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On the Qualitative Analysis of a Class of Volterra Quadratic Stochastic Operators with Continuous Time

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Abstract—In this paper, a continuous analog of one class of Volterra quadratic stochastic operators with continuous time is studied. A qualitative analysis of the operators is carried out, and numerical and analytical solutions are found. Analytical solutions are compared with the numerical one, and it is proved that the trajectory of the operator tends to the equilibrium. Fourteen extreme operators are also studied and the results are presented in the form of a table.

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1. INTRODUCTION

The representation of a mathematical model of a number of biological, physicochemical, and economic processes using quadratic stochastic operators (QSOs) is of interest to mathematicians in this field. At present, many scientific papers have been published in this area [1]–[12]. The theory of QSOs with discrete time was founded by Bernshtein [1]. Later, this area was developed and the concept of the theory of bisexual quadratic stochastic operators was introduced by Lyubich [5]. His monograph describes in detail the formulation of the problem and general methods of solution. In [6], bisexual QSOs, belonging to a subclass of QSOs [5], are investigated and fundamental results are obtained with a new approach. A special case of bisexual QSOs is also presented and analyzed—sixteen so-called extreme operators (the population dimension is four) that have the form

$$\begin{aligned}
&W_1(x_1,x_2,y_1,y_2)=(x_1+x_2y_1,x_2y_2,x_1+x_2y_1,x_2y_2),\\
&W_2(x_1,x_2,y_1,y_2)=(x_1+x_2y_1,x_2y_2,x_1,x_2),\\
&W_3(x_1,x_2,y_1,y_2)=(x_1+x_2y_1,x_2y_2,y_1,y_2),\\
&W_4(x_1,x_2,y_1,y_2)=(x_1+x_2y_1,x_2y_2,x_1y_1,x_2y_1+y_2),\\
&W_5(x_1,x_2,y_1,y_2)=(x_1,x_2,x_1+x_2y_1,x_2y_2),\\
&W_6(x_1,x_2,y_1,y_2)=(x_1,x_2,x_1,x_2),\\
&W_7(x_1,x_2,y_1,y_2)=(x_1,x_2,y_1,y_2),\\
&W_8(x_1,x_2,y_1,y_2)=(x_1,x_2,x_1y_1,x_2y_1+y_2),\\
&W_9(x_1,x_2,y_1,y_2)=(y_1,y_2,y_1+x_1y_2,x_2y_2),\\
&W_{10}(x_1,x_2,y_1,y_2)=(y_1,y_2,x_1,x_2),\\
&W_{11}(x_1,x_2,y_1,y_2)=(y_1,y_2,y_1,y_2),\\
&W_{12}(x_1,x_2,y_1,y_2)=(y_1,y_2,x_1y_1,x_2y_1+y_2),
\end{aligned}$$

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$$W_{13}(x_1, x_2, y_1, y_2) = (x_1y_1, x_2, x_1 + x_2y_1, x_2y_2),$$
$$W_{14}(x_1, x_2, y_1, y_2) = (x_1y_1, x_2y_1 + y_2, x_1, x_2),$$
$$W_{15}(x_1, x_2, y_1, y_2) = (x_1y_1, x_2y_1 + y_2, y_1, y_2),$$
$$W_{16}(x_1, x_2, y_1, y_2) = (x_1y_1, x_2y_1 + y_2, x_1y_1, x_2y_1 + y_2).$$

However, the continuous analogue of such extreme operators has not been investigated.

The implementation methods presented in this article are of interest for studying systems of nonlinear differential equations, since they provide some grounds for studying important classes of differential equations, as well as individual points of view for solving some classical problems.

The article has the following structure. First, continuous- and discrete-time QSOs are analyzed, their advantages and disadvantages are indicated. The problem statement is formulated and the first and fourteenth of the sixteen extreme operators are studied. In particular, general solutions to the system are found, equilibria are established, and the graphs of analytical and numerical solutions are constructed and compared. The analysis of the remaining fourteen extreme operators is presented in the form of a table.

It is well known that some economic processes or biological populations do not have the property of continuous measurement of numbers. For example, the growth in the numbers of their generations occurs at discrete moments in time. In addition, depending on the specific biological population, the length of the time interval can vary greatly. The magnitude of effects can also vary depending on the sampling frequency.

