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Kibriyo Amrilloyeva; Dilshoda Akramova; Shuhrat Juraev; Jamshid Ergashov

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Determination of a Diffusion Coefficient and a Source Control Term for a Time Fractional Diffusion Equation with Integral Type Overdetermining Conditions

Kibriyo Amrilloyeva,[1, a)] Dilshoda Akramova,[1, b)] Shuhrat Juraev,[1, c)] and Jamshid Ergashov[2, d)]

[1)]*Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan.*
[2)]*Jizzakh State Pedagogical University, 4, Sharof Rashidov str., Jizzakh 130100, Uzbekistan.*

[a)]*Corresponding author:kibriyo.amrilloyeva.98@mail.ru*
[b)]*akramova.shoda@mail.ru*
[c)]*sjuraev42@gmail.com*
[d)]*jamshid_egr@mail.ru*

Abstract. We consider a non-linear heat equation involving a fractional derivative in time, with a nonlocal boundary condition. We determine a diffusion and a source control term of the time variable, and the temperature distribution for a problem with two over-determining condition of integral type. We prove the existence and uniqueness of the solution, and its continuous dependence on the data.

INTRODUCTION

The fractional diffusion equation more accurately represents the anomalous process than the classical heat exchange equation. Therefore, scientific research in this direction is of interest to mathematical scientists and engineers [1]. This work is part of the above direction in which the character of the environment is expressed function determination, that is, the inverse issue has been studied [2, 3].

Inverse source problems are the problems that consist of finding the unknown source control term via an additional data. Some works on fractional inverse diffusion problems have been published. We refer to [4, 5].

Here, we consider the so-called fractional diffusion problem involving the linear nonhomogeneous equation:

$$u_t(x,t)+\partial_t^{\alpha}u(x,t)-ku_{xx}+q(t)u(x,t)=f(x,t), \quad x\in(0,1), t\in(0,T], \tag{1}$$

with initial and nonlocal boundary conditions

$$u(x,0)=a(x), \qquad x\in[0,1], \tag{2}$$

$$u(0,t)=u(1,t) \quad u_x(1,t)=0, \qquad t\in[0,T], \tag{3}$$

where ∂_t^{α} is the Caputo fractional derivative of order $\alpha\in(0,1)$ in the time variable, defined by

$$\partial_t^{\alpha}v(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}v'(\tau)d\tau,$$

and $q(t), t>0$ is the source control term [3], $f(x,t)$ is the known source term, $a(x)$ is the initial temperature.

Our inverse problem consists of determining the diffusion coefficient k and the time dependent unknown coefficient of the source control term $q(t)$ and the temperature distribution $u(x,t)$, from the initial temperature (2) and the boundary conditions (3). To be determined uniquely k and $q(t)$, we need the following two over-determination conditions:

$$\int_0^1 u(x,t)dx=g(t), \tag{4}$$

and

$$\int_0^1 x^2u(x,t)dx=\beta(t), \tag{5}$$

where $g(t)$, $\beta(t)$ are given thermal energies.

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PRELIMINARIES

In this section, we give some notations which will be repeatedly used in the sequent sections. The Mittag-Leffler function plays an important role in the theory of fractional differential equations; for any $z \in \mathbb{C}$ the Mittag-Leffler function with parameter α is

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \Re(\alpha) > 0. \tag{6}$$

In particular, $E_1(z) = e^z$.

The Mittag-Leffler function of two parameters $E_{\alpha,\beta}(z)$ which is a generalization of (6) is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$- denotes the real part of the complex number α.

Proposition 1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1+|z|}, \quad \kappa \leq |arg(z)| \leq \pi.$$

For the proof, we refer to [6] for example.

Proposition 2. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Proposition 3. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0.$$

Proposition 4. *Let $\alpha > 0$, $\beta > 0$ and $\lambda > 0$, then we have*

$$\frac{d}{dt} t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) = t^{\beta-2} E_{\alpha,\beta-1}(-\lambda t^\alpha), \quad t > 0.$$

The proof of these assertions come from the definition of Caputo fractional derivative and differentiation of the two-parameter Mittag-Leffler function.

Proposition 5. (see [7]) *For $0 < \alpha < 1$, $t > 0$, we have $0 < E_\alpha(-t) < 1$. Moreover, $E_\alpha(-t)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dt^n} E_\alpha(-t) \geq 0, \quad \forall n \in \mathbb{N}.$$

Proposition 6. *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leq E_{\alpha,\alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Lemma 1. (see [6]) *The following Laplace transform of a three-parameter Mittag-Leffler function is true:*

$$L\left[t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\pm\omega t^\alpha)\right](s) = \int_0^\infty e^{-st} t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\pm\omega t^\alpha) dt = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha \mp \omega)^\gamma},$$

where $|\omega/s^\alpha| < 1$ and

$$E_{\alpha,\beta}^{\gamma}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}$$

$(\gamma)_n$ *denotes the Pochhammer symbol or the shifted factorial defined by*

$$(\gamma)_0 = 1, \qquad (\gamma)_n = \gamma(\gamma+1)\cdot\ldots\cdot(\gamma+n-1), \quad \gamma \neq 0.$$

Definition 1. *The Fox $H-$function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral [8]*

$$H^{m,n}_{p,q}\left[z\Big|^{(a_j,A_j)_1^p}_{(b_j,B_j)_1^q}\right] = \frac{1}{2\pi i}\int_\Omega \mathscr{H}^{m,n}_{p,q}(s) z^{-s} ds,$$

where

$$\mathscr{H}^{m,n}_{p,q}(s) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_i s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\prod\limits_{i=n+1}^{p}\Gamma(a_i+A_i s)}$$

with complex variable $z \neq 0$ and a contour Ω in the complex domain; the orders (m,n,p,q) are non-negative integers so that $0 \le m \le q$, $0 \le n \le p$, the parameters $A_i > 0$, $B_j > 0$ are positive and a_i, b_j, $i = 1,...,p$; $j = 1,...,q$ are arbitrary complex such that

$$A_i(b_j + l) \neq B_j(a_i - l' - 1); \quad l, l' = 0,1,2,..., i = 1,...,n, j = 1,...,m. \tag{7}$$

The details on the properties of the Fox's $H-$function and types of contour Ω can be found in [8].

