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On the solvability of a boundary value problem for a quasilinear equation of mixed type with two degeneration lines

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Abstract. The article investigates the existence of a generalized solution to one boundary value problem for an equation of mixed type with two lines of degeneration in the weighted space of S.L. Sobolev. In proving the existence of a generalized solution, the spaces of functions $U(\Omega)$ and $V(\Omega)$ are introduced, the spaces $H_1(\Omega)$ and $H_1^*(\Omega)$ are defined as the completion of these spaces of functions, respectively, with respect to the weighted norms, including the functions $K(y)$ and $N(x)$. Using an auxiliary boundary value problem for a first order partial differential equation, Kondrashov's theorem on the compactness of the embedding of $W_2^1(\Omega)$ in $L_2(\Omega)$ and Vishik's lemma, the existence of a solution to the boundary value problem is proved.

1. Introduction. Formulation of the problem

Consider the equation

$$T(U) \equiv K(y)U_{xx} + N(x)U_{yy} + C(x,y)U = f(x,y,U), \tag{1}$$

where $K(y),\ N(x),\ C(x,y),\ f(x,y,U)$ - given functions, at that $K(t) > 0$ and $N(t) > 0$ for $t > 0$, $K(t) < 0$ and $N(t) < 0$ for $t < 0$, $K(0) = N(0) = 0$.

Let $\Omega-$ be a finite simply connected convex domain on the plane of variables (x, y), bounded for $x > 0,\ y > 0$ by a smooth curve σ with endpoints at the points $A(1, 0)$ and $B(0, 1)$ and for $x > 0,\ y < 0$ and $x < 0,\ y > 0$ - the characteristics OD, DA and OC, CB of equation (1), emerging from the points $O(0, 0), A(1, 0)$ and $O(0, 0), B(0, 1)$, respectively.

Suppose that the curve $DA \cup \sigma \cup BC$ satisfies the condition:

$$\left((\ x + a)n_1 + (y + a)n_2\right)\big|_{DA\cup\sigma\cup BC}\ < 0, \tag{2}$$

where (n_1, n_2) - is the internal normal vector to

$$\Gamma = DA \cup \sigma \cup BC,\ a = \max\left(|d_1|, |d_2|, |c_1|, |c_2|\right) + \delta,$$

here (d_1, d_2) and (c_1, c_2) - are the coordinates of points D and C, respectively, δ - is a small positive number.

Condition (2) means that the lines $y+a=\overline{k}(\ x+a\ )$, where $\overline{k}=const>0$, cannot intersect the curve twice Γ.

Problem T. Find a solution $U(x,y)$ of equation (1) in the domain Ω such that

$$U(x,y)|_{\Gamma}=0. \tag{3}$$

We introduce the function spaces:

$$U(\Omega)=\{U:U\in C^{\infty}(\overline{\Omega}),\ U|_{\Gamma}=0\}, \tag{4}$$

$$V(\Omega)=\{V:V\in C^{\infty}(\overline{\Omega}),\ U|_{CD\bigcup\sigma}=0\}. \tag{5}$$

Note that the classical solvability of boundary value problems for linear equations of mixed types with one and two lines of degeneration has been studied deeply. However, the generalized solvability of boundary value problems for quasilinear equations of mixed type has not been fully studied, since there is no general theory that can be applied to study such equations.

There are several articles devoted to boundary value problems for quasilinear equations of elliptic, hyperbolic, and mixed types with one line of degeneracy. Such problems were studied in the works [1-2]. They mainly investigated the classical solvability of the problems posed. In the works [3-4] studies the existence of a generalized solution of boundary value problems for quasilinear equations of mixed types.

At the same time, the generalized solvability of boundary value problems for quasilinear equations of mixed types with two lines of degeneracy has been little studied, we note the works [5-6].

Mainly, many authors have investigated boundary value problems for linear equations of mixed type with two lines of degeneration [7-9].

This work is a generalization of work [6]. A complete description is given of the proofs of the solvability of the boundary value problem **T**. In addition, concrete functions $K(y)$, $N(x)$, $C(x,y)$, $f(x,y,U)$ and examples of the considered domain Ω satisfying the conditions of the lemma and theorem on the solvability of the problem **T**.