When analyzing monthly data using a discrete-time model, we only obtain answers relative to the monthly interval and cannot be sure that the results extrapolate to other sampling intervals. In addition, discrete-time models can often have problems handling unevenly spaced observations or missing data. In these cases, we either need to start estimating different parameters for two different time intervals or find ways to create equal intervals.

In turn, continuous-time models take into account the constant influence of participants upon themselves and upon other participants, even if they are not considered (for example, the influence of a mother on a child). Thus, for primitive organisms, where the intervals between these time processes is sufficiently short, a continuous-time mathematical model can be a suitable idealization of the real process.

An essential distinguishing feature of this category of systems is the fact that the mathematical model is formulated as a system of differential equations with the addition of a system of algebraic equations. Basically, such models arise as a result of studying a process based on physical or biological laws that are known to control the state.

Note that Nagylaki [7] prefers discrete-time models for biological applications. Despite this, continuous-time dynamical systems are an actively developing and fruitful branch of modern mathematics.

Dynamical systems with continuous time are studied quite deeply. Thus, in the paper [8], using a target model in terms of differential equations, the control of ovulation in mammals was studied, in particular, the regulation of a given number of mature eggs.

In [9], using graphics processing units (GPUs) and the Computing Unified Device Architecture (CUDA), the characteristics of nonlinear dynamical systems are studied and discussed.

In [10], the dynamics of the Holling–Tanner predator–prey model, depending on Smith's diffusion coefficient with growth under the condition of zero flux, is studied. Also, some qualitative properties are discussed, including dissipation, inertia, and local and global stability of the solution with a positive constant. The bifurcation properties of the predator–prey system and the dynamics of the system are studied in the paper [11].

The paper [12] considers a Leslie-type "predator–prey" dynamical system with a generalized function (distribution)al response of the Holling III type. It is shown that the model exhibits a subcritical Hopf bifurcation and a Bogdanov–Takens bifurcation simultaneously in the corresponding small neighborhoods of two degenerate equilibria, respectively. Also, using computer numerical modeling, various phase portraits of the system are obtained.

In the paper [13], a continuous analogue of strictly non-Voltaire quadratic dynamical systems with continuous time and equilibrium points is investigated, a phase portrait of the system is constructed, numerical solutions are found, and a comparative analysis with a particular solution to the system is carried out. In [14], fixed points and general solutions to the continuous-time dynamical system are found and the asymptotic behavior of trajectories is studied.

To show the difference between the problem we are studying in this paper and other scientific works, we will first describe the terms and published articles used in this area.

2. STATEMENT OF THE PROBLEM

The general evolutionary equations of a bisexual population are introduced and studied in [5]. Let

$$S^{n-1} = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x_i \geq 0,\ i = 1, \dots, n,\ \sum_{i=1}^{n} x_i = 1 \right\}$$

be called an $(n-1)$-*symplex*.

The *state of a population* is a pair of probability distributions $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_\nu)$ on $\mathcal{F}$—the set of female types—and $\mathcal{M}$—the set of male types,

$$x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1, \qquad y_k \geq 0, \quad \sum_{k=1}^{\nu} y_k = 1, \qquad i = 1, \dots, n, \quad k = 1, \dots, \nu.$$

Let (x, y) be the state in the generation G and in the next generation G' at the moment of its origin; the probabilities of the types are found based on the total probability (the equations of evolution of a bisexual population),

$$\begin{cases} x'_j = \sum\limits_{i,k=1}^{n,\nu} p^{(f)}_{ik,j} x_i y_k, & 1 \leq j \leq n \\ y'_l = \sum\limits_{i,k=1}^{n,\nu} p^{(m)}_{ik,l} x_i y_k, & 1 \leq l \leq \nu, \end{cases} \tag{2.1}$$

where $p^{(f)}_{ik,j}$ and $p^{(m)}_{ik,l}$ are the inheritance coefficients. The value $p^{(f)}_{ik,j}$ is defined as the probability of having an offspring of the female type F_j, $1 \leq j \leq n$, from a mother of the type F_i, $1 \leq i \leq n$, and a father of the type M_k, $1 \leq k \leq \nu$. Similarly, we define $p^{(m)}_{ik,l}$, $1 \leq i \leq n$, $1 \leq k, l \leq \nu$. It is clear from the definition of the coefficients that

$$\begin{cases} p^{(f)}_{ik,j} \geq 0, & \sum\limits_{j=1}^{n} p^{(f)}_{ik,j} = 1 \\ p^{(m)}_{ik,l} \geq 0, & \sum\limits_{i=1}^{\nu} p^{(m)}_{ik,l} = 1. \end{cases} \tag{2.2}$$