Let

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \quad \gamma = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{8}$$

The Mittag-Leffler function can be expressed in terms of the Fox $H-$function as follows (see [8], p. 25)

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}[-z|^{(1-\gamma,1)}_{(0,1),(1-\beta,\alpha)}], \quad Re(\gamma) > 0.$$

Lemma 2. (see, [2]) *Let γ and μ be as given in (8) and let the condition (7) be satisfied. Then there holds the following result:*

If $\mu > 0$, $\gamma > 0$ then the $H-$function has the asymptotic expansion at infinity given by

$$H^{m,n}_{p,q}(z) = O\left(z^d \|\ln(z)\|^{M-1}\right), \quad |z| \to \infty,$$

where

$$d = \min\left[\frac{\Re(a_j)-1}{A_j}\right], 1 \le j \le n$$

and M is the order of the poles $\omega_{\lambda k} = \frac{1-a_\lambda+k}{A_\lambda}$, $\lambda = 1,...,n; k = 0,1,2,...$ to which some of the poles of $\Gamma(1-a_j - A_j s)$, $j = 1,2,...,n$ coincide.

EXISTENCE AND UNIQUENESS RESULT

First, note that for the non-selfadjoint operator $AX = -X'' = \lambda_n X$ with $X(0) = X(1)$, $X'(1) = 0$

$$X_0(x) = 2 \qquad X_{2n}(x) = 4\cos\lambda_n x \qquad X_{2n-1}(x) = 4(1-x)\sin\lambda_n x \tag{9}$$

and

$$Y_0(x) = x \qquad Y_{2n}(x) = x\cos\lambda_n x \qquad Y_{2n-1}(x) = \sin\lambda_n x \tag{10}$$

$$\lambda_n = 2\pi n, \quad n = 1,2,3...$$

which are Riesz bases in $L_2[0,1]$. For more details, the reader can consult [2], [9], [10]. Let us define the following space:

$$C^{2,\alpha}(\Omega) = \left\{u(x,t) : u(\cdot,t) \in C^2(0,1); t \in [0,T] \, \text{and} \, \partial_t^{\alpha} u(x,\cdot) \in C(0,T]; x \in (0,1)\right\}.$$

The functions a, f, g and β satisfy the following assumptions:

(A1) $f(x,t) \in C^{3,0}([0,1] \times [0,T])$, $f^{(4)}(\cdot,t) \in L_2[0,1]$ and $f(1,t) = f(0,t)$, $f_x(1,t) = 0$, $f_{xx}(0,t) = f_{xx}(1,t)$, $f_{xxx}(1,t) = 0$ for $t \in [0,T]$ and $\beta_1(t) = \int_0^1 f(x,t)dx \neq 0 (\in AC[0,T])$ for all $t \in [0,T]$;

(A2) $a(x) \in C^3[0,1]$, $a^{(4)}(x) \in L_2[0,1]$; $a(1) = a(0)$, $a'(1) = 0$, $a''(1) = a''(0)$, $a'''(1) = 0$;

(A3) $g(t) \in C^1[0,T]$ and $\int_0^1 a(x)dx = g(0)$;

(A4) $\beta(t) \in C^1[0,T]$ and $\int_0^1 x^2 a(x)dx = \beta(0)$. Let $\beta_2(t) = \int_0^1 x^2 f(x,t)dx \in AC[0,T]$.

By applying the Fourier method, the solution $u(x,t)$ of the problem (1)-(3) can be expanded in a uniformly convergent series in term of eigenfunctions of (9) in $L_2[0,1]$ of the form

$$u(x,t) = u_0(t)X_0(x) + \sum_{n=1}^{\infty} u_{2n}(t)X_{2n}(x) + \sum_{n=1}^{\infty} u_{2n-1}(t)X_{2n-1}(x) \tag{11}$$

The coefficients $u_0(t), u_{2n}(t), u_{2n-1}(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions of (9) and integrate over $(0,1)$. Recall that the inner product in $L_2[0,1]$ is defined by $(f,g) = \int_0^1 f(x)g(x)dx$. Let us note the expansion coefficients of $f(x,t)$ and $a(x)$ in the eigenfunctions of (10) for $n \geq 1$ respectively by

$$\begin{cases} (f(x,t), Y_0(x)) = f_0(t), \\ (f(x,t), Y_{2n-1}(x)) = f_{2n-1}(t), \\ (f(x,t), Y_{2n}(x)) = f_{2n}(t), \end{cases} \tag{12}$$

and

$$\begin{cases} (a(x), Y_0(x)) = a_0, \\ (a(x), Y_{2n-1}(x)) = a_{2n-1}, \\ (a(x), Y_{2n}(x)) = a_{2n}. \end{cases} \tag{13}$$

