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On a Volterra Dynamical System of a Two-Sex Population

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Abstract—In this paper, we study one class of Volterra quadratic stochastic operators with continuous time. Fixed points are investigated, numerical and analytical solutions are found, and, as a special case, the solution of the main problem is given. Numerical and analytical solutions of the problem at various initial values are analyzed using the MathCAD mathematical system.

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1. INTRODUCTION

It is known that physical, chemical, mathematical biology, economic, and other processes are represented by quadratic stochastic operators (QSOs) with discrete and continuous time, with the help of which the evolution of the system under consideration is studied.

There have been many studies on QSOs [1–3]. The concept of QSO was introduced by S.N. Bernsrein [1]. He also studied the issue of evolution of biological population in [1]. Moreover, the theory of two-sex QSOs was studied by Yu.I. Lyubich [2]. Further, the study of Volterra quadratic stochastic two-sex population operators is investigated in [3] and 16 special cases of two-sex QSO, the so-called extreme operators, were also presented. Moreover, continuous-time QSO studies were carried out in works [4] and [5] devoted to the study of chemical process and biological populations. The Volterra QSO of a two-sex population with continuous time was studied in [6].

It should be noted that this area is currently developing very intensively and the number of monographs and articles. For example, in [7, 8] the dynamics of two-sex biological populations and quadratic stochastic operators were studied, as well as the theory of random processes associated with QSO. In [9], a two-parameter QSO mapping a three-dimensional simplex into itself was considered, and the set of all limit points of trajectories was described and it was proved that the set of all two-periodic points is non-hyperbolic. A number of articles have been published on QSO with continuous time (in dynamic systems).

The asymptotic stability of the two-type stochastic Lotka–Volterra model was studied and it was found that when the random intensity changes, the random stability value of the Lotka–Volterra model differs from the stability of the deterministic system mentioned in [10]. The Lotka–Volterra model is considered, in [11], which describes the "predator–prey" dynamics when the prey population depends on environmental changes. Conditions for global asymptotic stability of the equilibrium position of the model are determined under certain restrictions in the coefficients of the system. In [12], a model of population dynamics was studied that takes into account the age structure of the population and a criterion for asymptotic stability was found, depending on the coefficients of a nontrivial equilibrium position. In [13], a strict non-Volterra quadratic dynamic system with continuous time was studied,

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a particular analytical solution of the system was found, a phase portrait was constructed, and a comparative analysis of numerical solutions with a particular solution of the system was carried out.

Also in works [14—16] "predator—prey" models in various modifications were studied. This article studies the continuous analogue of the fourth extreme operator Volterra quadratic stochastic two-sex population operators (VQSTSPO), presented in [3], which has the form

$$\begin{cases} \dot{x}_1(t) = x_2(t)y_1(t) \\ \dot{x}_2(t) = x_2(t)y_2(t) - x_2(t) \\ \dot{y}_1(t) = x_1(t)y_1(t) - y_1(t) \\ \dot{y}_2(t) = x_2(t)y_1(t). \end{cases} \tag{1}$$

The continuous analogue of such extreme operators has not been studied previously. The works of R.D. Jenks [17, 18] provide motivation for studying quadratic differential equations of a free population with continuous time. They were introduced (in particular, (1)) in Kesten's work (see formula (5.37) in [6]). Regardless of the type of time, these equations are obtained from the total probability formula (see [6]). Remark that system (1) is similar to the Lotka—Volterra equations, in addition, there are quadratic stochastic processes associated with system (1) (see [7, Chapter 6], [8, Chapters 3 and 4]).

Note that when studying specific quadratic operators, a very large number of calculations are required and it is difficult to create sufficiently developed analytical methods for studying their trajectories. The method outlined for implementing the system in this article is of interest for the study of more general systems of nonlinear differential equations, as well as for finding their analytical solutions and characterizing phase portraits.

2. FORMULATION OF THE PROBLEM

In [3], the definition of VQSTSPO is given (in the remainder of the article we rely on this definition and terms) with discrete time. Let's rewrite system (1) in vector form

$$\dot{X}(t) = F(X(t)),$$

where $X(t) = (x(t); y(t)) = (x_1(t), x_2(t); y_1(t), y_2(t))$ is the state of some system at a moment of continuous time at $t \geq 0$, $F(X(t)) = F(x_1(t), x_2(t); y_1(t), y_2(t)) = f_j(x_1(t), x_2(t); y_1(t), y_2(t))$, $j = \overline{1, 4}$. In the case under consideration, the population is autosomal [2, p. 16], i.e., male and female types are equal ($n = \nu = 2$). It's clear that

$$x_1(t) \geq 0, \quad x_2(t) \geq 0, \quad y_1(t) \geq 0, \quad y_2(t) \geq 0,$$

$$x_1(t) + x_2(t) = 1, \quad y_1(t) + y_2(t) = 1. \tag{2}$$

Let us study the qualitative properties of system (1) in the domains

$$\overline{\Omega}_1 = \{(x_1(t), x_2(t)) : \ x_1(t) \geq 0, \ x_2(t) \geq 0, \ x_1(t) + x_2(t) \leq 1\},$$

$$\overline{\Omega}_2 = \{(y_1(t), y_2(t)) : \ y_1(t) \geq 0, \ y_2(t) \geq 0, \ y_1(t) + y_2(t) \leq 1\},$$

and further, we will study the qualitative properties of this system in $S^1 \times S^1$, as a special case of system (1).