The definition of the Volterra quadratic stochastic operator of a bisexual operation is introduced in the paper [6] (further in the presentation of the article we rely on this definition).

Definition 1. The evolutionary operator (2.1) is called the *Volterra quadratic stochastic operator of a bisexual population* (VQSOBP) if the inheritance coefficients (2.2) satisfy the condition

$$\begin{cases} p^{(f)}_{ik,j} = 0, & j \overline{\in} \{i, k\},\ 1 \leq i,\ j \leq n,\ 1 \leq k \leq \nu \\ p^{(m)}_{ik,l} = 0, & l \overline{\in} \{i, k\},\ 1 \leq i \leq n,\ 1 \leq k, l \leq \nu. \end{cases} \tag{2.3}$$

Here the biological meaning of (2.3) is that any individual repeats the genotype of the father or mother. In the paper, fixed points are investigated and it is proved that the set of fixed points coincides with the set of fixed points of the Voltaire-type QSO [15]. A complete list of extreme operators is given (in the case under consideration, there are sixteen extreme operators), which are discrete-time QSOs.

In the present paper, we analyze a continuous analogue of these extreme operators, which are systems of nonlinear ordinary differential equations (nonlinear continuous-time dynamical systems). Thus, the first continuous analog of the VQSOBP ($\widetilde{W}_1$) has the form

$$\begin{cases} \dot{x}_1(t) = x_2(t)y_1(t) \\ \dot{x}_2(t) = x_2(t)y_2(t) - x_2(t) \\ \dot{y}_1(t) = x_1(t) + x_2(t)y_1(t) - y_1(t) \\ \dot{y}_2(t) = x_2(t)y_2(t) - y_2(t) \end{cases} \tag{2.4}$$

or in vector form

$$\dot{X}(t) = F(X(t)),$$

where $X(t) = (x(t), y(t)) = \big(x_1(t), x_2(t), y_1(t), y_2(t)\big)$ is the state of some system at a continuous-time moment for $t \geq 0$,

$$F(X(t)) = F\big(x_1(t), x_2(t), y_1(t), y_2(t)\big) = f_i\big(x_1(t), x_2(t), y_1(t), y_2(t)\big),$$

$i = 1, \dots, 4$. Note that in the case under consideration the population is autosomal, i.e., the male and female types are equal ($n = \nu$).

It follows from the statement of the problem that $x_1(t) \geq 0$, $x_2(t) \geq 0$, $y_1(t) \geq 0$, $y_2(t) \geq 0$, and

$$x_1(t) + x_2(t) = 1, \qquad y_1(t) + y_2(t) = 1. \tag{2.5}$$

Remark 1. The motivation for studying quadratic differential equations of a free population with continuous time was given by Jenks (see [2], [3]) . For a bisexual population, quadratic equations (in particular, (2.4) with continuous time) were introduced by Kesten (see formula (5.37) in [4]). Note that regardless of the type of time, these equations are obtained from the total probability formula (see [4]).

Remark 2. The systems of differential equations (2.4) are similar to the Lotka–Volterra equations. There are quadratic stochastic processes associated with Eqs. (2.4) (see [16, Chaps. 3, 4], [17, Chap. 6]).

First, we study the qualitative properties of system (2.4) in the domains

$$\overline{\Omega}_1 = \big\{(x_1(t), x_2(t))\colon x_1(t) \geq 0,\ x_2(t) \geq 0,\ x_1(t) + x_2(t) \leq 1\big\},$$
$$\overline{\Omega}_2 = \big\{(y_1(t), y_2(t))\colon y_1(t) \geq 0,\ y_2(t) \geq 0,\ y_1(t) + y_2(t) \leq 1\big\}.$$

Next, we will study the qualitative properties of system (2.4) in the simplex S^1 as a special case.

