 ds \geq C_1 \int_{U_\delta(x_\alpha)} \frac{ds}{|t-x_\alpha|^4} = +\infty.$$

This implies $g_\alpha \notin L_2(\mathbf{T}^3)$. On the other hand applying (11), (12), (13) and using the continuity property of $v_\alpha(\cdot)$ on $\mathbf{T}^3$ we have

$$\int_{\mathbf{T}^3} |g_\alpha(s)| ds = \mu_\alpha^0 C_1 \int_{U_\delta(x_\alpha)} \frac{|s-x_\alpha|^{n_\alpha}dt}{|s-x_\alpha|^2} + \mu_\alpha^0 C_2 \int_{\mathbf{T}^3\setminus U_\delta(x_\alpha)} ds < +\infty.$$

This means $g_\alpha \in L_1(\mathbf{T}^3)$, where by $L_1(\mathbf{T}^3)$ we denote the Banach space of integrable (in general complex valued) functions on $\mathbf{T}^3$. Therefore, $g_\alpha \in L_1(\mathbf{T}^3) \setminus L_2(\mathbf{T}^3)$, which contradicts the fact that $g_\alpha \in L_2(\mathbf{T}^3)$. So, $m_\alpha \notin \mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))$, $\alpha = 1,2$.

From Theorems 3 and 4 it follows the following assertion.

Corollary 2. *Let the condition* (3) *be fulfilled.*
(a) *If* $v_2(x_2) \neq 0$ *and* $v_1(x_1) = 0$, *then* $\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0)) = [m_1; m_2)$;

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(b) *If $v_2(x_2)=0$ and $v_1(x_1)\neq 0$, then $\mathscr{W}(\mathscr{A}(\mu_1^0,\mu_2^0))=(m_1;m_2]$.*

In the remainder of this section we study the virtual levels and threshold eigenvalues [16] of $\mathscr{A}(\mu_1^0,\mu_2^0)$.

The Banach space $C(\mathbf{T}^3)$ is contain all continuous functions on $\mathbf{T}^3$.

Definition 1. *Let the condition (3) be fulfilled and $\alpha=1,2$. If the quantity 1 is an discrete eigenvalue of the Fredholm operator*

$$(G_\alpha(\mu_1,\mu_2)\psi_\alpha)(x)=\int_{\mathbf{T}^3}\frac{\mu_1 v_1(x)v_1(s)-\mu_2 v_2(x)v_2(s)}{u(s)-m_\alpha}\psi_\alpha(s)ds,\quad \psi_\alpha\in C(\mathbf{T}^3)$$

and the corresponding eigen function $\psi_\alpha(\cdot)$ (up to constant factor) is satisfy the assertion $\psi_\alpha(p_\alpha)\neq 0$, then at the point $z=m_\alpha$ the Friedrichs model $\mathscr{A}(\mu_1,\mu_2)$ is said to have a virtual level.

Remark 2. *For $\alpha\in\{1,2\}$ the quantity 1 is a discrete eigenvalue of the Fredholm operator $G_\alpha(\mu_1,\mu_2)$ iff $\mu_\alpha=\mu_\alpha^0$. Consequently, at the point $z=m_\alpha$ the operator $\mathscr{A}(\mu_1,\mu_2)$ has a virtual level iff $\mu_\alpha=\mu_\alpha^0$. In this case the value of $\mu_\beta>0$ with $\beta\neq\alpha$ is an arbitrary.*

Definition 2. *Let $\alpha=1,2$. At the point $z=m_\alpha$ the Friedrichs model $\mathscr{A}_\alpha(\mu_\alpha)$ is said to have a virtual level, if the quantity 1 is a discrete eigenvalue of the Fredholm operator*

$$(\widehat{G}_\alpha(\mu_\alpha)\varphi_\alpha)(x)=(-1)^{\alpha+1}\mu_\alpha v_\alpha(x)\int_{\mathbf{T}^3}\frac{v_\alpha(s)\varphi_\alpha(s)ds}{u(s)-m_\alpha},\quad \varphi_\alpha\in C(\mathbf{T}^3)$$

and for the corresponding eigen function $\varphi_\alpha(\cdot)$ the assertion $\varphi_\alpha(p_\alpha)\neq 0$ is valid.

We mention that in [17] the Friedrichs models family $h_\mu(p)$, $p\in\mathbf{T}^3$, $\mu>0$ with rank one perturbations, is considered. Authors are proved that for all $p\neq 0$ there exists a unique eigenvalue of $h_\mu(p)$, lying on l.h.s. of the essential spectrum if $h_\mu(0)$ has either a threshold eigenvalue or a virtual level at the point $z=0$. In [18, 19] for a family of Friedrichs models with rank perturbation is used the threshold analysis to investigate the number of discrete eigenvalues of the corresponding three-particle discrete model Schrödinger operator. Similar results are discussed in [20, 21] for some 3×3 operator matrices.

Taking into account Definitions 1, 2 and Remark 2 we receive the statement.

Corollary 3. *Let the condition (3) be fulfilled and $\alpha=1,2$.*

(a) *At the point $z=m_\alpha$ the operator $\mathscr{A}(\mu_1,\mu_2)$ has a virtual level iff at the point $z=m_\alpha$ the operator $\mathscr{A}_\alpha(\mu_\alpha)$ has a virtual level.*

(b) *The quantity $z=m_\alpha$ is a threshold eigenvalue of $\mathscr{A}(\mu_1,\mu_2)$ iff the quantity $z=m_\alpha$ is a threshold eigenvalue of $\mathscr{A}_\alpha(\mu_\alpha)$.*

We remark that the spectrum of the Friedrichs model $\mathscr{A}(\mu_1,\mu_2)$ may consist of a union of one, two, or three sets. In this paper we study the field of values in the case where the spectrum of $\mathscr{A}(\mu_1,\mu_2)$ is a purely essential. Mainly we analyzed the problem of whether boundary points of the essential spectrum of $\mathscr{A}(\mu_1,\mu_2)$ belong to its numerical range. As a conclusion one can notice that if both the bounds $z=m_\alpha$, $\alpha=1,2$ are threshold eigenvalues of $\mathscr{A}(\mu_1,\mu_2)$ then the numerical range and spectrum of $\mathscr{A}(\mu_1,\mu_2)$ are coincide. Thus, in this paper we have established a relation between the numerical range and threshold eigenvalues or virtual levels of $\mathscr{A}(\mu_1,\mu_2)$.

For the generalized Friedrichs model the results, where both boundary points of its essential spectrum are either threshold eigenvalues or virtual levels, were studied in [16, 22, 23] and they used to proof the existence the so-called two-sided Efimov's effect in [22, 23].

CONCLUSION

In the present paper the Friedrichs model $\mathscr{A}(\mu_1,\mu_2)$, $\mu_1,\mu_2>0$ with rank two perturbation is considered. This operator is related with a system of two quantum particles on 3D lattice. Firstly an information (definition, main properties and examples) about the field of values of a linear operators are given. Using the Weyl theorem the essential spectrum of $\mathscr{A}(\mu_1,\mu_2)$ is found. The discrete spectrum of $\mathscr{A}(\mu_1,\mu_2)$ is described as a set of all zeros of the corresponding Fredholm determinant. The number and location of the discrete eigenvalues of $\mathscr{A}(\mu_1,\mu_2)$ are investigated. The sufficient and necessary conditions which guarantees the equality of the spectrum of $\mathscr{A}(\mu_1,\mu_2)$ and its field of values (or numerical range) are given. The relation of the threshold eigenvalues and virtual levels with the numerical range of $\mathscr{A}(\mu_1,\mu_2)$ are established.

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