3. MAIN RESULT

From the first and fourth equations of system (1) we find (for convenience, we omit the arguments t)

$$x_1 = y_2 + C_1, \tag{3}$$

where $C_1 = \text{const}$.

Note that in the future, when integrating the second equation of system (1), we denote the constant by C_2, the third by C_3 and so on. We find y_1, from the first equation of the system, differentiate with respect to t, then, taking into (3), from the third equation of the system, we obtain an equation of decreasing order [19, p. 36]

$$\ddot{x}_1 - (2x_1 - 2 - C_1)\dot{x}_1 = 0. \tag{4}$$

After the appropriate change of variable, we integrate (4) and, returning to the old variables, we obtain two equations

$$\dot{x_1} = \left(x_1 - 1 - \frac{C_1}{2}\right)^2 + \overline{C}_2, \tag{5}$$

$$\dot{x}_1 = 0, \tag{6}$$

where $\overline{C}_2 = \text{const}$.

Let us study equation (5) (equation (6) is studied further). Let's consider the cases separately: a) $\overline{C}_2 = (C_2)^2 > 0$; b) $\overline{C}_2 = -(C_2)^2 < 0$; c) $\overline{C}_2 = C_2 = 0$, here we assume that $C_2 > 0$. We find the following solutions

case a)

$x_1 = 1 + \frac{C_1}{2} + C_2 \tan(C_2(t + C_3))$, where $C_3 = \text{const}$.

case b)

$$x_1 = 1 + \frac{C_1}{2} + C_2 - \frac{2C_2}{1 + C_3 e^{-2C_2 t}};$$

case c):

$$x_1 = 1 + \frac{C_1}{2} \quad \text{and} \quad x_1 = 1 + \frac{C_1}{2} + \frac{1}{-t + C_2}.$$

After solving the remaining equations of (1) and taking into account cases a), b), and c), we obtain the following general solutions to system (1).

Case a)

$$\begin{cases} x_1 = 1 + \frac{C_1}{2} + C_2 \tan(C_2(t + C_3)), \quad x_2 = \frac{(C_2)^2}{C_4} \frac{e^{-\frac{C_1}{2}t}}{|\cos(C_2(t+C_3))|} \\ y_1 = \frac{(C_2)^2}{C_4} \frac{e^{\frac{C_1}{2}t}}{|\cos(C_2(t+C_3))|}, \quad y_2 = 1 - \frac{C_1}{2} + C_2 \tan(C_2(t + C_3)). \end{cases}$$

Case b)

$$\begin{cases} x_1 = 1 + \frac{C_1}{2} + C_2 - \frac{2C_2}{1+C_3 e^{-2C_2 t}}, \quad x_2 = C_4 \frac{e^{-\left(\frac{C_1}{2}+C_2\right)t}}{1+C_3 e^{-2C_2 t}} \\ y_1 = -\frac{4(C_2)^2 C_3}{C_4} \frac{e^{\left(\frac{C_1}{2}-C_2\right)t}}{1+C_3 e^{-2C_2 t}}, \quad y_2 = 1 - \frac{C_1}{2} + C_2 - \frac{2C_2}{1+C_3 e^{-2C_2 t}}. \end{cases} \tag{7}$$

In case c), three solutions of system (1) are obtained:

$$\begin{cases} x_1 = 1 + C_1/2, \quad x_2 = 0 \\ y_1 = C_2 e^{\frac{C_1}{2}t}, \quad y_2 = 1 - C_1/2; \end{cases} \qquad \begin{cases} x_1 = 1 + C_1/2, \quad x_2 = C_2 e^{-\frac{C_1}{2}t} \\ y_1 = 0, \quad y_2 = 1 - C_1/2; \end{cases}$$

$$\begin{cases} x_1 = 1 + C_1/2 + \frac{1}{-t+C_2}, \quad x_2 = C_3|-t + C_2|e^{-\frac{C_1}{2}t} \\ y_1 = C_4|-t + C_2|e^{\frac{C_1}{2}t}, \quad y_2 = 1 - C_1/2 + \frac{1}{-t+C_2}. \end{cases}$$

Note that the values of the solutions obtained in cases a) and c) go beyond the boundaries of the regions under consideration. Therefore, we further study solution (7), as the main solution of the system (1). Let us consider the Cauchy problems for the system (1). Let

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \quad y_1(0) = y_1^0, \quad y_2(0) = y_2^0.$$

Then, we obtain the following system of equations for C_1, C_2, C_3, and C_4:

$$\begin{cases} 1 + C_1/2 + C_2 - \frac{2C_2}{1+C_3} = x_1^0, \quad \frac{C_4}{1+C_3} = x_2^0 \\ -\frac{4(C_2)^2 C_3}{C_4}\frac{1}{1+C_3} = y_1^0, \quad 1 - C_1/2 + C_2 - \frac{2C_2}{1+C_3} = y_2^0. \end{cases} \tag{8}$$

This system allows two solutions (for $x_2^0 \neq 0$, $y_1^0 \neq 0$)

$$\begin{cases} C_1^{1,2} = x_1^0 - y_2^0, \quad C_2^{1,2} = \pm\frac{1}{2}\sqrt{a} \\ C_3^{1,2} = \mp\frac{(x_1^0 + y_2^0 - 2 + \sqrt{a})\sqrt{a}}{4x_2^0 y_1^0} - 1, \quad C_4^{1,2} = \mp\frac{(x_1^0 + y_2^0 - 2 + \sqrt{a})\sqrt{a}}{2y_1^0}, \end{cases} \tag{9}$$

where $a = (x_1^0 + y_2^0 - 2)^2 - 4x_2^0 y_1^0$, and for C_j^1, $j = \overline{1,4}$ the upper sign is taken, and for C_j^2, $j = \overline{1,4}$ the lower sign.