We obtain in view of (1) and with $(u(x,t), Y_0(x)) = u_0(t)$,

$$u_0'(t) + \partial_t^{\alpha} u_0(t) + q(t)u_0(t) = f_0(t),$$

and according to $F_0(t) = f_0(t) - q(t)u_0(t)$, and first component of (13), we may write

$$\begin{cases} u_0'(t) + \partial_t^{\alpha} u_0(t) = F_0(t), \\ u_0(0) = a_0. \end{cases} \tag{14}$$

For $u_{2n}(t) = (u_{2n}(x,t), Y_{2n}(x))$; $n \geq 1$, in view of (1) we have

$$\begin{cases} u_{2n}'(t) + \partial_t^{\alpha} u_{2n}(t) + k\lambda_n^2 u_{2n}(t) = F_{2n}(t), \\ u_{2n}(0) = a_{2n} \end{cases} \tag{15}$$

where $F_{2n}(t) = f_{2n}(t) - q(t)u_{2n}(t) - k\lambda_n u_{2n-1}(t)$. Also, the linear fractional differential equations satisfied by $u_{2n-1}(t)$; $n \geq 1$, are

$$\begin{cases} u_{2n-1}'(t) + \partial_t^{\alpha} u_{2n-1}(t) + k\lambda_n^2 u_{2n-1}(t) + q(t)u_{2n-1}(t) = f_{2n-1}(t), \\ u_{2n-1}(0) = a_{2n-1}. \end{cases} \tag{16}$$

Applying Laplace transform to (14), we get the following Volterra integral equation satisfied by the solution

$$u_0(t) = \int_0^t E_{1-\alpha}(-(t-\tau)^{1-\alpha})\big(f_0(\tau) - q(\tau)u_0(\tau)\big)d\tau$$

$$+a_0t^{1-\alpha}E_{1-\alpha,2-\alpha}(-t^{1-\alpha})+a_0E_{1-\alpha}(-t^{1-\alpha}). \tag{17}$$

This solution is bounded in $C[0,T]$ in view of (A1)-(A3). We have

$$|u_0(t)| \le \left(\|f_0\|_{C[0,T]} + \|q\|_{C[0,T]}\|u_0(t)\|\right)\left|\int_0^t E_{1-\alpha}(-(t-\tau)^{1-\alpha})d\tau\right|$$

$$+|a_0|\cdot|t^{1-\alpha}E_{1-\alpha,2-\alpha}|+|a_0|\cdot|E_{1-\alpha}(-t^{1-\alpha})|$$

$$\le c_1T^{\alpha}\|f_0\|_{C[0,T]}+c_2T^{\alpha}|a_0|+c_1T^{\alpha}\|q\|_{C[0,T]}\|u_0(t)\|,$$

here and hereinafter c_i are positive values independent of given functions and T. Hence,

$$\|u_0(t)\| \le \frac{c_1T^{\alpha}||f_0||_{C[0,T]}+|a_0|(c_1T^{\alpha}+c_2)}{1-\Psi_0} := \psi_0, \tag{18}$$

here

$$\Psi_0 = c_1T^{\alpha}\|q\|_{C[0,T]} < \frac{1}{2}. \tag{19}$$

Let $\mathscr{L}[u_{2n-1}(t)] := U_{2n-1}(s)$ be the Laplace transform of $u_{2n-1}(t)$ with respect to variable t. In sequence, applying to equation (16) the Laplace transform with respect to the time variable t, we obtain the following equation:

$$sU_{2n-1}(s)-U_{2n-1}(0)+s^{\alpha}U_{2n-1}(s)-s^{\alpha-1}U_{2n-1}(0)+\lambda_n^2kU_{2n-1}(s)=\Phi_{2n-1}(s),$$

where $\mathscr{L}[f_{2n-1}(t)] := \Phi_{2n-1}(s)$. After solve this with equation respect to $U(s)$ we get

$$U_{2n-1}(s) = \frac{1}{s+s^{\alpha}+\lambda_n^2k}\Phi_{2n-1}(s)+\frac{s^{\alpha-1}+1}{s+s^{\alpha}+\lambda_n^2k}a_{2n-1}. \tag{20}$$

We calculate the inverse Laplace transform of the function $U_{2n-1}(s)$ defined by (17). First, these operations we carry out for $\frac{1}{s+s^{\alpha}+\lambda_n^2k}\Phi_{2n-1}(s)$. It may be performed by using the equality

$$\frac{1}{s+s^{\alpha}+\lambda_n^2k} = \frac{1}{s+s^{\alpha}}\cdot\frac{1}{1+\frac{\lambda_n^2k}{s+s^{\alpha}}}, \tag{21}$$

$$\frac{1}{1+\frac{\lambda_n^2k}{s+s^{\alpha}}} = \sum_{j=0}^{\infty}\left(-\frac{\lambda_n^2k}{s+s^{\alpha}}\right)^j$$

for $\left|\frac{\lambda_n^2k}{s+s^{\alpha}}\right| < 1$. On bases of (21) from last equality we have

$$\frac{1}{s+s^{\alpha}+\lambda_n^2k} = \sum_{j=0}^{\infty}\frac{(-\lambda_n^2k)^js^{-\alpha(j+1)}}{(s^{1-\alpha}+1)^{j+1}}.$$