Subtracting the third equation from the first one in system (2.4) and the fourth equation from the second one, we find

$$\begin{cases} \dot{x}_1(t) - \dot{y}_1(t) = -(x_1(t) - y_1(t)) \\ \dot{x}_2(t) - \dot{y}_2(t) = -(x_2(t) - y_2(t)). \end{cases} \tag{2.6}$$

Consider the following cases separately:

(a) $x_1(t) - y_1(t) = 0$, $x_2(t) - y_2(t) \neq 0$,

(b) $x_1(t) - y_1(t) \neq 0$, $x_2(t) - y_2(t) = 0$,

(c) $x_1(t) - y_1(t) = 0$, $x_2(t) - y_2(t) = 0$,

(d) $x_1(t) - y_1(t) \neq 0$, $x_2(t) - y_2(t) \neq 0$.

In what follows, when integrating the first equation, we will denote the integration constant by C_1, and when integrating the second equation, by C_2, and so on.

Consider case (a). Then from (2.4) and (2.6) we obtain

$$\begin{cases} \dot{x}_1(t) = x_2(t)y_1(t) \\ \dot{x}_2(t) = x_2(t)y_2(t) - x_2(t) \\ \dot{y}_1(t) = x_2(t)y_1(t) \\ \dot{y}_2(t) = x_2(t)y_2(t) - y_2(t), \end{cases} \tag{2.7}$$

$$x_2(t) = y_2(t) + C_2e^{-t}. \tag{2.8}$$

Substituting (2.8) into the fourth equation in system (2.7), we find

$$\dot{y}_2(t) = (C_2e^{-t} - 1)y_2(t) + y_2^2(t). \tag{2.9}$$

If $y_2(t) = 0$, then the system admits the two solutions

$$\begin{cases} x_1(t) = 0 \\ x_2(t) = C_2e^{-t} \\ y_1(t) = 0 \\ y_2(t) = 0, \end{cases} \qquad \begin{cases} x_1(t) = C_3\exp(-C_2e^{-t}) \\ x_2(t) = C_2e^{-t} \\ y_1(t) = C_3\exp(-C_2e^{-t}) \\ y_2(t) = 0. \end{cases}$$

Let $y_2(t) \neq 0$. Using the method presented in [18, Sec. 1.1.5, p. 50], we find a solution to Eq. (2.9) in quadratures,

$$y_2(t) = \frac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})}. \tag{2.10}$$

From (2.8) we find

$$x_2(t) = \frac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})} + C_2e^{-t}. \tag{2.11}$$

From the first and third equations in system (2.7) we find the following two solutions:

$$\begin{cases} x_1(t) = 0 \\ x_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})} + C_2e^{-t} \\ y_1(t) = 0 \\ y_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})}, \end{cases} \qquad \begin{cases} x_1(t) = \dfrac{C_4}{1 - C_2C_3\exp(C_2e^{-t})} \\ x_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})} + C_2e^{-t} \\ y_1(t) = \dfrac{C_4}{1 - C_2C_3\exp(C_2e^{-t})} \\ y_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})}. \end{cases}$$

Similarly, as above, we find solutions to system (2.4) for cases (b) and (c), which have the form (case (b)):

$$\begin{cases} x_1(t) = C_1 \\ x_2(t) = 0 \\ y_1(t) = -C_1e^{-t} \\ y_2(t) = 0, \end{cases} \qquad \begin{cases} x_1(t) = \dfrac{C_1}{2}e^{-t} + C_3e^t \\ x_2(t) = 1 \\ y_1(t) = -\dfrac{C_1}{2}e^{-t} + C_3e^t \\ y_2(t) = 1, \end{cases}$$

$$\begin{cases} x_1(t) = \dfrac{2C_2e^t + C_1e^{-t}}{2(1 - C_3e^t)} \\ x_2(t) = \dfrac{1}{1 - C_3e^t} \\ y_1(t) = \dfrac{2C_2e^t + C_1e^{-t}}{2(1 - C_3e^t)} - C_1e^{-t} \\ y_2(t) = \dfrac{1}{1 - C_3e^t}. \end{cases}$$