The following lemma is valid.

Lemma 1. *If* $(x_1^0, x_2^0; y_1^0, y_2^0) \in \Omega_1 \times \Omega_2$ *and* $x_2^0 \neq 0, y_1^0 \neq 0$, *then the following inequalities are valid:*

$$a > 0; \quad C_3^{1,2} > -1; \quad C_4^1 > 0, C_4^2 < 0; \quad 0 < 1 + C_1^1/2 - C_2^1 < 1; \quad 0 < 1 - C_1^1/2 - C_2^1 < 1.$$

Proof. Since in $\Omega_1 \times \Omega_2$ the inequalities $x_2^0 < 1 - x_1^0$ and $y_1^0 < 1 - y_2^0$ are valid, then

$$x_2^0 y_1^0 < 1 - x_1^0 - y_2^0 + x_1^0 y_2^0,$$

which means

$$a = (x_1^0 + y_2^0 - 2)^2 - 4x_2^0 y_1^0 > (x_1^0 + y_2^0 - 2)^2 - 4(1 - x_1^0 - y_2^0 + x_1^0 y_2^0) = (x_1^0 - y_2^0)^2 \geq 0.$$

The remaining points of the lemma can be proved in a similar way. Lemma is proved. □

Based on the lemma, we have that $C_3^2 > -1$ and $C_4^2 < 0$, then the corresponding solution to system (7) x_2 takes a negative value, which contradicts the statement of the problem, i.e., the condition $x_2 \geq 0$ is not fulfilled. Note that if $x_2^0 = 0$ and $y_1^0 \neq 0$, then the solution to system (1) are the functions $x_1 = x_1^0$, $x_2 = 0$, $y_1 = y_1^0 e^{(x_1^0 - 1)t}$, and $y_2 = y_2^0$; if $x_2^0 \neq 0$ and $y_1^0 = 0$, then $x_1 = x_1^0$, $x_2 = x_2^0 e^{(y_2^0 - 1)t}$, and $y_1 = 0$, $y_2 = y_2^0$, and if $x_2^0 = 0$ and $y_1^0 = 0$, then the functions $x_1 = x_1^0$, $x_2 = 0$, $y_1 = 0$, and $y_2 = y_2^0$ satisfy system (1), with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, $y_1(0) = y_1^0$, and $y_2(0) = y_2^0$, which do not require deep analysis.

Next, we take the first solution of system (8) $C_1^1, C_2^1, C_3^1, C_4^1$ and for ease of notation we denote them again by C_1, C_2, C_3, and C_4.

Here are two well-known definitions that will be used below.

Definition 1 [20, p. 318]. *The solution* $(x_1(t), x_2(t); y_1(t), y_2(t))$ *of system (1) is called Lyapunov stable, if for any* $\varepsilon > 0$ *there is a* $\delta > 0$, *such that as soon as* $|x_i^0 - \overline{x}_i^0| < \delta$ *and* $|y_i^0 - \overline{y}_i^0| < \delta$, $i = 1, 2$, *then*

$$|x_i(t, x_1^0, x_2^0; y_1^0, y_2^0) - x_i(t, \overline{x}_1^0, \overline{x}_2^0; \overline{y}_1^0, \overline{y}_2^0)| < \varepsilon,$$

$$|y_i(t, x_1^0, x_2^0; y_1^0, y_2^0) - y_i(t, \overline{x}_1^0, \overline{x}_2^0; \overline{y}_1^0, \overline{y}_2^0)| < \varepsilon, \ i = 1, 2$$

for all values $0 \leq t < +\infty$. *Here* $\overline{x}_i^0$ *and* $\overline{y}_i^0$, $i = 1, 2$ *modified initial conditions.*

Definition 2 [21]. *The equilibrium positions of system (1) are those points* $X^*(t)$ *of the phase space such that* $F(X^*(t)) = 0$. *It is obvious that* $X^*(t)$ *is a solution to system (1), since* $\dot{X}^* = 0$.

The system has three equilibrium points $M_1(C_1^*, 0; 0, C_4^*)$, $M_2(C_1^*, C_2^*; 0, 1)$, and $M_3(1, 0; C_3^*, C_4^*)$, where $C_i^* = \text{const}$ and $C_i^* \in [0, 1]$, $i = \overline{1,4}$.

Theorem 1. *I. In the case b) the general solution of system (1) has the form (7) and it is uniformly continuous on* $t \in [0, +\infty)$, *and is also Lyapunov stable in the equilibrium position* $M_1(1 + C_1/2 - C_2, 0; 0, 1 - C_1/2 - C_2)$.

II. Let conditions (2) be satisfied. Then,

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty}(x_1(t), x_2(t); y_1(t), y_2(t)) = \begin{cases} (1, 0; 0, 1 - x_1^0 + y_2^0), & \text{if} \quad x_1^0 - y_2^0 > 0 \\ (1 + x_1^0 - y_2^0, 0; 0, 1), & \text{if} \quad x_1^0 - y_2^0 < 0. \end{cases} \tag{10}$$

Proof. I. Depended on the facts it follows that the functions $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ of (7) satisfy system (1). Let us show the uniform continuity of the functions (7) $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ on $t \in [0, +\infty)$ for any $(x_1^0(t), x_2^0(t); y_1^0(t), y_2^0(t)) \in \Omega_1 \times \Omega_2$ we have both solutions of (8) are real. Note that to study the system, the continuity of functions will be sufficient. According to part 2 of the Lemma 1, the function $1 + C_3 e^{-2C_2 t}$ does not vanish at $t \in [0, +\infty)$.

