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Analysis of Some Boundary Value Problems for Mixed-Type Equations with Two Lines of Degeneracy

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ABSTRACT

The article provides an overview of the practical significance of boundary value problems for mixed-type equations. methods for solving boundary value problems for a mixed-type equation with two degeneracy lines are analyzed and compared for cases when the degree of variables coefficients before second-order derivatives is the same and different. The problems arising in their solution are discussed.

Keywords: *Degeneration line, Elliptical type, Mixed type, Green's function and formula, Horn's function, Regularization, functional relation.*

1. Introduction

In the modern theory of partial differential equations, an important place is occupied by the study of degenerate hyperbolic and elliptic equations, as well as equations of mixed type. The increased interest in this class of equations is explained both by the great theoretical significance of the results obtained and by their numerous applications in gas dynamics, hydrodynamics, in the theory of infinitesimal surface bendings, in the momentless theory of shells, in various branches of continuum mechanics, acoustics, and in the theory of electron scattering, and many other fields of knowledge. The development of modern science and technology has shown that degenerate equations are a good model of real physical and biological processes. And this led to the relevance of setting and solving various boundary value problems for them, which are currently the subject of fundamental research by many mathematicians.

2. Literature Survey

It is known that S.A. Chaplygin in his work «About gas jets» [1] showed that the movement of gas under conditions of transition from subsonic to supersonic speed is described by a mixed type equation, which is currently called the equation of S.A. Chaplygin:

$$K(y)U_{xx} + U_{yy} = 0 \;\; (K(0) = 0, \;\; K'(y) > 0).$$

The systematic development of the theory of boundary value problems for degenerate equations of various types with a clear statement of problems, the proof of the existence and uniqueness of the solution, began in the 20-30s of the last century. During these years, Frankl F.I. [2] and S. Gellerstedt [3] obtained fundamental results. The beginning of a new stage in the development of the theory of equations of mixed type was the work of F.I. Frankl [2], in which he showed that the problem of the outflow of a supersonic jet from a vessel with flat walls (the velocity inside the vessel is subsonic) is reduced to a problem for the S.A. Chaplygin [1].

M.A.Lavrentiev together with AB Bitsadze in [4] considered the boundary value problem for the Lavrentiev-Bitsadze equation:

$U_{xx} + sgny \cdot U_{yy} = 0,$

in which, Lavrentiev M.A. the expediency of studying boundary value problems for equations of a simpler form, the study of which makes it possible to reveal the main properties of solutions to these equations, is noted.

Later, the theory of boundary value problems for a mixed type equation was developed in the works of T.Sh. Kalmenov, M.M. Smirnov, E.I. Moiseev, M.S. Salakhitdinov, T.D. Juraev and their students. The bibliography of the question can be found in the monograph [5].

It should be noted that the classical solvability of boundary value problems for linear equations of mixed types with one and two lines of degeneracy has been studied quite deeply. However, the solvability of boundary value problems for quasilinear equations of mixed type has not been fully studied, since there is no general theory that can be applied to study such equations. There are several papers devoted to boundary value problems for quasilinear equations of elliptic, hyperbolic and mixed types with one line of degeneracy. Such problems were studied in the works of J.K. Gvazava [6], M.I. Aliev [7], I.V. Mayorov [8]. They mainly study the classical solvability of the problems posed.

In the works of A.G. Podgaev [9] and K.F. Aziz [10], the existence of a generalized solution of boundary value problems for quasilinear equations of mixed types is studied.

At the same time, the classical and generalized solvability of boundary value problems for quasilinear equations of mixed types with two degeneracy lines has been little studied. In this direction, we note the works.

3. Proposed System

The first paper for a mixed-type equation with two lines of degeneracy was published. It explored the following problem.

Let $\Omega -$ a finite simply connected domain on the plane of variables (x, y), bounded at $x > 0, y > 0$ the curve σ_0: $x^3 + y^3 = 1$, and at $x < 0, y > 0$ and $x > 0, y < 0$ by the characteristics BC: $(-x)^{3/2} + y^{3/2} = 1$, CD: $x + y = 0$, DA: $x^{3/2} + (-y)^{3/2} = 1$ of the equation

$$yU_{xx} + xU_{yy} = 0 \qquad (1)$$

denoted by the Ω_0 elliptic part, and by Ω_1 and the $\Omega_2 -$ hyperbolic parts at $x > 0$ and $x < 0$, respectively, the domains Ω. Let further be a $OA(OB) -$ line $y = 0$, $(x = 0)$ segment $0 < x < 1$ $(0 < y < 1)$ on which the equation degenerates parabolically.

