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Boundary Value Problem in a Domain with Deviation from the Characteristics for One Nonlinear Equation with Mixed Type

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Abstract. In this paper, the existence of a generalized solution of the investigated boundary-value problem for a nonlinear equation of mixed type with two lines of degeneration in the weighted S. L. Sobolev space is proved. A particular case of an equation is given, in which a generalized solution exists in a weightless S. L. Sobolev space. Examples of functions, satisfying the conditions of the lemmas and theorems on the solvability of the problem, are constructed.

INTRODUCTION. FORMULATION OF THE PROBLEM

Relatively few works have been devoted to boundary-value problems for nonlinear equations of mixed type. In all these works, the considered region, where the equation belongs to the hyperbolic type, consists of a characteristic triangle [1, 2, 3, 4].

In this paper, a boundary-value problem in a domain with deviation from the characteristic for one nonlinear equation of mixed type with two lines of degeneracy is studied.

Consider the equation

$$T(U) \equiv K(y)U_{xx} + N(x)U_{yy} + C(x,y)U = f(x,y,U) \tag{1}$$

in the domain Ω on the plane of variables (x,y), bounded by $x>0$, $y>0$ a smooth curve σ with ends at the points $A(1,0)$ and $B(0,1)$, and by $x>0$, $y<0$ $(x<0, y>0)$ — the characteristics of the equation $OD_1(OC_1)$ and smooth curves $D_1A(BC_1)$ lying inside the characteristic triangle ODA (OBC), respectively. Here, $K(y)$, $N(x)$, $C(x,y)$, $f(x,y,U)$ — are given functions, and $K(t) \gtreqless 0$, $N(t) \gtreqless 0$ where $t \gtreqless 0$, OD, DA and OC, CB characteristics of the equation outgoing from points $O(0,0)$, $A(1,0)$ and $O(0,0)$, $B(0,1)$, respectively. Let $\Gamma D_1A \cup \sigma \cup BC_1$.

Suppose the given functions $K(y)$, $N(x)$, $C(x,y)$ are continuously differentiable and satisfy the conditions $N'(x) \geq \alpha|N(x)|$, $K'(y) \geq \alpha|K(y)|$, $C(x,y)\big|_{C_1D_1} \leq 0$, $C_x(x,y)+C_y(x,y) \leq -m < 0$ in $\overline{\Omega}$, where α, $m = \text{const} > 0$.

Problem T. Find a solution $U(x,y)$ of the equation (1) in the domain Ω, so that

$$U(x,y)\big|_{\Gamma} = 0. \tag{2}$$

EXISTENCE OF A GENERALIZED SOLUTION OF THE PROBLEM

Consider function spaces: $U(\Omega) = \{U : U \in C^{\infty}(\overline{\Omega}), U\big|_{\Gamma} = 0\}$, $V(\Omega) = \{V : V \in C^{\infty}(\overline{\Omega}), V\big|_{\partial\Omega} = 0\}$.
Denote by $H_1(\Omega)$ and $H_1^*(\Omega)$ the closure in the norm of the function spaces $U(\Omega)$ and $V(\Omega)$, respectfully:

$$\|U\|_{H_1(\Omega)} = \left(\int_{\Omega} \left(|K(y)|U_x^2 + |N(x)|U_y^2 + U^2\right) d\Omega\right)^{1/2},$$

$$\|V\|_{H_1^*(\Omega)} = \left(\int_{\Omega} \left(|K(y)|V_x^2 + |N(x)|V_y^2 + V^2\right) d\Omega\right)^{1/2}.$$

Definition 1 *A generalized solution of the problem* (1), (2) *is a function* $U(x,y) \in H_1(\Omega)$, *satisfying the identity*

$$B(U,V) \equiv -\int_{\Omega} \left(K(y)U_xV_x + N(x)U_yV_y - C(x,y)UV\right) d\Omega = \int_{\Omega} f(x,y,U)V\,d\Omega$$

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for any function $V(x,y) \in H_1^*(\Omega)$.

Lemma 1 *Assume*

a) $\{\psi_n(x,y)\}_{n\in N}$ *— is a complete system of smooth functions in the space* $H_1^*(\Omega)$*, belonging to* $V(\Omega)$*;*

b) $-n_1+n_2 \geq 0$ *on* C_1O*;* $n_1-n_2 \geq 0$ *on* OD_1*;* $n_1+n_2 \geq 0$ *on* C_1D_1*;*

c) $n_1+n_2 < 0$ *on* Γ*, where* (n_1,n_2) *— the inner normal vector.*

Then there are functions $\{\varphi_n(x,y)\}_{n\in N} \in H_1(\Omega)$*, that are also solutions of the following boundary-value problem:*

$$l(\varphi_n) \equiv \varphi_{nx}(x,y)+\varphi_{ny}(x,y) = \psi_n(x,y),\ \varphi_n(x,y)|_\Gamma = 0,\ n\in N. \tag{3}$$

Note that the condition $n_1+n_2<0$ on Γ means lines $y=x+C$, can not intersect the curve Γ twice, where $C=\text{const}$.