2. Existence of a generalized solution of the problem

We define the spaces $H_1(\Omega)$ and $H_1^*(\Omega)$ as the completion of the spaces of functions (4) and (5), respectively, with respect to the weighted norms, including the functions $K(y)$ and $N(x)$:

$$\|U\|_{H_1(\Omega)}=\left(\int\limits_{\Omega}(|K(y)|U_x^2+|N(x)|\,U_y^2+U^2)d\Omega\right)^{\frac{1}{2}}, \tag{6}$$

$$\|V\|_{H_1^*(\Omega)}=\left(\int\limits_{\Omega}(|K(y)|\,V_x^2+|N(x)|\,V_y^2+V^2)d\Omega\right)^{\frac{1}{2}}. \tag{7}$$

Definition. A function $U(x,y)\in H_1(\Omega)$ is called a generalized solution to problem **T**, if

$$B\left(U,V\right)=-\int\limits_{\Omega}(K(y)U_xV_x+N(x)U_yV_y-C(x,y)UV)d\Omega=\int\limits_{\Omega}f(x,y,U)Vd\Omega \tag{8}$$

for all $V(x,y)\in H_1^*(\Omega)$.

First prove an auxiliary lemma.

Lemma. Let condition (2) be satisfied and
a) $N(x), K(y) \in C^1(\overline{\Omega})$, $(x+a)N'(x) \geq \alpha\,|N(x)|\,, (y+a)K'(y) \geq \alpha\,|K(y\,)|$ in $\overline{\Omega}$,
$-(\,x+a)n_1 + (y+a)n_2 \geq 0$ on CO,
$(\,x+a)n_1 - (y+a)n_2 \geq 0$ on OD,
$(\,x+a)n_1 + (y+a)n_2 \geq 0$ on CD,
where $\alpha = const > 0$, $(n_1,\ n_2)$ - internal normal vector to CD;
b) $C(x,y) \in C^1(\overline{\Omega})$, $\quad C(x,y)|_{CD} \leq 0$,
$((x+a)\,C\,(x,y))_x + ((y+a)\,C\,(x,y))_y \leq -m < 0$ in $\overline{\Omega}$,
where $m = const > 0$.

Then there exist functions $\{\varphi_n(x,y)\}_{n\in N} \in H_1(\Omega)$ that are a solution to the boundary value problem

$$l(\varphi_n) \equiv (x+a)\varphi_{nx}(x,y) + (y+a)\varphi_{ny}(x,y) = \psi_n(x,y), \tag{9}$$

$$\varphi_n(x,y)|_{\Gamma} = 0,\ n \in N, \tag{10}$$

where $\{\psi_n(x,y)\}_{n\in N} -$ a complete system of smooth linearly independent functions in the space $H_1^*(\Omega)$, belonging to $V(\Omega)$.

Proof. By virtue of condition (2) the characteristics of equation (9) cannot intersect the curve Γ twice. Introduce new independent variables:

$$\xi = x,\ \eta = \frac{x+a}{y+a}.$$

Then equation (9) and boundary condition (10) take the form (for convenience, we omit the indices n):

$$\varphi_\xi = \frac{1}{\xi+\alpha}\psi, \tag{11}$$

$$\varphi|_{\widetilde{\Gamma}} = 0, \tag{12}$$

where $\widetilde{\Gamma} -$ is the image of Γ on the plane (ξ,η).

Solving problem (11)–(12) and returning to variables (x,y), we obtain

$$\varphi(x,y) = \int\limits_{\chi(\frac{x+a}{y+a})}^{x} \frac{1}{t+a}\psi(t, \frac{t(a+y)+a(y-x)}{a+x})dt \equiv I(\psi),$$

where $x = \chi(\frac{x+a}{y+a})-$ equation of the curve Γ.

Consider the integral:

$$2\int\limits_{\Omega} I(\psi)T(\psi)d\Omega = 2\int\limits_{\Omega} \varphi T(l(\varphi))d\Omega =$$

$$= 2\lim_{\varepsilon\to 0}\int\limits_{\Omega_0^\varepsilon} \varphi T(l(\varphi))d\Omega + 2\lim_{\varepsilon\to 0}\int\limits_{\Omega_1^\varepsilon} \varphi T(l(\varphi))d\Omega + 2\lim_{\varepsilon\to 0}\int\limits_{\Omega_2^\varepsilon} \varphi T(l(\varphi))d\Omega, \tag{13}$$