Then, according to Lemma 2, we note

$$\frac{1}{s+s^{\alpha}+\lambda_n^2k} = \mathscr{L}\Big[\sum_{j=0}^{\infty}t^j(-\lambda_n^2k)^jE_{1-\alpha,j+1}^{j+1}(-t^{1-\alpha})\Big], \tag{22}$$

and

$$\frac{s^{\alpha-1}}{s+s^{\alpha}+\lambda_n^2k} = \mathscr{L}\Big[\sum_{j=0}^{\infty}t^{j+1-\alpha}(-\lambda_n^2k)^jE_{1-\alpha,j-\alpha+2}^{j+1}(-t^{1-\alpha})\Big]$$

$$\frac{s^{\alpha-1}+1}{s+s^{\alpha}+\lambda_n^2k} = \mathscr{L}\Big[\sum_{j=0}^{\infty}t^j(-\lambda_n^2k)^jE_{1-\alpha,j+1}^{j+1}(-t^{1-\alpha})+\sum_{j=0}^{\infty}t^{j+1-\alpha}(-\lambda_n^2k)^jE_{1-\alpha,j+2-\alpha}^{j+1}(-t^{1-\alpha})\Big]. \tag{23}$$

We can get

$$\sum_{j=0}^{\infty} t^j(-\lambda_n^2 k)^j E_{1-\alpha,j+1}^{j+1}(-t^{1-\alpha}) = \sum_{j=0}^{\infty}(-\lambda_n^2 k)^j t^j \sum_{i=0}^{\infty}\frac{(j+1)_i}{\Gamma((1-\alpha)i+j+1)}\frac{(-t^{1-\alpha})^i}{i!}$$

$$= \sum_{i=0}^{\infty}(-t^{1-\alpha})^i \sum_{j=0}^{\infty}\frac{(i+1)_j}{\Gamma((1-\alpha)i+j+1)}\frac{(-\lambda_n^2 kt)^j}{j!} = \sum_{i=0}^{\infty}(-t^{1-\alpha})^i E_{1,(1-\alpha)i+1}^{i+1}(-\lambda_n^2 kt), \tag{24}$$

and

$$\sum_{j=0}^{\infty} t^{j+1-\alpha}(-\lambda_n^2 k)^j E_{1-\alpha,j+2-\alpha}^{j+1}(-t^{1-\alpha}) = \sum_{j=0}^{\infty}(-\lambda_n^2 k)^j t^{j+1-\alpha} \sum_{i=0}^{\infty}\frac{(j+1)_i}{\Gamma((1-\alpha)i+j-\alpha+2)}\frac{(-t^{1-\alpha})^i}{i!}$$

$$= \sum_{i=0}^{\infty} t^{1-\alpha}(-t^{1-\alpha})^i \sum_{j=0}^{\infty}\frac{(i+1)_j}{\Gamma((1-\alpha)i+j-\alpha+2)}\frac{(-\lambda_n^2 kt)^j}{j!} = \sum_{i=0}^{\infty} t^{1-\alpha}(-t^{1-\alpha})^i E_{1,(1-\alpha)i+2-\alpha}^{i+1}(-\lambda_n^2 kt). \tag{25}$$

Taking into account (24) and (25), from (22) and (23), we obtain solution of (16):

$$u_{2n-1}(t) = \int_0^t \sum_{i=0}^{\infty}\left(-(t-\tau)^{1-\alpha}\right)^i E_{1,(1-\alpha)i+1}^{i+1}(-\lambda_n^2 k(t-\tau))\Big[f_{2n-1}(\tau)$$

$$-q(\tau)u_{2n-1}(\tau)\Big]d\tau + a_{2n-1}\Big[\sum_{i=0}^{\infty}(-1)^i t^{(1-\alpha)(i+1)} E_{1,(1-\alpha)i+2-\alpha}^{i+1}(-\lambda_n^2 kt)$$

$$+\sum_{i=0}^{\infty}(-1)^i t^{(1-\alpha)i} E_{1,(1-\alpha)i+1}^{i+1}(-\lambda_n^2 kt)\Big]. \tag{26}$$

According to proposition 4

$$E_{1,(1-\alpha)i+2-\alpha}^{i+1}(-\lambda_n^2 kt) = \frac{1}{\Gamma(i+1)} H_{1,2}^{1,1}[\lambda_n^2 kt|_{(0,1),(1-((1-\alpha)i+2-\alpha),1)}^{(1-(i+1),1)}]$$

$$E_{1,(1-\alpha)i+1}^{i+1}(-\lambda_n^2 kt) = \frac{1}{\Gamma(i+1)} H_{1,2}^{1,1}[\lambda_n^2 kt|_{(0,1),(1-((1-\alpha)i+1),1)}^{(1-(i+1),1)}]$$

Then, according to Lemma 2, we have

$$H_{1,2}^{1,1}[\lambda_n^2 kt|_{(0,1),(1-((1-\alpha)i+2-\alpha),1)}^{(1-(i+1),1)}] \le \frac{c_3}{(\lambda_n^2 kt)^{i+1}},$$

and

$$H_{1,2}^{1,1}[\lambda_n^2 kt|_{(0,1),(1-((1-\alpha)i+1),1)}^{(1-(i+1),1)}] \le \frac{c_4}{(\lambda_n^2 kt)^{i+1}}.$$

From the last estimates, we can get aprior estimate for $u_{2n-1}(t)$:

$$\|u_{2n-1}(t)\| \le \frac{c_5}{k}\Big[\|f_{2n-1}\| + |a_{2n-1}|\Big]\Big[1-\Psi_1\Big]^{-1} := \psi_1 \tag{27}$$

where $c_5 = \max\{c_3, c_4\}$ and

$$\Psi_1 = \frac{c_4}{\lambda_n^2 k} e^{-\frac{1}{4\lambda_n^2 kT^\alpha}}\|q\|_{C[0,T]} < \frac{1}{2}. \tag{28}$$