Proof. The characteristics of equation (3) due to condition c) cannot intersect the curve Γ twice. $\{\psi_n(x,y)\}_{n\in N}$ — smooth functions, then the solution of this boundary-value problem (3) exists and is a smooth function, excluding, perhaps, points $(1,0)$ and $(0,1)$.

In new independent variables $\xi=(x+y)/2,\ \ \eta=(x-y)/2$ of this boundary-value problem (3) take the form (omit the subscripts n):

$$\varphi_\xi = \psi,\ \ \varphi\big|_{\widetilde{\Gamma}} = 0,$$

where $\widetilde{\Gamma}$ — the image of the curve Γ on the plane (ξ,η).

Returning to the old variables (x,y), the solution of the problems has the form:

$$\varphi(x,y) = \int\limits_{\chi((x-y)/2)}^{(x+y)/2} \psi(t+(x-y)/2, t-(x-y)/2)\,dt \equiv I(\psi),$$

where $x+y=2\chi\left((x-y)/2\right)$ is an equation of the curve Γ. Now let us prove that $\{\varphi_n(x,y)\}_{n\in N}\in H_1(\Omega)$.

Denote by $\Omega_0=\Omega\cap\{(x,y): x>0, y>0\}$, $\Omega_1=\Omega\cap\{(x,y): x>0, y<0\}$ and $\Omega_2=\Omega\cap\{(x,y): x<0, y>0\}$.

Cut out a part of the circle centered at points $(1,0)$ and $(0,1)$ of small radius $\varepsilon>0$ from the domain Ω_0. The remaining part of the domain denote by $\Omega_0^\varepsilon = (\Omega\cap\{(x,y): x>0, y>0\})\setminus\left(\overline{S}_{1\varepsilon}\cup\overline{S}_{2\varepsilon}\right)$, where $S_{1\varepsilon}=\{(x,y): (x-1)^2+y^2<\varepsilon^2\}$, $S_{2\varepsilon}=\{(x,y): x^2+(y-1)^2<\varepsilon^2\}$.

By the smoothness of the function $\varphi_n(x,y)$ and the equality $l(\varphi_n)=\psi_n$, the following integral can be integrated (omitting the index n):

$$2\int\limits_{\Omega_0} I(\psi)T(\psi)d\Omega = 2\int\limits_{\Omega_0}\varphi T(l(\varphi))d\Omega = 2\lim_{\varepsilon\to 0}\int\limits_{\Omega_0^\varepsilon}\varphi T(l(\varphi))d\Omega. \tag{4}$$

Integrating by parts and using the following identities:

$$2K(y)\varphi_{xx}l(\varphi) = K'(y)\varphi_x^2 + \left(K(y)\varphi_x^2\right)_x + \left(2K(y)\varphi_x\varphi_y\right)_x - \left(K(y)\varphi_x^2\right)_y,$$

$$2N(x)\varphi_{yy}l(\varphi) = N'(x)\varphi_y^2 - \left(N(x)\varphi_y^2\right)_x + \left(2N(x)\varphi_x\varphi_y\right)_y - \left(N(x)\varphi_y^2\right)_y,$$

$$2C(x,y)\varphi l(\varphi) = -\left(C_x(x,y)+C_y(x,y)\right)\varphi^2 + \left(C(x,y)\varphi^2\right)_x + \left(C(x,y)\varphi^2\right)_y,$$

and using Green's formula, the integral on the right-hand side of the equality (4) takes the form:

$$2\int\limits_{\Omega_0^\varepsilon}\varphi T\left(l(\varphi)\right)d\Omega = \int\limits_{\Omega_0^\varepsilon}\Pi d\Omega + \int\limits_0^{1-\varepsilon}\left(N(x)\varphi_y^2 - 2N(x)\varphi\psi_y\right)dx$$

$$+\int\limits_0^{1-\varepsilon}\left(K(y)\varphi_x^2 - 2K(y)\varphi\psi_x\right)dy + \int\limits_{\partial\Omega_0^\varepsilon\setminus((x=0)\cup(y=0))} Pdx+Qdy$$