where

$$\Omega_0^\varepsilon = \Omega \cap \{(x,y) : x>0,\ y>0\}\backslash(\overline{S}_{1\varepsilon} \cap \overline{S}_{2\varepsilon}),$$

$$\Omega_1^\varepsilon = \Omega \cap \{(x,y) : x>0,\ y<0\}\backslash\overline{S}_{1\varepsilon},\ \Omega_2^\varepsilon = \Omega \cap \{(x,y) : x<0,\ y>0\}\backslash\overline{S}_{2\varepsilon},$$

$$S_{1\varepsilon} = \left\{(x,y) : (x-1)^2 + y^2 < \varepsilon^2\right\},\ S_{2\varepsilon} = \left\{(x,y) : x^2 + (y-1)^2 < \varepsilon^2\right\},$$

$\varepsilon -$ enough small positive number.

Next, we consider separately the integrals on the right-hand side of equality (13). We integrate twice by parts using the following identities:

$$2K(y)\varphi_{xx}l(\varphi) = (y+a)K^{'}(y)\varphi_x^2 + 2((y+a)K(y)\varphi_x\varphi_y)_x + \\ +((x+a)K(y)\varphi_x^2)_x - ((y+a)K(y)\varphi_x^2)_y, \tag{14}$$

$$2N(x)\varphi_{yy}l(\varphi) = (x+a)N^{'}(x)\varphi_y^2 + 2((x+a)N(x)\varphi_x\varphi_y)_y - \\ -((x+a)N(x)\varphi_y^2)_x + ((y+a)N(x)\varphi_y^2)_y, \tag{15}$$

$$2C(x,y)\varphi((x+a)\varphi_x + (y+a)\varphi_y) = -(((x+a)C(x,y))_x + \\ +((y+a)C(x,y))_y)\varphi^2 + ((x+a)C(x,y)\varphi^2)_x + ((y+a)C(x,y)\varphi^2)_y \tag{16}$$

and Green's formula, the first integral on the right-hand side of equality (13) can be written in the form

$$2\int\limits_{\Omega_0^\varepsilon} \varphi T(l(\varphi))d\Omega = \int\limits_{\Omega_0^\varepsilon} \Pi d\Omega - \int\limits_0^{1-\varepsilon} (2N(x)\varphi(l(\varphi))_y - aN(x)\varphi_y^2)dx - \\ - \int\limits_0^{1-\varepsilon} (2K(y)\varphi(l(\varphi))_x - aK(y)\varphi_x^2)dy + \int\limits_{\partial\Omega_0^\varepsilon \backslash ((x=0)\cup(y=0))} Pdx + Qdy - \\ - \int\limits_{\partial\Omega_0^\varepsilon} C(x,y)\varphi^2((x+a)n_1 + (y+a)n_2)dS, \tag{17}$$

where

$$\Pi = (y+a)\,K^{'}\,(y)\,\varphi_x^2 + (x+a)\,N^{'}\,(x)\,\varphi_y^2 - (\,((x+a)\,C\,(x,y))_x + ((y+a)\,C\,(x,y))_y\,)\varphi^2,$$

$$P = (y+a)\,K\,(y)\,\varphi_x^2 - 2\,(x+a)\,N\,(x)\,\varphi_x\varphi_y - (y+a)\,N\,(x)\,\varphi_y^2 - 2N\,(x)\,\varphi\,(l\,(\varphi))_y + 2N\,(x)\,l\,(\varphi)\,\varphi_y,$$

$$Q = (x+a)\,K\,(y)\,\varphi_x^2 + 2\,(y+a)\,K\,(y)\,\varphi_x\varphi_y - (x+a)\,N\,(x)\,\varphi_y^2 + 2K\,(y)\,\varphi\,(l(\varphi))_x - 2K\,(y)\,l\,(\varphi)\,\varphi_x.$$