Using the later cases for the function $x_1(t)$, we have

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} \left(1 + C_1/2 + C_2 - \frac{2C_2}{1 + C_3 e^{-2C_2 t}}\right) = 1 + C_1/2 - C_2 = C < 1.$$

Since the function $x_1(t)$ is continuous on $t \in [0, +\infty)$ and $\lim\limits_{t\to+\infty} x_1(t) = C$. It follows $x_1(t)$ is uniformly continuous on $t \in [0, +\infty)$ [22, p. 6]. The uniform continuity of the remaining functions of (7) can be considered in a similar manner.

Now, we prove the stability of the solution to system (1) at the fixed point $M_1(1 + C_1/2 - C_2, 0; 0, 1 - C_1/2 - C_2)$.

Suppose that there are two solutions to system (1) $X(x_1(t), x_2(t); y_1(t), y_2(t))$ and $\Phi(\varphi_1(t), \varphi_2(t); \psi_1(t), \psi_2(t))$ with initial data $t = t_0 : x_1(t_0) = x_1^0$, $x_2(t_0) = x_2^0$, $y_1(t_0) = y_1^0$, $y_2(t_0) = y_2^0$ and $\varphi_1(t_0) = \varphi_1^0$, $\varphi_2(t_0) = \varphi_2^0$, $\psi_1(t_0) = \psi_1^0$, $\psi_2(t_0) = \psi_2^0$, respectively. According to Definition 1, we have

$$|x_1^0 - \varphi_1^0| < \delta, \quad |x_2^0 - \varphi_2^0| < \delta, \quad |y_1^0 - \psi_1^0| < \delta, \quad |y_2^0 - \psi_2^0| < \delta.$$

From the first relation (9) we find

$$|C_1 - \overline{C}_1| = |x_1^0 - \varphi_1^0 - (y_2^0 - \psi_2^0)| < 2\delta. \tag{11}$$

Similarly, as above, we obtain that

$$|C_2 - \overline{C}_2| < \alpha\delta, \quad |C_3 - \overline{C}_3| < \alpha\delta, \quad |C_4 - \overline{C}_4| < \alpha\delta, \tag{12}$$

where α is known finite constant, $\overline{C}_1, \overline{C}_2, \overline{C}_3$, and $\overline{C}_4$ are corresponding constants depending on φ_1^0, φ_2^0, ψ_1^0, and ψ_2^0, according to relations (9), respectively. Now we will show that if (11) and (12) are satisfied, then for $\forall\varepsilon > 0$ the inequalities are valid

$$|x_i(t) - \varphi_i(t)| < \varepsilon, \quad |y_i(t) - \psi_i(t)| < \varepsilon, \quad i = 1, 2. \tag{13}$$

Let's consider the difference $|x_1(t) - \varphi_1(t)|$. From (7) we find

$$\begin{aligned}\big|x_1(t) - \varphi_1(t)\big| &= \Big|1 + \frac{C_1}{2} + C_2 - \frac{2C_2}{1 + C_3 e^{-2C_2 t}} - 1 - \frac{\overline{C}_1}{2} - \overline{C}_2 + \frac{2\overline{C}_2}{1 + \overline{C}_3 e^{-2\overline{C}_2 t}}\Big| \\ &= \left|\frac{C_1 - \overline{C}_1}{2} + (C_2 - \overline{C}_2) - \frac{2(C_2 - \overline{C}_2) + 2\left(C_2\overline{C}_3 e^{-2\overline{C}_2 t} - \overline{C}_2 C_3 e^{-2C_2 t}\right)}{(1 + C_3 e^{-2C_2 t})(1 + \overline{C}_3 e^{-2\overline{C}_2 t})}\right|.\end{aligned}$$

Since the function $e^{-2\overline{C}_2 t}$ is uniformly continuous and bounded, then by virtue of (11) and (12) we find $|x_1(t) - \varphi_1(t)| < \beta\delta$, where β is a known finite constant. If $\delta = \varepsilon/\gamma$, $\gamma = \max(2, \alpha, \beta)$, then $|x_1(t) - \varphi_1(t)| < \varepsilon$. The remaining inequalities (13), are proved similarly, which are valid for $t \in [0, +\infty)$. By virtue of the Lemma 1 from (7), we find

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} (x_1(t), x_2(t); y_1(t), y_2(t)) = (1 + C_1/2 - C_2, 0; 0, 1 - C_1/2 - C_2).$$

Hence, we conclude that the solutions of the system are Lyapunov stable [20, p. 318].

II. Let the conditions be satisfied (2) and $x_1^0 - y_2^0 > 0$. Then,

$$C_2 = \frac{1}{2}\sqrt{a} = \frac{1}{2}\sqrt{(x_1^0 + y_2^0 - 2)^2 - 4x_2^0 y_1^0} = \frac{1}{2}\sqrt{(x_1^0 + y_2^0 - 2)^2 - 4(1 - x_1^0)(1 - y_2^0)} = \frac{1}{2}(x_1^0 - y_2^0).$$

Means,

$$1 + C_1/2 - C_2 = 1, \quad 1 - C_1/2 - C_2 = 1 - 2C_2 = 1 - (x_1^0 - y_2^0).$$

From this we get the validity of the first limit (10). Similarly, we get the validity of the second limit (10).