$$-\int\limits_{\partial\Omega_0^\varepsilon} C(x,y)\varphi^2(n_1+n_2)dS,$$

where

$$\Pi = K'(y)\varphi_x^2 + N'(x)\varphi_y^2 - \left(C_x(x,y)+C_y(x,y)\right)\varphi^2,$$

$$P = K(y)\varphi_x^2 - 2N(x)\varphi_x\varphi_y - N(x)\varphi_y^2 - 2N(x)\psi\varphi_y + 2N(x)\varphi\psi_y,$$

$$Q = K(y)\varphi_x^2 + 2K(y)\varphi_x\varphi_y - N(x)\varphi_y^2 + 2K(y)\varphi\psi_x - 2K(y)\psi\varphi_x.$$

Similarly, cut out a part of the circle centered at points $(1,0)$ and $(0,1)$ of small radius $\varepsilon > 0$ from the domains Ω_1 and Ω_2, respectively. The remaining part of domains denote by $\Omega_1^\varepsilon = \Omega_1 \backslash \overline{S}_{1\varepsilon}$ and $\Omega_2^\varepsilon = \Omega_2 \backslash \overline{S}_{2\varepsilon}$, and consider the integrals:

$$I_1^\varepsilon = 2\int\limits_{\Omega_1^\varepsilon} \varphi T\left(l(\varphi)\right)d\Omega \text{ and } I_2^\varepsilon = 2\int\limits_{\Omega_2^\varepsilon} \varphi T\left(l(\varphi)\right)d\Omega.$$

Given the equality $-\sqrt{-K(y)}dy = \sqrt{N(x)}dx$ on OD_1, the right-hand side of integral I_1^ε can be rewritten as

$$2\int\limits_{\Omega_1^\varepsilon} \varphi T\left(l(\varphi)\right)d\Omega = \int\limits_{\Omega_1^\varepsilon} \Pi d\Omega - \int\limits_0^{1-\varepsilon} \left(N(x)\varphi_y^2 - 2N(x)\varphi\psi_y\right)dx$$

$$-\int\limits_{OD_1} \left[\sqrt{-K(y)}\varphi_x - \sqrt{N(x)}\varphi_y\right]^2 (dx+dy) + 2\int\limits_{OD_1} \varphi d_m\psi - \psi d_m\varphi$$

$$+\int\limits_{(y<0)\cap(\partial S_{1\varepsilon}\cup D_1A'_\varepsilon)} Pdx + Qdy - \int\limits_{\partial\Omega_1^\varepsilon} C(x,y)\varphi^2(n_1+n_2)dS,$$

where A'_ε is the intersection point of the curve D_1A with the curve $\partial S_{1\varepsilon}$ and the right-hand side I_2^ε as

$$2\int\limits_{\Omega_2^\varepsilon} \varphi T\left(l(\varphi)\right)d\Omega = \int\limits_{\Omega_2^\varepsilon} \Pi d\Omega - \int\limits_0^{1-\varepsilon} \left(K(y)\varphi_x^2 - 2K(y)\varphi\psi_x\right)dy$$

$$+\int\limits_{C_1O} \left[\sqrt{K(y)}\varphi_x - \sqrt{-N(x)}\varphi_y\right]^2 (dx+dy) + 2\int\limits_{C_1O} \varphi d_m\psi - \psi d_m\varphi$$

$$+\int\limits_{(x<0)\cap(\partial S_{2\varepsilon}\cup B'_\varepsilon C_1)} Pdx + Qdy - \int\limits_{\partial\Omega_2^\varepsilon} C(x,y)\varphi^2(n_1+n_2)dS,$$

where B'_ε is the intersection point of the curve C_1B with the curve $\partial S_{2\varepsilon}$.

Note that on the part Γ_ε of the curve Γ where $n_1+n_2<0$, $\varphi_x+\varphi_y$ is the nontangential derivative, which, due to the condition $\psi|_\Gamma = 0$, is equal to zero together with the function φ. Therefore, $\varphi_x = \varphi_y = 0$ is on Γ_ε.

Taking this into account and conditions b), c) of Lemma 1 adding integrals $I_0^\varepsilon, I_1^\varepsilon, I_2^\varepsilon$ discarding non-negative terms and calculating the limits at $\varepsilon \to 0$

$$2\int_\Omega \varphi T(\psi) d\Omega \geq \beta \int_\Omega \left(|K(y)|\,\varphi_x^2 + |N(x)|\,\varphi_y^2 + \varphi^2\right) d\Omega = \beta \|\varphi\|^2_{H_1(\Omega)}, \tag{5}$$

where $\beta = \min(\alpha, m)$ is obtained.