Taking into account (14)–(16) and the equalities $\sqrt{-K(y)}dy = -\sqrt{N(x)}dx$ on OD, $\sqrt{-K(y)}dy = \sqrt{N(x)}dx$ on DA, the second integral on the right-hand side of equality (13) can be rewritten as

$$2\int\limits_{\Omega_1^\varepsilon} \varphi T(l(\varphi))d\Omega = \int\limits_{\Omega_1^\varepsilon} \Pi d\Omega + \int\limits_0^{1-\varepsilon} (2N(x)\varphi(l(\varphi))_y - aN(x)\varphi_y^2)dx - \\ - \int\limits_{OD} [\sqrt{-K(y)}\varphi_x - \sqrt{N(x)}\varphi_y]^2((x+a)dy + (y+a)dx) +$$

$$+\int\limits_{DA'_\varepsilon}[(-(x+a)\sqrt{-N(x)K(y)}+(y+a)K(y))\varphi_x-((y+a)\sqrt{-N(x)K(y)}+$$

$$(y+a)K(y))\varphi_y]d\varphi+2\int\limits_{OD\cup DA'_\varepsilon}\varphi d_m\psi-\psi d_m\varphi+$$

$$\int\limits_{(y<0)\cap\partial S_{1\varepsilon}}Pdx+Qdy-\int\limits_{\partial\Omega_1^\varepsilon}C(x,y)((x+a)n_1+(y+a)n_2)\varphi^2dS, \tag{18}$$

where $d_m\varphi=K(y)\varphi_xdy-N(x)\varphi_ydx,\ A'_\varepsilon-$ point of intersection of the DA characteristic with the curve $\partial S_{1\varepsilon}$.

Similarly, as in (18), we rewrite the third integral on the right-hand side of (13) in the form

$$2\int\limits_{\Omega_2^\varepsilon}\varphi T\left(l\left(\varphi\right)\right)d\Omega=\int\limits_{\Omega_2^\varepsilon}\Pi d\Omega+\int\limits_0^{1-\varepsilon}(2K(y)\varphi(l(\varphi))_x-aK(y)\varphi_x^2)dy+$$

$$+\int\limits_{CO}[\sqrt{K(y)}\varphi_x-\sqrt{-N(x)}\varphi_y]^2((x+a)dy+(y+a)dx)+$$

$$+\int\limits_{B'_\varepsilon C}[((x+a)\sqrt{-N(x)K(y)}+(y+a)K(y))\varphi_x-((y+a)\sqrt{-N(x)K(y)}-$$

$$-(x+a)N(x))\varphi_y]d\varphi+2\int\limits_{B'_\varepsilon C\cup CO}\varphi d_m\psi-\psi d_m\varphi+\int\limits_{(x<0)\cap\partial S_{2\varepsilon}}Pdx+Qdy-$$

$$-\int\limits_{\partial\Omega_2^\varepsilon}C(x,y)\varphi^2((x+a)n_1+(y+a)n_2)dS, \tag{19}$$

where $B'_\varepsilon-$ the intersection of the CB characteristic with the curve $\partial S_{2\varepsilon}$.

Taking into account the conditions $\psi(x,y)|_\sigma=0$ and (2), by direct calculations we obtain that

$$\varphi_x\left(x,y\right)=\varphi_y\left(x,y\right)=0 \quad \text{on} \quad \sigma. \tag{20}$$

Thus, by virtue of conditions a), b) of the lemma and (20), adding the integrals in (17), (18) and (19), discarding non-negative terms and calculating the limits, we obtain

$$2\int\limits_\Omega\varphi T\left(\psi\right)d\Omega\geq\beta\int\limits_\Omega(|K\left(y\right)|\varphi_x^2+|N\left(x\right)|\varphi_y^2+\varphi^2)d\Omega=\beta\left\|\varphi\right\|^2_{H_1(\Omega)}, \tag{21}$$

where $\beta=\min\left(\alpha,m\right)$.

Integrating by parts the left-hand side of inequality (21), we have

$$\int\limits_\Omega\varphi T(\psi)d\Omega=-\int\limits_\Omega(K(y)\varphi_x\psi_x+N(x)\varphi_y\psi_y-C(x,y)\varphi\psi)d\Omega\equiv B(\varphi,\psi). \tag{22}$$

Applying Hölder's inequality [10, p. 11] and the Cauchy inequality with ε [11, p. 67]

$$ab\leq\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2,\ a,b\geq0, \tag{23}$$

to the right-hand side of (22), we obtain

$$2\int\limits_{\Omega} \varphi T(\psi) d\Omega \leq \varepsilon C_1 \|\varphi\|^2_{H_1(\Omega)} + \frac{C_1}{\varepsilon} \|\psi\|^2_{H_1^*(\Omega)}, \tag{24}$$

where $C_1 = \max(1, \max\limits_{\overline{\Omega}} |C(x,y)|)$.