Similarly does operation to the problem of (15) like (16), we obtain its solution

$$u_{2n}(t) = \int_0^t \sum_{i=1}^{\infty}(-(t-\tau)^{1-\alpha})^i E_{1,(1-\alpha)i+1}^{i+1}(-\lambda_n^2 k(t-\tau))\Big[f_{2n}(\tau)$$

$$-q(\tau)u_{2n}(\tau)-2\lambda_n k u_{2n-1}(\tau)\Big]d\tau + a_{2n}\Big[\sum_{i=0}^{\infty} t^{1-\alpha}(-t^{1-\alpha})^i E^{i+1}_{1,(1-\alpha)i+2-\alpha}(-\lambda_n^2 kt)$$

$$+\sum_{i=0}^{\infty}(-t^{1-\alpha})^i E^{i+1}_{1,(1-\alpha)i+1}(-\lambda_n^2 kt)\Big]. \tag{29}$$

Under the conditions (A1)-(A3) the function $u_{2n}(t)$ is bounded in $C[0,T]$ as follows

$$\|u_{2n}(t)\| \le \frac{c_6}{k}\Big[\|f_{2n}\| + |a_{2n}| + \lambda_n k \psi_1\Big]\Big[1-\Psi_1\Big]^{-1} := \psi_2. \tag{30}$$

Let us use the topological product Banach spaces $Y = C[0,T]\times C[0,T]\times C[0,T]$ endowed with its norm to prove the existence and uniqueness of the solution under this form $(u_0(t), u_{2n}(t), u_{2n-1}(t)) \in Y$. Define the operator Γ on Y by $\Gamma(u_0, u_{2n}, u_{2n-1})(t) = (P_0u_0(t), P_{2n}u_{2n}(t), P_{2n-1}u_{2n-1}(t))$ where the operators P_0, P_{2n}, P_{2n-1} are defined on $C[0,T]$ by the right side of (17), (26) and (29) respectively. In view of (18), (27) and (30) $\Gamma : Y \to Y$.

Prove that Γ is a contraction on Y. So, for each

$$(u_0(t), u_{2n}(t), u_{2n-1}(t)); (v_0(t), v_{2n}(t), v_{2n-1}(t)) \in Y$$

we have

$$\|\Gamma(u_0, u_{2n}, u_{2n-1}) - \Gamma(v_0, v_{2n}, v_{2n-1})\|_Y$$

$$\le \max(\|P_0u_0 - P_0v_0\|_{C[0,T]}; \|P_{2n}u_{2n} - P_{2n}v_{2n}\|_{C[0,T]}; \|P_{2n-1}u_{2n-1}\|_{C[0,T]}).$$

First, we get easily

$$\|P_0u_0 - P_0v_0\|_{[C[0,T]]^3} \le c_1 T^{\alpha}\|q\|_{C[0,T]}\|u_0 - v_0\|_{C[0,T]} \le \Psi_0\|u_0 - v_0\|_{C[0,T]},$$

for $n \ge 1$

$$\|P_{2n-1}u_{2n-1} - P_{2n-1}v_{2n-1}\|_Y \le \Psi_1||u_{2n-1} - v_{2n-1}||_{C[0;T]}.$$

Similarly, for each $t \in [0,T]$

$$\|P_{2n}u_{2n} - P_{2n}v_{2n}\|_Y \le \Psi_1\|u_{2n} - v_{2n}\|_{C[0,T]} + 2\lambda_n k\Psi_2\|u_{2n-1} - v_{2n-1}\|_{C[0,T]},$$

which gives for $n \ge 1$, here Ψ_i, $(i = 0,1)$ are given in (19), (28) and

$$\Psi_2 = \frac{c_7}{\lambda_n^2 k} e^{-\frac{1}{\lambda_n^2 k T^{\alpha}}}. \tag{31}$$

Consequently,

$$\|\Gamma(u_0, u_{2n}, u_{2n-1}) - \Gamma(v_0, v_{2n}, v_{2n-1})\|_Y$$

$$\le \max\Big[\Big(\Psi_0\|u_0 - v_0\|_{C[0,T]}; \Psi_1\|u_{2n-1} - v_{2n-1}\|_{C[0,T]}; \Psi_1\|u_{2n} - v_{2n}\|_{C[0,T]}\Big)$$

$$+2\lambda_n k\Psi_2\Big(0; 0; \|u_{2n-1} - v_{2n-1}\|_{C[0,T]}\Big)\Big]$$

$$\Rightarrow \|\Gamma(u_0, u_{2n}, u_{2n-1}) - \Gamma(v_0, v_{2n}, v_{2n-1})\|_Y \le [\max(\Psi_0; \Psi_1) + \frac{c_7}{\pi n} e^{-\frac{1}{\lambda_n^2 k T^{\alpha}}}]$$

$$\times\|(u_0, u_{2n}, u_{2n-1}) - (v_0, v_{2n}, v_{2n-1})\|$$

According to (19) and (26)

$$\max(\Psi_0; \Psi_1) + \tfrac{c_7}{\pi n} e^{-\frac{1}{\lambda_n^2 k T^{\alpha}}} < 1 \text{ for}$$

$$\frac{c_7}{\pi n} e^{-\frac{1}{\lambda_n^2 k T^{\alpha}}} < \frac{1}{2}. \tag{32}$$

Then, Γ is a contraction on Y and has a unique fixed point which is the coefficients (u_0, u_{2n}, u_{2n-1}) of the solution (11). Then, there exists a unique solution of (1)-(3) for arbitrary $q(t)$ bounded in $C[0,T]$.

DETERMINATION OF THE DIFFUSION COEFFICIENT k AND $q(t)$ IN $C[0,T]$

Firstly, we recover the unknown diffusion coefficient k. For this, we use the over-determination conditions (4) and (5).