Case (c):

$$\begin{cases} x_1(t) = 0 \\ x_2(t) = 1 \\ y_1(t) = 0 \\ y_2(t) = 1, \end{cases} \qquad \begin{cases} x_1(t) = C_1e^t \\ x_2(t) = 1 \\ y_1(t) = C_1e^t \\ y_2(t) = 1, \end{cases}$$

$$\begin{cases} x_1(t) = 0 \\ x_2(t) = \dfrac{1}{1 - C_2e^t} \\ y_1(t) = 0 \\ y_2(t) = \dfrac{1}{1 - C_2e^t}, \end{cases} \qquad \begin{cases} x_1(t) = \dfrac{C_1e^t}{1 - C_2e^t} \\ x_2(t) = \dfrac{1}{1 - C_2e^t} \\ y_1(t) = \dfrac{C_1e^t}{1 - C_2e^t} \\ y_2(t) = \dfrac{1}{1 - C_2e^t}. \end{cases}$$

Note that the solutions in case (c) are solutions in an unstable equilibrium of system (2.4) (further proved).

Let us now consider a more general case (d). If the right-hand sides in Eq. (2.6) are nonzero, then we find

$$x_1(t) = y_1(t) + C_1e^{-t}, \tag{2.12}$$

$$x_2(t) = y_2(t) + C_2e^{-t}, \tag{2.13}$$

where $C_1, C_2 = \text{const}$.

Using the method [18, Sec. 1.1.5, p. 50], we find two solutions for $y_2 = 0$,

$$\begin{cases} x_1(t) = -\dfrac{C_1}{C_2} + C_1e^{-t} \\ x_2(t) = C_2e^{-t} \\ y_1(t) = -\dfrac{C_1}{C_2} \\ y_2(t) = 0, \end{cases} \qquad \begin{cases} x_1(t) = -\dfrac{C_1}{C_2} + C_3\exp(C_2e^{-t}) + C_1e^{-t} \\ x_2(t) = C_2e^{-t} \\ y_1(t) = -\dfrac{C_1}{C_2} + C_3\exp(C_2e^{-t}) \\ y_2(t) = 0. \end{cases}$$

For $y_2 \neq 0$ we find the general solution to system (2.4),

$$\begin{cases} x_1(t) = \dfrac{C_4 - C_1e^{-t} + C_1C_3\exp(C_2e^{-t})}{1 - C_2C_3\exp(C_2e^{-t})} + C_1e^{-t} \\ x_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})} + C_2e^{-t} \\ y_1(t) = \dfrac{C_4 - C_1e^{-t} + C_1C_3\exp(C_2e^{-t})}{1 - C_2C_3\exp(C_2e^{-t})} \\ y_2(t) = \dfrac{-C_2e^{-t}}{1 - C_2C_3\exp(C_2e^{-t})}, \end{cases} \tag{2.14}$$

where $C_3, C_4 = \mathrm{const}$.

Let us study the functions (2.14). Similar reasoning can also be carried out with respect to the functions in cases (a)–(c).

Let the following initial conditions be given at $t = t_0$: $x_1(t_0) = x_1^0$, $x_2(t_0) = x_2^0$, $y_1(t_0) = y_1^0$, $y_2(t_0) = y_2^0$. Then the unknown constants C_1, C_2, C_3, C_4 are determined from (2.14) as follows:

$$\begin{cases} C_1 = x_1^0 - y_1^0 \\ C_2 = x_2^0 - y_2^0 \\ C_3 = \dfrac{x_2^0 \exp(y_2^0 - x_2^0)}{(y_2^0 - x_2^0) y_2^0} \\ C_4 = \dfrac{x_2^0 - y_2^0}{.} \end{cases} \tag{2.15}$$

Let us give two definitions.