From the above arguments and from (7) it follows that the equilibrium points are $M_2(C_1^*, C_2^*; 0, 1)$ and $M_3(1, 0; C_3^*, C_4^*)$ are unstable, if $C_2^* \neq 0$ and $C_3^* \neq 0$.

Now, we consider the system (1) on $S^1 \times S^1$. Obviously, relation (2) is satisfied. Then, system (1) takes the following form

$$\begin{cases} \dot{x}_1(t) = (1 - x_1(t))y_1(t) \\ \dot{y}_1(t) = (x_1(t) - 1)y_1(t). \end{cases} \tag{14}$$

Taking into account the relation $\dot{x}_1(t) + \dot{y}_1(t) = 0$, we obtain

$$\dot{x}_1(t) = (1 - x_1(t))(C_5 - x_1(t)), \tag{15}$$

when C_5 is constant of integration and $C_5 \in (0, 1]$. If $C_5 = 0$, then we get the solution $x_1(t) = 0$, $x_2(t) = 1$, $y_1(t) = 0$, and $y_2(t) = 1$.

Equation (15) with $-\infty < x_1(t) < C_5$ and $1 < x_1(t) < +\infty$ has the following solution

$$x_1(t) = \frac{C_5 - C_6 e^{(C_5-1)t}}{1 - C_6 e^{(C_5-1)t}}, \tag{16}$$

and when $C_5 < x_1(t) < 1$

$$x_1(t) = \frac{C_5 + C_6 e^{(C_5-1)t}}{1 + C_6 e^{(C_5-1)t}}. \tag{17}$$

Note that for a fixed positive C_5, formula (16) gives two solutions: one defined on the interval $-\infty < t < -\ln C_5$ and located in the half-plane $x_1(t) > 1$, and the second, defined on the interval $-\ln C_5 < t < +\infty$ and located in the half-plane $x_1(t) < C_5$. Equation (15) also has two more solutions $x_1(t) \equiv C_5$ and $x_1(t) \equiv 1$, which can be formally obtained from (17), by substituting $C_6 = 0$ and $C_6 = \infty$, located in these above areas. Hence, we conclude that the phase portrait of equation (15) consists of five phase trajectories: two infinite intervals $(-\infty, C_5)$ and $(1, +\infty)$, interval $(C_5, 1)$, as well as two singular points C_5 and 1. Here point $x_1(t) \equiv C_5$ is stable, and point $x_1(t) \equiv 1$ is an unstable equilibrium position. This means that the general solution of system (14) on $S^1 \times S^1$ for a fixed $0 < C_5 < 1$ has the form

$$\text{d)} \begin{cases} x_1(t) = \frac{C_5 - C_6 e^{(C_5-1)t}}{1 - C_6 e^{(C_5-1)t}} \\ x_2(t) = 1 - x_1(t) \\ y_1(t) = \frac{C_6(1-C_5) e^{(C_5-1)t}}{1 - C_6 e^{(C_5-1)t}} \\ y_2(t) = 1 - y_1(t); \end{cases} \qquad \text{e)} \begin{cases} x_1(t) = \frac{C_5 + C_6 e^{(C_5-1)t}}{1 + C_6 e^{(C_5-1)t}} \\ x_2(t) = 1 - x_1(t) \\ y_1(t) = \frac{C_6(C_5-1) e^{(C_5-1)t}}{1 + C_6 e^{(C_5-1)t}} \\ y_2(t) = 1 - y_1(t), \end{cases} \tag{18}$$

where d) is a solution of the system (1) for $0 < x_1(t) < C_5$, and e) for $C_5 < x_1(t) < 1$.

We find a solution to the Cauchy problem for system (14). Let

$$x_1(0) = x_1^0, \quad y_1(0) = y_1^0. \tag{19}$$

Substituting them into (18), we obtain a system of equations for C_5 and C_6, the solution of which for case d) has the form

$$\begin{cases} C_5 = x_1^0 + y_1^0 \\ C_6 = y_1^0 / (1 - x_1^0), \end{cases} \tag{20}$$

and for case e)

$$\begin{cases} C_5 = x_1^0 + y_1^0 \\ C_6 = y_1^0 / (x_1^0 - 1). \end{cases} \tag{21}$$

Comparing (16) and (17) taking into account (20) and (21), we make sure that the solutions to system (14) formally have the same form for $0 < x_1(t) < C_5$ and $C_5 < x_1(t) < 1$.

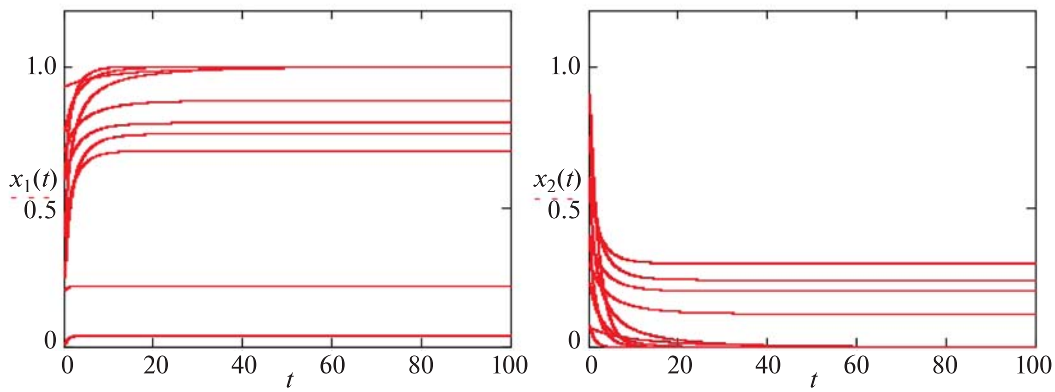

Fig. 1. Graphs of numerical solutions $x_1(t)$ and $x_2(t)$ of the system (1) in $\Omega_1 \times \Omega_2$.