So, applying over-determination conditions (4) and (5) to equation (1), we obtain the following system of equations with respect to the unknowns:

$$g'(t)+(\partial_t^{\alpha}g)(t)-k\int_0^1 u_{xx}dx+g(t)q(t)=\beta_1(t), \tag{33}$$

$$\beta'(t)+(\partial_t^{\alpha}\beta)(t)-k\int_0^1 x^2u_{xx}dx+\beta(t)q(t)=\beta_2(t). \tag{34}$$

If we multiply equation (33) by $\beta(t)$, equation (34) by $g(t)$, and also taking into account that

$$\int_0^1 u_{xx}dx=-u_x(0,t), \quad \text{and} \quad \int_0^1 x^2u_{xx}dx=-2u(1,t)+2g(t),$$

then, we obtain

$$g(t)[\beta'(t)+\partial_t^{\alpha}\beta(t)]+\beta(t)\Big[\int_0^1 f(x,t)dx-g'(t)-\partial_t^{\alpha}g(t)\Big]-k\big(2g^2(t)$$

$$+4u_0(t)g(t)+8\sum_{n=1}^{\infty}u_{2n}(t)g(t)+\beta(t)\sum_{n=1}^{\infty}4\lambda_nu_{2n-1}(t)\big)=\beta_1(t)g(t). \tag{35}$$

Considering the value of (35) in $t=0$, we have

$$k=\frac{1}{2}\left[\frac{g(0)[\beta'(0)+\partial_t^{\alpha}\beta(0)-\beta_1(0)]}{\nu}+\frac{\beta(0)[\beta_1(0)-g'(0)-\partial_t^{\alpha}g(0)]}{\nu}\right] \tag{36}$$

where

$$\nu:=g^2(0)+2a_0g(0)+2\beta(0)\sum_{n=1}^{\infty}\lambda_na_{2n-1}+4g(0)\sum_{n=1}^{\infty}a_{2n}. \tag{A5}$$

In subsequent calculations, we take k as a known number.

Then, we derive (11) with respect to x and get $u_x(0,t)=\sum\limits_{n=1}^{\infty}8\pi nu_{2n-1}(t)$ where $u_{2n-1}(t);n\geq 1$ are given by (26). Under (A2)-(A3) the last series is convergent.

From (33) we obtain an integral equation with respect to $q(t)$ as follows

$$q(t)=\frac{1}{g(t)}\left[\beta_1(t)-g'(t)-\partial_t^{\alpha}g(t)-k\sum_{n=1}^{\infty}4\lambda_nu_{2n-1}(t)\right]$$

or

$$q(t)=q_0(t)-\frac{8\pi k}{g(t)}\sum_{n=1}^{\infty}n\int_0^t\Big[\sum_{i=0}^{\infty}\big(-(t-\tau)^{1-\alpha}\big)^iE^{i+1}_{1,(1-\alpha)i+1}(-\lambda_n^2k(t-\tau))q(\tau)u_{2n-1}(\tau)\Big]d\tau, \tag{37}$$

where

$$q_0(t)=\frac{1}{g(t)}\Bigg[\beta_1(t)-g'(t)-\partial_t^{\alpha}g(t)-\sum_{n=1}^{\infty}4\lambda_nk\int_0^t\sum_{i=0}^{\infty}\big(-(t-\tau)^{1-\alpha}\big)^iE^{i+1}_{1,(1-\alpha)i+1}(-\lambda_n^2k(t-\tau))f_{2n-1}(\tau)d\tau$$

$$-\sum_{n=1}^{\infty}4\lambda_nka_{2n-1}\Big[\sum_{i=0}^{\infty}(-1)^it^{(1-\alpha)(i+1)}E^{i+1}_{1,(1-\alpha)i+2-\alpha}(-\lambda_n^2kt)+\sum_{i=0}^{\infty}(-1)^it^{(1-\alpha)i}E^{i+1}_{1,(1-\alpha)i+1}(-\lambda_n^2kt)\Big]\Bigg].$$

The solution of integral equation (26) depends on q, i.e. $u_{2n-1} = u_{2n-1}(t;q)$. We introduce an operator G defining it by the right hand side of (37)

$$G[q](t) = q_0(t) - \frac{8\pi k}{g(t)} \sum_{n=1}^{\infty} n \int_0^t \Big[\sum_{i=0}^{\infty} \left(-(t-\tau)^{1-\alpha}\right)^i E^{i+1}_{1,(1-\alpha)i+1}(-\lambda_n^2 k(t-\tau)) q(\tau) u_{2n-1}(\tau;q) \Big] d\tau. \tag{38}$$

Then equation (38) is written in a more convenient form as

$$q(t) = G[q](t) \tag{39}$$

Let $q_{00} := \max_{t\in[0,T]}$ .Fix a number $\rho > 0$ and consider the ball

$$\Phi^T(q_0,\rho) := \{q(t) : q(t) \in C[0,T], \|q - q_0\| \le \rho\}.$$

Theorem 1. *Let (A1)-(A5) be satisfied. Then the inverse problem has a unique solution $\{u(x,t), k, q(t)\}$ for some small T.*

Proof. Let us first prove that for an enough small $T > 0$ the operator G maps the ball $\Phi^T(q_0,\rho)$ into itself, i.e., the condition $q(t) \in \Phi^T(q_0,\rho)$ implies that $G[q](t) \in \Phi^T(q_0,\rho)$. Indeed, for any continuous function $q(t)$, the function $G[q](t)$ calculated using formula (38) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|G[q](t) - q_0(t)\| \le \frac{C\|q\|}{\pi g_0} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{\lambda_n^2 k T^\alpha}}}{n} \cdot \frac{|a_{2n-1}| + \|f_{2n-1}\|}{1 - \Psi_1}$$