Choosing ε small enough, from (21) and (24) we find

$$\|\varphi\|_{H_1(\Omega)} \leq C_2 \|\psi\|_{H_1^*(\Omega)},$$

where C_2 depends on $\alpha,\ m,\ C_1$ and ε.

Hence it follows that

$$\|\varphi_n\|_{H_1(\Omega)} \leq C_2 \|\psi_n\|_{H_1^*(\Omega)} \tag{25}$$

and $\varphi_n(x,y) \in H_1(\Omega),\ n \in N$.

The lemma is proved.

It is clear that the functions $\varphi_n(x,y),\ n \in N$ are linearly independent. Indeed, if

$$\sum_{i=1}^{k} c_i^k \varphi_i \equiv 0$$

for some collection $\varphi_1, \varphi_2, \ldots, \varphi_k$, then, acting on this sum by the operator l, we have

$$\sum_{i=1}^{k} c_i^k l(\varphi_i) = \sum_{i=1}^{k} c_i^k \psi_i \equiv 0.$$

This implies that $c_i^k \equiv 0$ for all $i = \overline{1,k}$.

Note that without loss of generality, the system of functions $\{\varphi_n(x,y)\}_{n\in N}$, can be considered orthogonalized so that

$$(\varphi_i, \varphi_j)_{H_1(\Omega)} \equiv \int\limits_{\Omega} (|K(y)|\varphi_{ix}\varphi_{jx} + |N(x)|\varphi_{iy}\varphi_{jy} + \varphi_i\varphi_j) d\Omega = \delta_{ij}, \tag{26}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{at } i = j, \\ 0, & \text{at } i \neq j. \end{cases}$$

Since the system $\{\varphi_n(x,y)\}_{n\in N}$ is linearly independent, then using the Schmidt orthogonalization process [12, p. 40]

$$\overline{\varphi}_i = \frac{\widetilde{\varphi}_i}{\|\widetilde{\varphi_i}\|_{H_1(\Omega)}},\ \widetilde{\varphi}_1 = \varphi_1,\ \widetilde{\varphi}_{i+1} = \varphi_{i+1} - \sum_{k=1}^{i} (\overline{\varphi}_k, \varphi_{i+1})_{H_1(\Omega)} \overline{\varphi}_k,\ i = 1, 2, \ldots,$$

we obtain the required condition for orthonormality. Moreover, $\overline{\varphi}_i$, and, hence $\overline{\psi}_j$ generate the same variety of functions. This implies the completeness of the system $\{\psi_j(x,y)\}_{j\in N}$. Acting on $\widetilde{\varphi}_i$, $\overline{\varphi}_i$ by a linear operator l, we obtain a recurrence relation for $\overline{\psi}_j$:

$$\overline{\psi}_j = \frac{\widetilde{\psi}_j}{\|\widetilde{\varphi_j}\|_{H_1(\Omega)}},\ \widetilde{\psi}_1 = \psi_1,\ \widetilde{\psi}_{j+1} = \psi_{j+1} - \sum_{k=1}^{j} (\overline{\varphi}_k, \varphi_{j+1})_{H_1(\Omega)} \overline{\psi}_k,\ j = 1, 2, \ldots.$$

Theorem. Let conditions (2), a), b) of the lemma be satisfied and c) the function $f(x,y,U)$ is continuous by U and

$$f(x,y,U) = |N(x)K(y)|^{\frac{1}{2}} f_1(x,y,U),$$

where

$$\|f_1(x,y,U)\|_{L_2(\Omega)} \leq const$$

uniformly by U for any U from the ball

$$\|U\|_{L_2(\Omega)} \leq const.$$

Then there is a generalized solution to problem **T** from the class $H_1(\Omega)$.

Proof. We will seek an approximate solution to problem **T** in the form

$$U_r\left(x,y\right) = \sum_{i=1}^{r} C_{ir}\varphi_i\left(x,y\right), \quad r \in N, \tag{27}$$

where C_{ir} are determined from the system of nonlinear equations:

$$B(U_r,\psi_j) = \int\limits_{\Omega} f(x,y,U_r)\psi_j d\Omega,\ j = \overline{1,r}. \tag{28}$$

Taking into account (9) from (27)we find

$$l(U_r) \equiv \sum_{i=1}^{r} C_{ir} l(\varphi_i) = \sum_{i=1}^{r} C_{ir}\psi_i \in H^*{}_1(\Omega). \tag{29}$$