From d) (18) taking into account (20), we find

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} (x_1(t), x_2(t); y_1(t), y_2(t)) = \left(x_1^0 + y_1^0, 1 - x_1^0 - y_1^0; 0, 1\right).$$

Here the solution $(y_1(t), y_2(t))$ individually converges exponentially quickly to the equilibrium point $(0, 1)$.

Now consider the case e). So, from e) (18) and (21) we find $C_5 = x_1^0 + y_1^0 < x_1(t) < 1$. It follows that the solution to system (14) satisfies the initial conditions (19) only in the case when $x_1(t) \equiv x_1^0 = C_5$ and $y_1(t) = y_1^0 = 0$.

Let us now consider when $C_5 = x_1^0 + y_1^0 = 1$, which is equivalent to the equality $y_2^0 = x_1^0$ $(x_2^0 = y_1^0)$. In this case, system (14) has two solutions

$$\begin{cases} x_1(t) = 1, & x_2(t) = 0 \\ y_1(t) = 0, & y_2(t) = 1; \end{cases} \qquad \begin{cases} x_1(t) = \frac{x_1^0 + y_1^0 t}{1 + y_1^0 t}, & x_2(t) = \frac{y_1^0}{1 + y_1^0 t} \\ y_1(t) = \frac{y_1^0}{1 + y_1^0 t}, & y_2(t) = \frac{x_1^0 + y_1^0 t}{1 + y_1^0 t}. \end{cases}$$

Considering that in this case there is

$$\lim_{t\to\infty} |x_i(t) - \varphi_i(t)| = 0, \quad \lim_{t\to\infty} |y_i(t) - \psi_i(t)| = 0, \; i = 1, 2,$$

we conclude that the solution is asymptotically stable at the equilibrium point $M_4(1, 0; 0, 1)$. □

Let us consider numerical solutions of system (1) in the region $\Omega_1 \times \Omega_2$ in the following ten initial conditions (the graphs of these solutions are shown in Figs. 1 and 2):

$$(x_1^0, x_2^0; y_1^0, y_2^0)^T = \begin{pmatrix} 0.01 & 0.2 & 0.4 & 0.6 & 0.7 & 0.25 & 0.77 & 0.61 & 0.93 & 0.43 \\ 0.9 & 0.08 & 0.6 & 0.4 & 0.3 & 0.75 & 0.23 & 0.39 & 0.07 & 0.57 \\ 0.05 & 0.4 & 0.3 & 0.2 & 0.18 & 0.51 & 0.41 & 0.78 & 0.11 & 0.62 \\ 0.5 & 0.06 & 0.7 & 0.8 & 0.82 & 0.49 & 0.59 & 0.22 & 0.89 & 0.38 \end{pmatrix}$$

We present graphs of phase trajectories of $(t, x_1(t), x_2(t))$ and $(t, y_1(t), y_2(t))$ in Fig. 3.

Using MathCAD, numerical solutions of system (1) were compared with analytical ones (7). In the course of research, it was found that their difference does not exceed 0.001, and at $t \geq 1$ they coincide and tend to the equilibrium position $M_1(0.705, 0; 0, 0.875)$ (see part I of the Theorem 1), where $x_1^0 = 0.5$, $x_2^0 = 0.4$, $y_1^0 = 0.32$, $y_2^0 = 0.67$ (see Figs. 4 and 5).

In Fig. 4, the graphs $x_1(t)$, $x_2(t)$ and $X_1(t)$, $X_2(t)$ denote the values of analytical and numerical solutions of system (1), respectively.

In Fig. 5, $y_1(t)$, $y_2(t)$ and $Y_1(t)$,$Y_2(t)$ denote the values of analytical and numerical solutions of system (1), respectively.

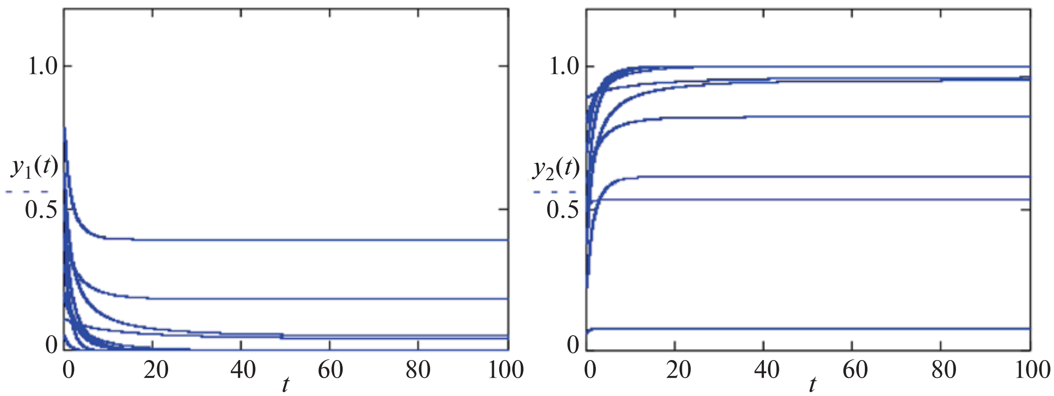

Fig. 2. Graphs of numerical solutions $y_1(t)$ and $y_2(t)$ of the system (1) in $\Omega_1 \times \Omega_2$.