Here we have used the estimate (27). Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(t)$ belongs to the ball $\Phi^T(q_0,\rho)$ implies the inequality

$$\|q\| \le \rho + \|q_0\| \tag{40}$$

Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\rho + \|q_0\|$. Performing these replacements, we obtain the estimate

$$\|G[q](t) - q_0(t)\| \le \frac{C(\rho + \|q_0\|)}{g_0} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{\lambda_n^2 k T^\alpha}}}{\pi n} \cdot \frac{|a_{2n-1}| + \|f_{2n-1}\|}{1 - (\rho + \|q_0\|)\Psi_2}.$$

Where

$$\Psi_q = (\rho + \|q_0\|)\Psi_2 < 1. \tag{41}$$

According to the Abel's test and (A1)-(A5), the following series are convergent

$$K_1 := \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{\lambda_n^2 k T^\alpha}}}{n} \cdot \frac{|a_{2n-1}|}{1 - (\rho + \|q_0\|)\Psi_2}$$

$$K_2 := \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{\lambda_n^2 k T^\alpha}}}{n} \cdot \frac{\|f_{2n-1}\|}{1 - (\rho + \|q_0\|)\Psi_2}.$$

Let T_1 be a positive root of the equation

$$m_1(T) = \frac{C(\rho + \|q_0\|)}{g_0} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{\lambda_n^2 k T^\alpha}}}{\pi n} \cdot \frac{|a_{2n-1}| + \|f_{2n-1}\|}{1 - (\rho + \|q_0\|)\Psi_2} = \rho$$

Then for $T \in [0,T_1]$ we have $G[q](t) \in \Phi^T(q_0,\rho)$: Now consider two functions $q(t)$ and $\tilde{q}(t)$ belonging to the ball $\Phi^T(q_0,\rho)$ and estimate the distance between their images $G[q](t)$ and $G[\tilde{q}](t)$ in the space $C[0,T]$. The function $\tilde{u}_{2n-1}(t)$ corresponding to $\tilde{q}(t)$ satisfies the integral equation (26) with the functions $a_{2n-1} = \tilde{a}_{2n-1}$ and $f_{2n-1}(t) =$

$\tilde{f}_{2n-1}(t)$ Composing the difference $G[q](t)-G[\tilde{q}](t)$ with the help of equations (27) and then estimating its norm, we obtain

$$\|G[q](t)-G[\tilde{q}](t)\| \leq \frac{C\|q\|}{\pi g_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{\lambda_n^2 kT^\alpha}}}{n}\Big[\|q\|\|u_{2n-1}\|+\|\tilde{q}\|\|\tilde{u}_{2n-1}\|\Big].$$

Using inequality (26) and the estimate (27) with $a_{2n-1}=\tilde{a}_{2n-1},\ f_{2n-1}(t)=\tilde{f}_{2n-1}(t)$, we continue the previous inequality in the following form:

$$\|G[q](t)-G[\tilde{q}](t)\| \leq \frac{C\|q\|}{\pi g_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{\lambda_n^2 kT^\alpha}}}{n}\frac{|a_{2n-1}|+\|f_{2n-1}\|}{1-\|q\|\Psi_2}(1+\|\tilde{q}\|)\|\tilde{q}\| \tag{42}$$

The functions $q(t)$ and $\tilde{q}(t)$ belong to the ball $\Phi^T(q_0,\rho)$, and hence for each of these functions one has inequality (37). Note that the function on the right-hand side in inequality (38) at the factor$\|\bar{q}\|$ is monotone increasing with $\|q\|$,$\|\bar{q}\|$ and$T$. Consequently, replacing $\|q\|$ and $\|\bar{q}\|$ in inequality (38) (including in λ) with $\rho+\|q\|$ will only strengthen the inequality. Thus, we have

$$\|G[q](t)-G[\tilde{q}](t)\| \leq \frac{C\|q\|}{\pi g_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{\lambda_n^2 kT^\alpha}}}{n}\frac{|a_{2n-1}+\|f_{2n-1}\||}{1-(\rho+\|q\|)\Psi_2}(1+(\rho+\|q\|))\|\tilde{q}\| \leq m_2(T)\|\tilde{q}\|$$

Let T_1 be a positive root of the equation

$$m_2(T)=\frac{C\|q\|}{\pi g_0}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{\lambda_n^2 kT^\alpha}}}{n}\frac{|a_{2n-1}+\|f_{2n-1}\||}{1-(\rho+\|q\|)\Psi_2}(1+(\rho+\|q\|))=1$$

Then for $T\in[0,T_2]$ we have that the distance between the functions $G[q](t)$ and $G[\tilde{q}](t)$ in the function space $C[0,T]$ is not greater than the distance between the functions $q(t)$ and $\tilde{q}(t)$ multiplied by $m_2(T)<1$. Consequently, if we choose $T^*<min(T_1,T_2)$, then the operator G is a contraction in the ball $\Phi^T(q_0,\rho)$. However, in accordance with the Banach theorem, the operator G has a unique fixed point in the ball $\Phi^T(q_0,\rho)$, i.e., there exists a unique solution of equation (38). Theorem 1 is proven.

