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Содержание

NUMBER AND LOCATION OF EIGENVALUES OF GENERALIZED FRIEDRICHS MODEL WITH FINITE RANK PERTURBATIONS

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Abstract: *in the present paper we study a generalized Friedrichs model A with finite rank perturbations. This model (Hamiltonian) is associated with the operator energy of non-conserved bounded number of particles on a d-dimensional lattice. The Fredholm determinant corresponding to the operator A is constructed. We choose the finite system of the bounded self-adjoint operators $\{A_\alpha\}$ such that the union of discrete spectrum of A_α is coincide with the discrete spectrum of A. The number and location of the eigenvalues of A is found.*

Keywords: *generalized Friedrichs model, non-local potential, molecular-resonance model, essential spectrum, eigenvalue.*

Operators known as Friedrichs operators [1] and generalized Friedrichs operators [2] appear in a series of problems in analysis, mathematical physics, and probability theory. The latter operators act in the Hilbert space

$$H := C \oplus L_2(T^d),$$

where T^d is the d-dimensional torus, according to the rule

$$A := \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^* & A_{11}^0 - \sum_{k=1}^{d} V_k \end{pmatrix}.$$

Here

$$A_{00} f_0 = w_0 f_0, \quad A_{01} f_1 = \alpha \int_{T^d} v(t) f_1(t) dt,$$

$$(A_{11}^0 f_1)(x) = w_1(x) f_1(x), \quad (V_k f_1)(x) = \beta \sin x_k \int_{T^d} \sin t_k f_1(t) dt, \quad k = 1,2,\ldots,d,$$

$f = (f_0, f_1) \in H$, $x = (x_1,\ldots,x_d) \in T^d$; w_0 is a constant, $v(\cdot)$ and $w_1(\cdot)$ are real-valued continuous functions on T^d, $\alpha, \beta > 0$ are the coupling parameters.

It is easy to verify that under these assumptions the model operator A is bounded and self-adjoint in H. In modern mathematical physics the operator A_{01} is called annihilation operator and A_{01}^* is called creation operator.

We note that the character of the spectrum, the structure of the resolvent, the form of the eigenvectors for the discrete and continuous spectra, and the existence and completeness of the wave operators naturally related to the ordinary Friedrichs model, i.e. to a self-adjoint operator of the form

+,

was completely or partly studied in many works, see, e.g., the pioneering work [1] and also [3] and [4], where $M \subset R^d$ is a manifold and D is a function of two variables on the M^2. It was established in [1] that in the case where $M = [-1,1] \subset R$, $w_1(x) = x$, and $\beta > 0$ is small, the operator B up to finitely many eigenvalues has an absolutely continuous spectrum and that this operator in its absolutely continuous subspace is unitarily equivalent to the operator B_0 such that

$$(B_0 f)(x) = w_1(x) f(x), \quad f \in L_2(M, dx).$$

Generalized Friedrichs model itself was introduced in [2], where its eigenvalues and "resonances" (i.e., the singularities of the analytic continuation of the resolvent) were studied. This model also considered in some other publications, among which we mention [5]. The threshold resonance, threshold and usual eigenvalues of A in the case $\beta = 0$ were discussed in many works, see e.g. [6-16]. The number and location of the eigenvalues of the generalized model with rank 3 perturbations were studied in [17-22] and used to define the number of closed bounded intervals and also to study the structure of the essential spectrum of a corresponding 3×3 operator matrices. More general case were studied in [23, 24]. In contrast above mentioned papers here the study of the discrete spectrum of a generalized Friedrichs model is reduces to the investigation of the discrete spectrum of the finite system of operators simpler that considered one.

To study the essential and discrete spectrum of A, we introduce the following operators:

$$\tilde{A}_0 : H \to H, \quad \tilde{A}_0 := \begin{pmatrix} 0 & 0 \\ 0 & A_{11}^0 \end{pmatrix};$$

$$A_0 : H \to H, \quad A_0 := \begin{pmatrix} A_{00} & A_{01} \\ A_{01}^* & A_{11}^0 \end{pmatrix};$$

$$A_k : L_2(T^d) \to L_2(T^d), \quad A_k := A_{11}^0 - V_k, \quad k = 1, \ldots, d.$$

It is clear that the perturbation $A - \tilde{A}_0$ of the operator $\tilde{A}_0$ is a self-adjoint operator of rank $d+2$. Therefore, in accordance with the Weyl theorem about the invariance of the essential spectrum under the finite rank perturbations, the essential spectrum of the operator A coincides with the essential spectrum of the operator $\tilde{A}_0$. It is evident that $\sigma_{ess}(\tilde{A}_0) = [m; M]$, where the numbers m and M are defined by

$$m := \min_{x \in T^d} w_1(x), \quad M := \max_{x \in T^d} w_1(x).$$

This yields $\sigma_{ess}(A) = [m; M]$.