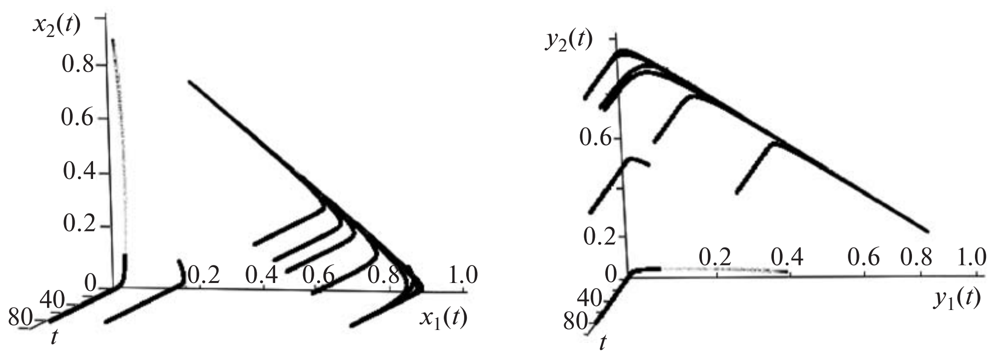

Fig. 3. Phase trajectories of system (1) in space $(t, x_1(t), x_2(t))$ and $(t, y_1(t), y_2(t))$.

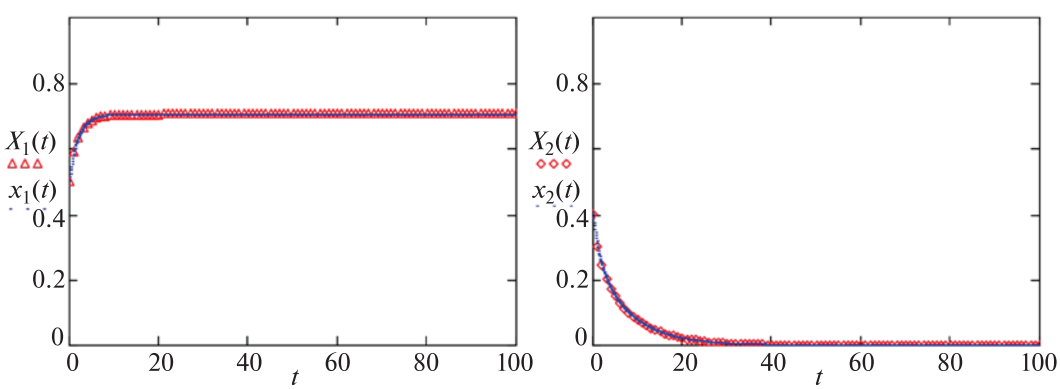

Fig. 4. Graphs of comparison of analytical and numerical solutions $x_1(t)$ and $X_1(t)$, $x_2(t)$ and $X_2(t)$ of the system (1) in $\Omega_1 \times \Omega_2$.

Now, let's compare the analytical and numerical solutions of the system on $S^1 \times S^1$. Let $x_1^0 = 0.635$, $x_2^0 = 365$, $y_1^0 = 0.443$, $y_2^0 = 0.557$ (see Figs. 6 and 7).

In Figs. 6 and 7, $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ and $X_1(t)$, $X_2(t)$, $Y_1(t)$, $Y_2(t)$ denote the values of analytical and numerical solutions of system (1), respectively.

4. CONCLUSIONS

The solution of system (1) when relation (2) is satisfied, tends to the equilibrium position $M_1^1(1, 0; 0, 1 - x_1^0 + y_2^0)$, if $x_1^0 - y_2^0 > 0$. Thus, under different initial conditions, the entire solution

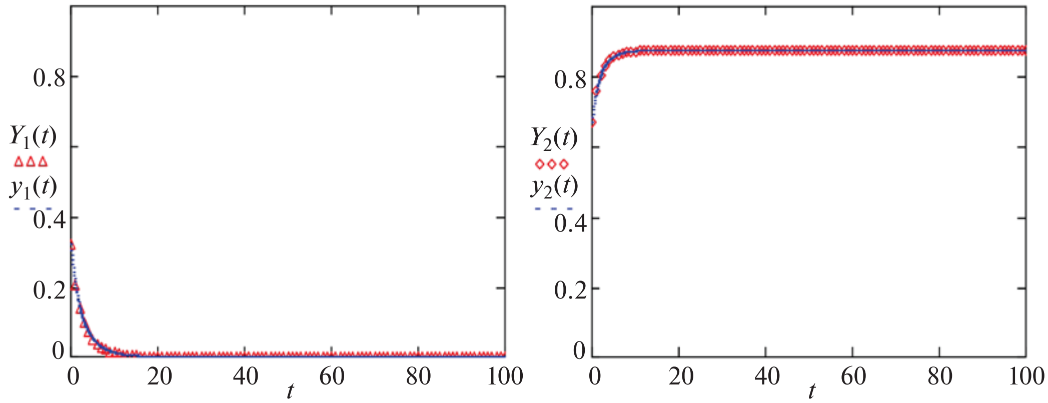

Fig. 5. Graphs of comparison of analytical and numerical solutions $y_1(t)$ and $Y_1(t)$, $y_2(t)$ and $Y_2(t)$ of the system (1) in $\Omega_1 \times \Omega_2$.