Problem 1

Find a function $U(x, y)$, with the following properties: $U(x, y) \in C(\overline{\Omega}) \cap C^2(\Omega)$; 2) $\left.\frac{\partial U}{\partial x}\right|_{x=0}$

and $\left.\frac{\partial U}{\partial y}\right|_{y=0}$

can go to infinity of order less than 1 at the point $O(0,0)$ and order less $2/3$ at the points B and A; 3) $U(x,y) -$ solution of equation (1) in the area Ω, satisfying the boundary conditions,

$U|_{AB} = \varphi(\theta), \qquad 0 \le \theta \le \pi/2,$

$U|_{OD} = \psi_1(x), \qquad 0 \le x \le 1/\sqrt[3]{4},$

$U|_{OC} = \psi_2(y), \qquad 0 \le y \le 1/\sqrt[3]{4},$

where $\varphi(\theta), \psi_1(x)$ and $\psi_2(y) -$ are given functions, and $\psi_1(0) = \psi_2(0)$.

The uniqueness of the solution of problem 1 is proved using the A.V. Bitsadze principle.

The proof of the existence of a solution is carried out by the method of integral equations. So, for this, the Cauchy-Goursat problem is first solved with the boundary conditions

$\lim_{y\to -0} U_y(x,y) = \nu_1(x), 0 < x < 1,$

U $|_{OD} = \psi_1(x), \;\; 0 \le x \le 1/\sqrt[3]{4}$

$\left(\lim_{x\to -0} U_x(x,y) = \nu_2(y), \;\; 0 < y < 1, U|_{OC} = \psi_2(y), \;\; 0 \le y \le 1/\sqrt[3]{4}\right).$

Here and below, the value of the function is assumed to be known $U(x,y)$ and its partial derivatives with respect to x and y first order on segments $OA(OB)$ (at the end of the study, these values of the function and derivatives are found). Denote $U(x,0) = \tau_1(x), U(0,y) = \tau_2(y)$.

Next, the form of functional relations between the functions $\tau_1(x)$ and $\nu_1(x)$, as well as $\tau_2(y)$ and $\nu_2(y)$, brought from the hyperbolic part of the $\Omega_1(\Omega_2)$ mixed region Ω.

Similarly, as above, problems ***N*** with boundary conditions are solved,

$U(x,y) = 0, (x,y) \in \sigma_0,$

$$\lim_{y\to +0} \frac{\partial U}{\partial y} = \nu_1(x), 0 < x < 1, \lim_{x\to +0} \frac{\partial U}{\partial x} = \nu_2(y), 0 < y < 1.$$

Then, assuming in the solution of the problem ***N*** first $y = 0$, and then the $x = 0$, relations between $\tau_i(x)$ and $\nu_i(x) (i = 1,2)$ are found, brought from the elliptic part of the Ω_0 mixed domain Ω.

As a result of exclusion $\tau_1(x)$ and $\tau_2(x)$ from the relations obtained from the elliptic and hyperbolic parts of the domain Ω, after some transformations, we obtain a system of singular integral equations with respect to $\nu_1(x)$ and $\nu_2(x)$. After the regularization of the system of equations by the Carleman-Vekua method, we obtain an integral Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution of the problem.

Note that with an exception $\tau_1(x)$ and $\tau_2(x)$ from relations and regularization of systems of singular integral equations, the operator is relatively $\nu_1(x)$ and $\nu_2(x)$ widely used

$$D_{x1}^{1-2\beta}[\cdot] = -\frac{1}{\Gamma(2\beta)} \frac{d}{dx} \int_0^1 (t-x)^{2\beta-1} [\cdot] dt$$

properties of Gaussian hypergeometric functions $F(a,b;c;z)$, the theory of series and systems of singular integral equations, and the Fredholm integral equation of the second kind.

Note that equation (1) $x_1=\frac{1}{3}(x^3-y^3),\ \ y_1=xy$ is reduced by transformation to the Tricomi equation $y_1U_{x_1x_1}+U_{y_1y_1}=0$, and the problem is reduced to the Gellerstedt problem [3].

However, such a reduction does not simplify the study of problem 1 for an equation of type (1) with lower terms, and the method presented here without can be used for equations more general than (1). In addition, the modified problem 1 for equation (1), with the help of some transformations, is not reduced to problems for the Tricomi equation $y_1U_{x_1x_1}+U_{y_1y_1}=0$. This means that problems for equation (1) cannot always be reduced to problems for an equation with one line of degeneracy.

Also, a similar boundary value problem for the equation,

$$signy|y|^mU_{xx}+signx|x|^mU_{yy}=0,m=const>0 \qquad (2)$$

In [5], a boundary value problem, an analogue of the Tricomi problem for the equation

Consider the equations,

$$signy|\boldsymbol{y}|^{\boldsymbol{m}}\ U_{xx}\ +\ signx|x|^n\ U_{yy}\ +\ a(x,y)U_x\ +$$

$$b(x,y)U_y+c(x,y)U=d(x,y) \qquad (3)$$

where $m,n-$ piecewise continuous functions of (x,y) and

$$m(n)=\begin{cases} m(n) & at \quad x>0, \quad y>0,\\ m_1(n) & at \quad x>0, \quad y<0,\\ m(n_1) & at \quad x<0, \quad y>0.\end{cases}$$