Estimation of the time of the local existence. According to (19), (28) and (32) and (41) T^* must satisfy this approximation

$$T^*<\inf\Big[\big(\frac{1}{2c_1M}\big)^{\frac{1}{\alpha}};\big(\lambda_n^2 k\ln\big(\frac{2c_4M}{\lambda_n^2 k}\big)\big)^{-\frac{1}{\alpha}};\big(\lambda_n^2 k\ln\big(\frac{2c_7}{\pi n}\big)\big)^{-\frac{1}{\alpha}};$$

$$\big(\lambda_n^2 k\ln\big(\frac{2c_7((\|q_0\|+\rho)}{\lambda_n^2 k}\big)\big)^{-\frac{1}{\alpha}}\Big]$$

to ensure the existence of the solution on $[0,T]$ for each $T<T^*$, here $\|q\|=M$.

Convergence of the solution series (9). As it was proved, in view of (A1)-(A5), the coefficients $u_0(t),u_{2n-1}(t)$ and $u_{2n}(t);\, n\geq 1$ are bounded in $C[0,T]$. Thus, the series expression (6) of $u(x,t)$ gives

$$\sup_{x\in[0,1]}|u(x,t)|=|u(t)|\leq 2|u_0(t)|+4\sum_{n=1}^{\infty}|u_{2n-1}(t)|+4\sum_{n=1}^{\infty}|u_{2n}(t)|. \tag{43}$$

(A1)-(A3) imply that $\sum\limits_{n=1}^{\infty}n^3|a_{2n-1}|$, $\sum\limits_{n=1}^{\infty}n^3|f_{2n-1}|$, $\sum\limits_{n=1}^{\infty}n^2|f_{2n}|$ and $\sum\limits_{n=1}^{\infty}n^2|a_{2n}|$ are convergent. In consequent, by (27) and (30) the series $u(x,t)$ and its partial derivative $u_x(x,t)$ are uniformly convergent in $[0,1]\times[\varepsilon,T]$ for any $\varepsilon>0$. Therefore, $u(x,\cdot)$ and $u_x(x,\cdot)$ are in $C[0,T]$ for $x\in[0,1]$. Also, its second partial derivative $u_{xx}(x,t)$ is uniformly convergent in $[0,1]\times[\varepsilon,T]$ for any $\varepsilon>0$ by the Cauchy-Schwartz inequality and the Bessel's inequality in view of the fact that $a_{2n-1}=\frac{a^{(4)}_{2n-1}}{\lambda_n^4}$ and $f_{2n-1}=\frac{f^{(4)}_{2n-1}}{\lambda_n^4}$, $a_{2n}=\frac{a^{(2)}_{2n}}{\lambda_n^2}$ and $f_{2n}=\frac{f^{(2)}_{2n}}{\lambda_n^2}$. Then, the uniformly convergence of $\sum\limits_{n=1}^{\infty}n^3u_{2n-1}(t)$, $\sum\limits_{n=1}^{\infty}n^2u_{2n}(t)$ obtained from (14), (15) and (16). Besides, under the conditions (A2)- (A5), the fractional derivative $(\partial_t^\alpha u)(x,t)$ and $u_t(x,t)$ are uniform convergent in $[0,1]\times[\varepsilon,T]$. Thus, $u(x,t)\in C^{2,\alpha}[0,1]\times(0,T]\bigcap C([0,1]\times[0,T])$ and satisfies the conditions (2), (3) for arbitrary $q(t)\in C[0,T]$.

Step 5: Uniqueness of the solution $(u(x,t),q(t))$. Assume that the pairs of functions $(u(x,t),q(t))$ and $(v(x,t),b(t))$ are solutions of the inverse problem (1)-(5). Let us use the product Banach space $\left[C[0,T]\right]^4$ endowed with its norm to prove the uniqueness of the solutions under this form $\left(u_0(t),u_{2n-1}(t),u_{2n}(t),q(t)\right)\in\left[C[0,T]\right]^4$. We have

$$\|\left(u_0(t),u_{2n-1}(t),u_{2n}(t),q(t)\right)-\left(v_0(t),v_{2n-1}(t),v_{2n}(t),b(t)\right)\|_{\left[C[0,T]\right]^4}$$

$$\leq\max(\Psi_0,\Psi_1,\Psi_q)\|\left(u_0(t),u_{2n-1}(t),u_{2n}(t),q(t)\right)-\left(v_0(t),v_{2n-1}(t),v_{2n}(t),b(t)\right)\|_{\left[C[0,T]\right]^4}$$

In view of (19), (28) and (41)

$$\|\left(u_0(t),u_{2n-1}(t),u_{2n}(t),q(t)\right)-\left(v_0(t),v_{2n-1}(t),v_{2n}(t),b(t)\right)\|_{\left[C[0,T]\right]^4}=0$$

This implies that $u(x,t)=v(x,t)$ and $q(t)=b(t)$, $t\in[0,T]$. This completes the proof.

CONTINUOUS DEPENDENCE ON THE DATA

Theorem 2. *Under assumptions (A1)-(A5), the solution* $\{u(x,t),q(t)\}$ *of the problem (1)-(4) depends continuously upon the data of* $\{k,f(x,t),a(x),g(t),\beta(t),\beta_i(t),\}$, $i=1,2$.

The theorem is proved similar to [3].

CONCLUSION

The inverse problem regarding the simultaneous identification of the diffusion coefficient and time-dependent source control coefficient with the temperature distribution in a one-dimensional sub-diffusion equation with nonlocal boundary and two integral over-determination conditions has been considered. The nonlocal boundary conditions, the Caputo fractional derivative and the control coefficient made our problem more difficult. The conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established by using the Fourier method with some bi-orthogonal system, an associated Caputo fractional derivative which contains an initial data and the Banach fixed point theorem for a product of Banach spaces.

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