Multiplying (28) by C_{jr} and summing over j from 1 to r, we get

$$B(U_r, l(U_r)) = \int\limits_{\Omega} f(x,y,U_r) l(U_r) d\Omega. \tag{30}$$

Integrating the left side of (30), by parts, taking into account

$$U_r(x,y)|_{\Gamma} = 0$$

we have

$$B(U_r, l(U_r)) = \int\limits_{\Omega} U_r T(l(U_r)) d\Omega. \tag{31}$$

Using the reasoning given in the proof of the lemma, we obtain

$$\frac{1}{2}\beta\|U_r\|^2_{H_1(\Omega)} \leq \int\limits_{\Omega} f(x,y,U_r) l(U_r) d\Omega. \tag{32}$$

Since

$$f(x,y,U) = |N(x)K(y)|^{\frac{1}{2}} f_1(x,y,U_r),$$

then, applying Hölder's inequality [10, p. 11] to the right-hand side of (32), taking into account (23), we have

$$\int\limits_{\Omega} f(x,y,U_r) l(U_r) d\Omega \leq C_3 \|f_1\|_{L_2(\Omega)} \|U_r\|_{H_1(\Omega)} \leq$$
$$\leq \frac{C_3\varepsilon}{2}\|U_r\|^2_{H_1(\Omega)} + \frac{C_3}{2\varepsilon}\|f_1\|^2_{L_2(\Omega)}. \tag{33}$$

Choosing ε small enough, from (32) and (33) we find

$$\|U_r\|_{H_1(\Omega)} \leq const, \quad r \in N. \tag{34}$$

It follows from the well-known theorem on weak compactness [13, p. 83] that there exist a subsequence (we denote it again by U_r) and a function $U(x,y) \in H_1(\Omega)$ such that

$$U_r(x,y) \to U(x,y) \text{ weak in } H_1(\Omega). \tag{35}$$

From (35) it follows that in the linear part of (28) one can go to the limit, that is, the relation holds

$$-\int_\Omega (K(y)U_{rx}\psi_{jx} + N(x)U_{ry}\psi_{jy} - C(x,y)U_r\psi_j)d\Omega \overset{r\to\infty}{\to}$$

$$-\int_\Omega (K(y)U_x\psi_{jx} + N(x)U_y\psi_{jy} - C(x,y)U\psi_j)d\Omega. \tag{36}$$

Consider the function

$$w^r(x,y) = |K(y)N(x)|\, U_r(x,y).$$

Then for the function $w^r(x,y)$, by definition, the equality

$$\|w^r\|^2_{W_2^1(\Omega)} = \int_\Omega ((w_x^r)^2 + (w_y^r)^2 + (w^r)^2)d\Omega, \tag{37}$$

where $W_2^1(\Omega)-$ S.L. Sobolev space [10, p. 60].

Taking into account conditions a) of the lemma, (34) and using Hölder's inequality [10, p. 11], from (37) we have

$$\|w^r\|_{W_2^1(\Omega)} \leq C_4 \int_\Omega (|K(y)|U_{rx}^2 + |N(x)|U_{ry}^2 + U_r^2)d\Omega \leq C_5,$$

where $C_4, C_5 = const > 0$, dependent on known parameters.

This implies that $w^r(x,y) \in W_2^1(\Omega)$ for all $r \in N$.

Thus, by Kondrashov's theorem [10, p. 95], the embedding $W_2^1(\Omega)$ is compact in $L_2(\Omega)$, therefore, there exists a subsequence (which we again denote by $w^r(x,y)$ and the function $w(x,y) \in W_2^1(\Omega)$, such that

$$w^r(x,y) \to w(x,y) \,\text{strongly in}\, L_2(\Omega) \,\text{and almost everywhere in}\, \Omega. \tag{38}$$

It follows from (35) and (38) that $w(x,y) = |N(x)K(y)|\, U(x,y)$ almost everywhere. Consequently,

$$U_r\,(x,y) \to U\,(x,y) \text{ almost everywhere in } \Omega \text{ for } r \to \infty. \tag{39}$$

Since the function $f(x,y,U)$ is continuous with respect to U and it follows from (39)that

$$f(x,y,U_r) \to f(x,y,U) \,\text{almost everywhere in}\, \Omega \text{ as } r \to \infty. \tag{40}$$