For $k=0,1,\ldots,d$ we define an analytic function $\Delta_k(\cdot)$ (the Fredholm determinant associated with the operator A_k) in $C\setminus[m;M]$ by

$$\Delta_0(z):=w_0-z-\alpha^2\int_{T^d}\frac{v^2(t)dt}{w_1(t)-z};$$

$$\Delta_k(z):=1-\beta\int_{T^d}\frac{(\sin t_k)^2dt}{w_1(t)-z},\quad k=1,\ldots,d.$$

A simple consequence of the Birman-Schwinger principle and the Fredholm theorem imply that for any $k=0,1,\ldots,d$ the operator A_k has an eigenvalue $z\in C\setminus[m;M]$ if and only if $\Delta_k(z)=0$. Therefore,

$$\sigma_{disc}(A_k)=\{z\in C\setminus[m;M]:\ \Delta_k(z)=0\}.$$

In the rest part of this paper we assume that the functions $v(\cdot)$ and $w_1(\cdot)$ are the even functions on each variable. For example, the functions

$$w_1(x)=\sum_{k=1}^{d}(1-\cos x_k),\quad v(x)=\prod_{k=1}^{d}\cos(kx_k)$$

satisfy such conditions.

The following theorem describes the relation between the discrete spectrum of the operators A and A_k, $0,1,\ldots,d$.

Theorem 1. *The number* $z\in C\setminus[m;M]$ *is an eigenvalue of* A *if and only if* z *is an eigenvalue one of the operators* A_k, $0,1,\ldots,d$.

From Theorem 1 it follows that

$$\sigma_{disc}(A)=\bigcup_{k=0}^{d}\sigma_{disc}(A_k)$$

and hence

$$\sigma_{disc}(A)=\{z\in C\setminus[m;M]:\ \bigcap_{k=0}^{d}\Delta_k(z)=0\}.$$

Usually the function $\Delta(\cdot)$ defined in $C\setminus[m;M]$ by

$$\Delta(z):=\bigcap_{k=0}^{d}\Delta_k(z)$$

is called the Fredholm determinant associated with the operator A.

The next result establishes the number and location of the eigenvalues of the operator A.

Theorem 2. *For all values of the coupling parameters* $\alpha,\beta>0$ *the operator* A *has at most* $d+2$ *discrete eigenvalues (counting with the multiplicities) such that* $d+1$ *of them are located on the l.h.s. of* m *and one of them is located on the r.h.s. of* M.

Since the operators A_k, $0,1,\ldots,d$ have the simple structure than A, Theorems 1 and 2 plays crucial role in the investigation of the location and structure of the essential and discrete spectrum of the corresponding operator matrices in the truncated Fock space.