Integrating the left-hand side of (5) by parts and using inequality Hölder's ([5], p. 11) and inequality Cauchy ([6], p. 67) with ε

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \;\; a,b \geq 0,$$

we get

$$2\int_\Omega \varphi T(\psi) d\Omega \leq C_1 \varepsilon \|\varphi\|^2_{H_1(\Omega)} + \frac{C_1}{\varepsilon}\|\psi\|^2_{H_1^*(\Omega)} \tag{6}$$

is derived where $C_1 = \max\left(1, \max|C(x,y)| \text{ in } \overline{\Omega}\right)$.

Choosing ε small enough, from (5) and (6) find $\|\varphi\|^2_{H_1(\Omega)} \leq C_2\|\psi\|^2_{H_1^*(\Omega)}$, where C_2 depends on α, m, C_1 and ε. Hence it follows that $\|\varphi_n\|^2_{H_1(\Omega)} \leq C_2\|\psi_n\|^2_{H_1^*(\Omega)}$, and $\varphi_n(x,y) \in H_1(\Omega),\; n \in N$.

Theorem 2 *Suppose the conditions of Lemma 1 are satisfied, and the function $f(x,y,U)$ is continuous for U and $f(x,y,U) = |K(y)N(x)|^{1/2} f_1(x,y,U)$, where $\|f_1(x,y,U)\|_{L_2(\Omega)} \leq const$ is uniform in U for any U from the ball $\|U\|_{L_2(\Omega)} \leq const$.*

Then there exists a generalized solution of the problem **T** *from the class $H_1(\Omega)$.*

Proof. First of all, note that the system of functions $\{\varphi_n(x,y)\}_{n\in N}$ is independent and can be considered normalized ([7], p. 159, [4]) so that

$$(\varphi_i, \varphi_j)_{H_1(\Omega)} \equiv \int_\Omega (|K(y)|\varphi_{ix}\varphi_{jx} + |N(x)|\varphi_{iy}\varphi_{jy} + \varphi_i\varphi_j)\, d\Omega = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Seek an approximate solution to the boundary value problem **T**in the form

$$U_r(x,y) = \sum_{i=1}^r C_{ir}\varphi_i(x,y) \in H_1(\Omega),\; r \in N,$$

where C_{ir} are determined from the system

$$B(U_r, \psi_j) = \int_\Omega f(x,y,U_r)\psi_j d\Omega, \;\; j = 1,2,\ldots,r. \tag{7}$$

Multiplying (7) by C_{jr} summing over j from 1 to r.

$$B(U_r, l(U_r)) = \int_\Omega f(x,y,U_r) l(U_r) d\Omega$$

are drawn.

By analogous reasoning as in the proof of Lemma 1, find $\|U_r\|_{H_1(\Omega)} \leq \text{const}, \;\; r \in N$.

Therefore, there exists a subsequence ([8], p. 83) (denote it again by U_r) and a function $U(x,y) \in H_1(\Omega)$ such that

$$U_r(x,y) \to U(x,y) \text{ is weak in } H_1(\Omega).$$

It follows that in the linear terms on the left in (7), it is possible to pass to the limit:

$$-\int_\Omega (K(y)U_{rx}\psi_{jx} + N(x)U_{ry}\psi_{jy} - C(x,y)U_r\psi_j)\,d\Omega \longrightarrow [r\to\infty]$$

$$-\int_\Omega (K(y)U_x\psi_{jx} + N(x)U_y\psi_{jy} - C(x,y)U\psi_j)\,d\Omega.$$

By the condition of the Theorem 2 and the lemma ([9], p. 25)

$$f(x,y,U_r) \to f(x,y,U) \text{ is weak in } L_2(\Omega).$$

This implies that $U(x,y)$ — a generalized solution of the problem **T**.

To prove the solvability of the system (7). Put $C = (C_{1r},...C_{rr})$, $A(C) = (A^1(C),...,A^r(C))$.

$$A^j(C) = -\sum_{i=1}^r C_{ir}\int_\Omega (K(y)\varphi_{ix}\psi_{jx} + N(x)\varphi_{iy}\psi_{jy} - C(x,y)\varphi_i\psi_j)\,d\Omega$$

$$-\int_\Omega f\left(x,y,\sum_{i=1}^r C_{ir}\varphi_i\right)\psi_j d\Omega, \quad j=\overline{1,r}.$$

The properties of the functions $f(x,y,U)$, $\varphi_i(x,y)$, $\psi_j(x,y)$, imply the continuity of $A^j(C)$. Using the orthogonality of $\varphi_i(x,y)$ and $\psi_j(x,y)$, the linear part $(A(C),C)$ will give $|C|^2$. By Lemma ([10], p. 134) the system (7) has at least one solution.