But since

$$\|f(x,y,U_r)\|_{L_2(\Omega)} \leq const,$$

then, by virtue of the lemma [14, p. 25, Lemma 1.3] we obtain

$$f(x,y,U_r) \to f(x,y,U) \text{ weak in } L_2(\Omega).$$

Passing to the limit as $r \to \infty$ in (28) taking into account that $\psi_j(x,y) \in H_1^*(\Omega)$, $j \in N$ is a complete system in $H_1^*(\Omega)$, for $U(x,y) \in H_1(\Omega)$ we have

$$-\int\limits_{\Omega} (K(y)U_xV_x + N(x)U_yV_y - C(x,y)UV)\, d\Omega = \int\limits_{\Omega} f(x,y,U)V d\Omega,$$

for all $V(x,y) \in H_1^*(\Omega)$. This implies that $U(x,y)$ is a generalized solution to problem $\mathbf{T}$.

Now let us prove the solvability of system (28). We put

$$C = (C_{1r},\dots,C_{rr}),\ A(C) = (A^1(C),\dots,A^r(C)).$$

$$A^j(C) = -\sum_{i=1}^{r} C_{ir} \int\limits_{\Omega} (K(y)\varphi_{ix}\psi_{jx} + N(x)\varphi_{iy}\psi_{jy} - C(x,y)\varphi_i\psi_j)d\Omega -$$

$$-\int\limits_{\Omega} f(x,y,\sum_{i=1}^{r} C_{ir}\varphi_i)\psi_j d\Omega,\ j = \overline{1,r}. \tag{41}$$

If we show that $A^j(C)$ are continuous by C and there are constants $a_0 > 0$, $a_1 \geq 0$, such that for $\varepsilon_1 > 0$

$$(A(C), C) \geq a_0|C|^{1+\varepsilon_1} - a_1,$$

then, by Vishik's lemma [15] system (28) has at least one solution.

Based on the properties of the functions $f(x,y,U)$, $\varphi_i(x,y)$ and $\psi_j(x,y)$ it follows that $A^j(C)$ are continuous by C.

Taking into account (29), from (41) we find

$$\sum_{j=1}^{r} A^j(C)C_{jr} = -\int\limits_{\Omega} (K(y)U_{rx}(l(U_r))_x + N(x)U_{ry}(l(U_r))_y - C(x,y)U_r l(U_r))\, d\Omega -$$

$$-\int\limits_{\Omega} f(x,y,U_r)l(U_r)d\Omega \equiv J_1 + J_2. \tag{42}$$

Using the previously indicated orthogonality $\varphi_i(x,y)$, taking into account (27) and (32), for J_1 from (42) we obtain the following estimate:

$$J_1 \geq \frac{\beta}{2}\|U_r\|^2_{H_1(\Omega)} = \frac{\beta}{2}\sum_{i,j=1}^{r} C_{ir}C_{jr}\int\limits_{\Omega}(|K(y)|\varphi_{ix}\varphi_{jx} + |N(x)|\varphi_{iy}\varphi_{jy} + \varphi_i\varphi_j)d\Omega =$$

$$= \frac{\beta}{2}\sum_{i,j=1}^{r} C_{ir}C_{jr}\delta_{ij} = a_0|C|^2,$$

where $a_0 = \frac{\beta}{2}$.

Taking into account (33), (34) and condition c) of the theorem for J_2 from (42) we have

$$J_2 = -\int\limits_{\Omega} f(x,y,U_r)l(U_r)d\Omega \leq \frac{1}{2}C_3\varepsilon\|U_r\|^2_{H_1(\Omega)} + \frac{C_3}{2\varepsilon}\|f_1\|^2_{L_2(\Omega)} \leq a_1.$$

Means,

$$(A(C),C)=\sum_{j=1}^{r}A^{j}(C)C_{jr}\geq a_0|C|^2-a_1.$$

Therefore, it follows from Vishik's lemma [15] that system (28) has at least one solution. What was to be proven.

Example. Let $\sigma: x^{m+2}+y^{m+2}=1$, $K(y)=sgny|y|^m$, $N(x)=sgnx|x|^m$, at that $m\geq 1$, $a=1$, $\alpha-$ is a small positive number, $C(x,y)=-15-x-y$, $f(x,y,U)=|N(x)K(y)|^{\frac{1}{2}}/\left(1+U^2\right)^2$. Then conditions (2) a), b) of the lemma and c) of the theorem are satisfied.

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