References

1. *Friedrichs K.O.* Uber die Spectralzerlegung einee Integral operators // Math. Ann., 115:1, 1938. Pp. 249-272.
2. *Lakaev S.N.* Some spectral properties of a generalized Friedrichs model // Trudy Sem. Petrovsk. 1986. № 11. Pp. 210-238; English transl. in J. Soviet Math. 45, 1989.
3. *Friedrichs K.O.* Perturbation of spectra in Hilbert space // Amer. Math. Soc. Providence. Rhole Island, 1965.
4. *Friedrichs K.O.* On the perturbation of continuous spectra // Comm. Pure Appl. Math. 1:4 (1948). Pp. 361-406.
5. *Lakshtanov E.L., Minlos R.A.* Two-Particle Bound State Spectrum of Transfer Matrices for Gibbs Fields (Fields on the Two-Dimensional Lattice. Adjacent Levels) // Funct. Anal. Appl. 39:1 (2005). P. 31-45.
6. *Albeverio S., Lakaev S.N., Rasulov T.H.* On the spectrum of an Hamiltonian in Fock space. Discrete spectrum asymptotics // J. Stat. Phys. 127:2 (2007). Pp. 191-220.
7. *Albeverio S., Lakaev S.N., Rasulov T.H.* The Efimov effect for a model operator associated with the Hamiltonian of a non conserved number of particles // Methods Funct. Anal. Topology, 13:1 (2007). P.1-16.
8. *Rasulov T.Kh.* On the number of eigenvalues of a matrix operator // Siberian Math. J. 52:2 (2011). P. 316-328.
9. *Rasulov T., Tosheva N.* New branches of the essential spectrum of a family of 3x3 operator matrices // Journal of Global Research in Math. Archive. 6:9 (2019). P. 18-21.
10. *Rasulov T.H., Dilmurodov E.B.* Eigenvalues and virtual levels of a family of 2x2 operator matrices // Methods of Functional Analysis and Topology. 25:1 (2019). P. 273-281.
11. *Rasulov T.H., Dilmurodov E.B.* Investigations of the numerical range of a operator matrix. J. Samara State Tech. Univ., Ser. Phys. and Math. Sci. 35:2 (2014). P. 50-63.
12. *Rasulov T.H., Dilmurodov E.B.* Threshold analysis for a family of 2x2 operator matrices // Nanosystems: Physics, Chemestry, Mathematics. 10:6 (2019). P. 616-622.
13. *Rasulov T.H., Dilmurodov E.B.* Threshold effects for a family of 2x2 operator matrices // Journal of Global Research in Mathematical Archives. 6:10 (2019). P. 4-8.
14. *Muminov M.I., Rasulov T.Kh.* An eigenvalue multiplicity formula for the Schur complement of a 3x3 block operator matrix // Siberian Math. J., 56:4 (2015). P. 878.
15. *Muminov M., Neidhardt H., Rasulov T.* On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case // Journal of Mathematical Physics. 56 (2015), 053507.
16. *Rasulov T.Kh.* Branches of the essential spectrum of the lattice spin-boson model with at most two photons // Theoretical and Mathematical Physics. 186:2 (2016), 251-267.
17. *Rasulov T.Kh.* Study of the essential spectrum of a matrix operator // Theoret. and Math. Phys. 164:1 (2010). P. 883-895.
18. *Rasulov T.H.* On the finiteness of the discrete spectrum of a 3x3 operator matrix // Methods of Functional Analysis and Topology, 22:1 (2016). P. 48-61.
19. *Muminov M.I., Rasulov T.H.* On the eigenvalues of a 2x2 block operator matrix // Opuscula Mathematica. 35:3 (2015). P. 369-393.
20. *Muminov M.I., Rasulov T.H.* Embedded eigenvalues of an Hamiltonian in bosonic Fock space // Comm. in Mathematical Analysis. 17:1 (2014). P. 1-22.
21. *Rasulov T.H.* The finiteness of the number of eigenvalues of an Hamiltonian in Fock space // Proceedings of IAM, 5:2 (2016). P. 156-174.

22. *Muminov M.I., Rasulov T.H.* Infiniteness of the number of eigenvalues embedded in the essential spectrum of a 2x2 operator matrix // Eurasian Mathematical Journal. 5:2 (2014). P. 60-77.
23. *Rasulov T.Kh.* Investigation of the spectrum of a model operator in a Fock space // Theoret. and Math. Phys. 161:2 (2009). P. 1460-1470.
24. *Muminov M.I., Rasulov T.H.* The Faddeev equation and essential spectrum of a Hamiltonian in Fock Space // Methods Funct. Anal. Topology. 17:1 (2011). P. 47-57.

ESSENTIAL AND DISCRETE SPECTRUM OF THE THREE-PARTICLE MODEL OPERATOR HAVING TENSOR SUM FORM

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Abstract: *this paper is devoted to the spectral analysis of a model operator (Hamiltonian) H_μ, $\mu > 0$ associated to a system of three quantum particles on a two-dimensional lattice. The operator H_μ can be represented as a tensor sum of two linear bounded self-adjoint Friedrichs models h_μ. For all values of the parameter $\mu > 0$ the existence of the unique eigenvalue of the operators h_μ and H_μ are shown. Using the spectrum of h_μ the essential spectrum of H_μ is described. The location of the branches of the essential spectrum of H_μ is identified.*

Keywords: *Hamiltonian, quantum particles, lattice, dispersion function, tensor sum, Friedrichs model, eigenvalue, essential spectrum.*

In models of solid state physics [1,2] and also in lattice quantum field theory [3], one considers discrete Schroedinger operators, which are lattice analogs of the three-particle Schroedinger operator in the continuous space. One of the important problem in the spectral analysis of Schroedinger operators (in both cases) is to find whether the discrete spectrum is finite or infinite set. In the present paper we consider the Hamiltonian H_μ which is related with the system of three quantum particles on a two dimensional lattice and describe its spectrum. We remark that Hamiltonian H_μ can be represented as a tensor sum of two linear bounded self-adjoint Friedrichs models h_μ.

For the convenience of the reader, first we give some information about the spectrum of tensor sum of operators [4]. Tensor sum and tensor product of Hilbert space operators can be thought as an extension to infinite dimensional spaces of the traditional Kronecker sum and Kronecker product of matrices on finite dimensional spaces. Let H_1 and H_2 be the Hilbert spaces and H be the tensor product product of H_1 and H_2, that is,