Note, that if $K(y)=y$, $N(x)=x$, then the problem **T** for the equation

$$\overline{T}(U) = yU_{xx} + xU_{yy} + C(x,y)U = f(x,y,U) \tag{8}$$

can be considered in the weightless space of S.L. Sobolev $W_2^1(\Omega)$ ([5], p. 60) and, in addition, some conditions of Lemma 1 and Theorem 2 can be weakened.

Denote by $W_2^1(\Omega)$ and $W_2^{*1}(\Omega)$ the closure in the norm ([5], p. 60) of the function spaces $U(\Omega)$ and $V(\Omega)$, respectively.

Definition 2 *A generalized solution of the problem (8), (2) is a function $U(x,y)\in W_2^1(\Omega)$, satisfying the identity*

$$B_1(U,V) \equiv -\int_\Omega (yU_xV_x + xU_yV_y - C(x,y)UV)\,d\Omega = \int_\Omega f(x,y,U)V\,d\Omega$$

for any functions $V(x,y)\in W_2^{*1}(\Omega)$.

Lemma 3 *Assume*

a) $\{\psi_n(x,y)\}_{n\in N}$ — a complete system of smooth functions in the space $W_2^{*1}(\Omega)$, belonging to $V(\Omega)$;

b) $C(x,y)\in C^1(\overline{\Omega})$, $C(x,y)|_{C_1D_1}\le 0$, $C_x(x,y)+C_y(x,y)\le 0$ in $\overline{\Omega}$;

c) $n_1+n_2<0$ on Γ, where (n_1,n_2) — inner normal vector.

Then there are functions $\{\varphi_n(x,y)\}_{n\in N}\in W_2^1(\Omega)$, which are solutions of the boundary-value problem

$$\varphi_{nx}(x,y)+\varphi_{ny}(x,y) = \psi_n(x,y),\ \varphi_n(x,y)|_\Gamma = 0,\ n\in N.$$

Lemma 3 is proved similarly to the proof of Lemma 1.

Theorem 4 *Assume the conditions of Lemma 3 are satisfied and the function $f(x,y,U)$ is continuous for U and $\|f(x,y,U)\|_{L_2(\Omega)}\le const$ is uniform in U from any U from the ball $\|U\|_{L_2(\Omega)}\le const$.*

*Then there exists a generalized solution to problem **T** for the equation (8) from the class $W_2^1(\Omega)$.*

Theorem 4 is proved similarly to the proof of Theorem 2.

Example. If $K(y) = \mathrm{sign}y|y|^m$, $N(x) = \mathrm{sign}x|x|^n$, and $m \geq 1$, $n \geq 1$, $\alpha = 1/10$, $\sigma : x + y = 1$, $D_1A : y = (x-1)/(2^{4/5}-1)$, $C_1B : x = (y-1)/(2^{4/5}-1)$, $C(x,y) = -24 - x - y$, $f(x,y,U) = |N(x)K(y)|^{1/2}/(1+U^2+U^4)^4$, then the conditions b), c) of Lemma 1 and Theorem 2 are satisfied. Therefore, in this case, there exists a generalized solution of the problem **T** for the equation (1).

Comment. The next example shows that under the conditions of Theorem 4, there may not be the unique solution.

Assume $C(x,y) \equiv 0$,

$$f(x,y,U) = \left[(6y^3 + 6x^3)(1-x-y)^2 + 6(x^2y^3 + x^3y^2)\right] U^{1/3}$$

$$-18(x^2+y^2)U^{2/3}, \quad \sigma : x+y = 1, \; D_1A : y = \frac{x-1}{2^{4/5}-1}, \; C_1B : x = \frac{y-1}{2^{4/5}-1}.$$

Then the problem (8), (2) has at least two solutions: $U(x,y) \equiv 0$,

$$U(x,y) = \begin{cases} x^3y^3(1-x-y)^3, & \text{for } x > 0, \; y > 0, \\ 0, & \text{for } x > 0, \; y < 0, \\ 0, & \text{for } x < 0, \; y > 0. \end{cases}$$

CONCLUSION

In conclusion, if the function $f(x,y)$ is linear, then the proposed method makes it possible to numerically solve the problem **T** for the equation (1).

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