Let Ω be a finite simply connected domain in the plane of variables (x,y), bounded by $x>0,y>0$ curve σ with ends at points $A(h_1,0)$ and $B(0,h_2)$, and with $x>0,y<0$ and $x<0,y>0$ characteristics,

$$OC:\ \frac{1}{q}x^q-\frac{1}{p_1}(-y)^{p_1}=0, \qquad AC:\ \frac{1}{q}x^q+\frac{1}{p_1}(-y)^{p_1}=0$$

and

$$\boldsymbol{OD}:\frac{1}{q_1}x^{q_1}-\frac{1}{p}y^p=0, \quad \boldsymbol{BD}:\frac{1}{q_1}x^{q_1}-\frac{1}{p}y^p=1$$

equations (3) respectively, $2q=n+2,\ 2q_1=n_1+2,\ 2p=m+2,$

$2p_1=m_1+2, \qquad h_1=q^{1/q},\ h_2=p^{1/p}.$

It is assumed that the coefficients $a(x,y),b(x,y),c(x,y)$ and the free term $d(x,y)$ equations (3) are sufficiently smooth functions.

Problem TB. Find a function $u(x,y)$ with the following properties:

(1) $u(x,y)\epsilon\ C(\overline{\Omega})\cap C^1(\Omega_0\cup\Omega_1\cup\Omega_2)$;

(2) $u_x \epsilon C(J_2), u_y \epsilon C(J_1)$ moreover $u_x(x,0)$ and $u_x \epsilon (0,y)$ may have a feature of the order of less $(1-2\beta_1)/(1-2\alpha)$ and $(1-2\alpha_1)/(1-2\beta)$ at the ends of the intervals J_1 and J_2 respectively:

(3) $u(x,y) -$ a twice continuously differentiable solution of equation (3) in the domain $\Omega/(J_1 \cup J_2)$, here $J_1 = \{(x,y): y = 0,\ \ 0 < x < 1\}, J_2 = \{(x,y):\ x = 0,\ \ 0 < y < 1\}$;

(4) $u(x,y)$ satisfies the boundary conditions

$$u|_\sigma = \varphi(x,y), \quad (x,y) \in \bar{\sigma}$$

$$u|_{OC} = \psi_1(x), \quad 0 \le x \le (q\ /\ 2)^{1/q},$$

$$u|_{OD} = \psi_2(x), \quad -(q_1\ /\ 2)^{1/q_1} \le x \le 0,$$

(5) On the lines of the parabolic degeneracy of the equation (3) the gluing conditions are fulfilled

$$u_y(x,-0) = u_y(x,+0)\ , \quad (x,0) \in J_1,$$

$$u_x(-0,y) = u_x(+0,y)\ , \quad (0,y) \in J_2,$$

where $\varphi(x,y),\ \psi_1(x),\ \psi_2(x)$ - the given functions,

$$\varphi(x,y) = xy\bar{\varphi}(x,y), \quad \bar{\varphi}(x,y) \in C(\bar{\sigma}), \psi_1(x) \in C^3\left[0, (\tfrac{q}{2})^{1/q}\right],$$

$\psi_2(x) \in C^3\left[-\left(\frac{q_1}{2}\right)^{\frac{1}{q_1}}, 0\right]$ and $\psi_1(0) = \psi_2(0)$.

Note that the methodology for studying the solvability of this boundary value problem is theoretically similar to the previous problem, but in practice they are very different. For example, the underlying solutions used are different. Thus, when studying boundary value problems for equation (3), the theory of series, derivatives and integrals of fractional order, the generalized fractional order integral (see below), Green's formulas, curvilinear integrals of the first and second kind, improper integrals, properties of two variables of hypergeometric Gaussian functions, the Horn equation and its solutions. The Green's function of the problem $\boldsymbol{N}$ for equation (3) is a very complex function. The generalized fractional order integral of a function $g(x)$ has the form:

$$F_{0x}^{-c}[a,b;c;x^k]g(x) =$$

$$\frac{1}{\Gamma(c)}\int_0^x g(x)\left(x^k - t^k\right)^{c-1} F\left(a,b;c;\frac{x^k - t^k}{x^k}\right) kt^{k-1}dt, k > 0.$$

It should be noted that the degree of degeneracy $m = const > 0$ into equation (2) can be the same in all regions $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ simultaneously. However, when studying the boundary value problem for equation (3), the degrees of degeneracy n and m will be different in each area $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ and has the form:

$$m(n) = \begin{cases} m(n) & at \quad x > 0, \quad y > 0, \\ m_1(n) & at \quad x > 0, \quad y < 0, \\ m(n_1) & at \quad x < 0, \quad y > 0. \end{cases}$$

This is necessary for the solvability of the boundary value problem for equation (3), in particular, when the resulting systems of singular integral equations are regularized, it becomes necessary to require such conditions.

4. Conclusion

It can be shown that equation (3) after some transformations of the independent variables reduces to a mixed-type equation with one line of degeneracy. However, the studied local and nonlocal boundary value problems for equation (3) cannot be reduced to problems for equations with one line of degeneracy.

The method used here for solving the boundary value problem for equation (3) is new and universal, which, without fundamental difficulties, can also be used for equations more general than (1) and (2).

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Competing Interests Statement

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Consent for publication

Authors declare that they consented for the publication of this research work.

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