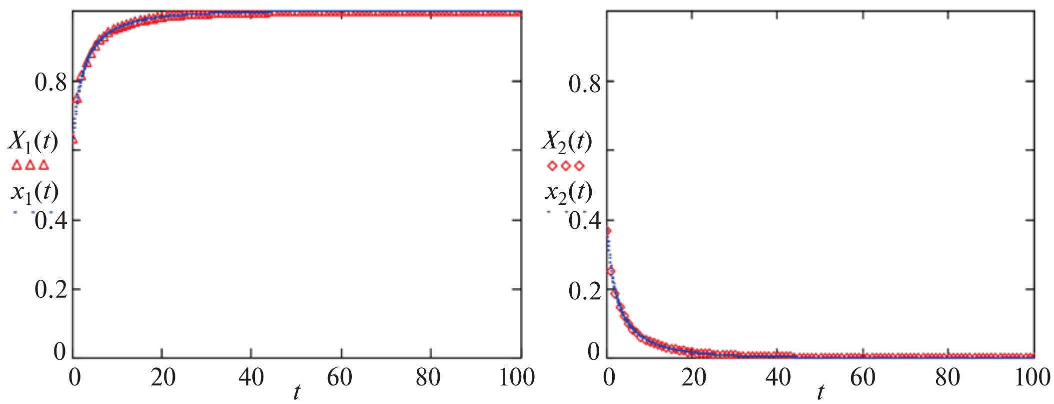

Fig. 6. Graphs of comparison of analytical and numerical solutions $x_1(t)$ and $X_1(t)$, $x_2(t)$ and $X_2(t)$ of the system (13) in $S^1 \times S^1$.

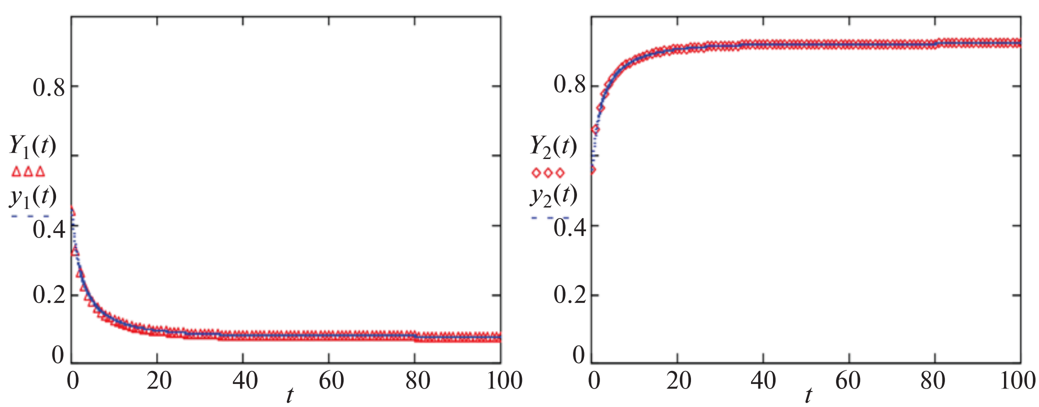

Fig. 7. Graphs of comparison of analytical and numerical solutions $y_1(t)$ and $Y_1(t)$, $y_2(t)$ and $Y_2(t)$ of the system (13) in $S^1 \times S^1$.

$x_1(t)$, $x_2(t)$ and $y_1(t)$ ends to a fixed limit, i.e.,

$$\lim_{t\to+\infty} x_1(t) = 1, \quad \lim_{t\to+\infty} x_2(t) = 0, \quad \lim_{t\to+\infty} y_1(t) = 0.$$

The speed of their convergence to the equilibrium position depends on the value of the degree of the function $e^{-(\frac{C_1}{2}+C_2)t}$ and $e^{-(\frac{C_1}{2}-C_2)t}$ (see solution (7)).

As noted in [13], indeed, the study of a continuous analogue of QSO provides some advantage. Thus, with the help of computer calculations, it was established that for $t \geq 1$ the analytical solutions of system (1) coincide with the numerical solutions (7) and (18).

Note that numerical solutions of a system of ordinary differential equations do not provide a good qualitative understanding of the behavior of the system. Therefore, we used graphical methods that make it possible to understand the behavior of dynamic systems.

MathCAD has powerful tools for solving linear and nonlinear ODE systems, and with their help, only numerical solutions are found, without specifying the range of permissible values of the solution arguments. In other words, there is no exception handling. Because the tools for solving ODE systems have certain limitations, i.e., it contains approximately twenty built-in functions for various types of equations and solution methods. In this regard, it is necessary to first conduct a qualitative analysis of the ODE system under consideration.

Having studied the resulting graphs (see Figs. 4–7), we note that using the built-in function "rkfixed" (where the 4th order Runge–Kutta method is used) when finding numerical solutions to system (1) gives a more accurate solution (see comparison numerical and analytical solutions). Although the Runge–Kutta method provides increased accuracy, it requires a larger amount of calculations, which should be taken into account when compiling a computer program that does not involve many time-consuming numerical calculations for time-saving reasons.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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