Definition 2 [19, p. 318]**.** A solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ to system (2.4) is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that as soon as $|x_i^0 - \overline{x}_i^0| < \delta$ and $|y_i^0 - \overline{y}_i^0| < \delta$, $i = 1, 2$, we have

$$\left|x_i(t, x_1^0, x_2^0, y_1^0, y_2^0) - x_i(t, \overline{x}_1^0, \overline{x}_2^0, \overline{y}_1^0, \overline{y}_2^0)\right| < \varepsilon,$$
$$\left|y_i(t, x_1^0, x_2^0, y_1^0, y_2^0) - y_i(t, \overline{x}_1^0, \overline{x}_2^0, \overline{y}_1^0, \overline{y}_2^0)\right| < \varepsilon, \qquad i = 1, 2,$$

for all values $0 \le t < +\infty$. Here $\overline{x}_i^0$ and $\overline{y}_i^0$, $i = 1, 2$, are modified initial conditions.

Definition 3 [20, p. 20]**.** The *equilibria* of system (2.4) are points $X^*(t)$ of the phase space such that $F(X^*(t)) = 0$.

It is obvious that $X^*(t)$ is a solution to system (2.4), since $\dot{X}^* = 0$.

It can readily be established that system (2.4) has infinitely many equilibria of the form $M_1^1(C, 0, C, 0)$, where $C = \mathrm{const}$ and $C \in [0, 1]$ and $M_1^2(0, 1, 0, 1)$.

The following theorem holds true.

Theorem 1.

1. *The general solution to system* (2.4) *(case (d)) has the form* (2.14) *and it is uniformly continuous on* $t \in [0, +\infty)$ *as well as Lyapunov stable at the equilibrium* $M_1^1(C, 0, C, 0)$*, where* $C \in [0, 1]$.

2. *The equilibrium* $M_2^1(0, 1, 0, 1)$ *is unstable.*

3. *If conditions* (2.5) *are satisfied, then* $C = 1$ *and one has*

$$\lim_{t\to+\infty} X(t) = \lim_{t\to+\infty} \big(x_1(t), x_2(t), y_1(t), y_2(t)\big) = (1, 0, 1, 0)$$

and the equilibrium $M_1^1(1, 0, 1, 0)$ *is asymptotically stable.*

Proof.

1. It follows from the above reasoning that the functions $x_1(t), x_2(t), y_1(t), y_2(t)$ in (2.14) satisfy system (2.4).

We will prove their uniform continuity on $t \in [0, +\infty)$, although the continuity of the functions will be sufficient when studying the system. It is easy to show that the function $(1 - C_2 C_3 e^{C_2 e^{-t}})$, which is in the denominator in (2.14), does not vanish on $t \in [0, +\infty)$.

We calculate the limit of the function (2.14) taking into account (2.15), for example, the limit of the function $x_1(t)$ as $t \to +\infty$,

$$\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} \left(\frac{C_4 + C_1 C_3 \exp(C_2 e^{-t}) - C_1 e^{-t}}{1 - C_2 C_3 \exp(C_2 e^{-t})} + C_1 e^{-t} \right) = \frac{C_4 + C_1 C_3}{1 - C_2 C_3}$$

$$= \frac{y_2^0 y_1^0}{y_2^0 - x_2^0 \exp(y_2^0 - x_2^0)} - \frac{x_1^0 - y_1^0}{x_2^0 - y_2^0} \frac{(y_2^0)^2 - x_2^0 \exp(y_2^0 - x_2^0)}{y_2^0 - x_2^0 \exp(y_2^0 - x_2^0)} = C.$$

The function $x_1(t)$ is continuous on $t \in [0, +\infty)$ and

$$\lim_{t \to +\infty} x_1(t) = C.$$

This implies the uniform continuity of $x_1(t)$ [21, p. 6, Assertion 4]. The uniform continuity of the remaining functions in (2.14) is proved similarly.

Now we will show that the solutions to system (2.7) are stable at the equilibrium $M_1^1(C, 0, C, 0)$.

Assume that we have two solutions $X(x_1(t), x_2(t), y_1(t), y_2(t))$ and $\Phi(\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t))$ of the system with the following initial data at $t = t_0$:

$$x_1(t_0) = x_1^0, \qquad x_2(t_0) = x_2^0, \qquad y_1(t_0) = y_1^0, \qquad y_2(t_0) = y_2^0$$

and

$$\varphi_1(t_0) = \varphi_1^0, \qquad \varphi_2(t_0) = \varphi_2^0, \qquad \psi_1(t_0) = \psi_1^0, \qquad \psi_2(t_0) = \psi_2^0,$$

respectively.

By Definition 2,

$$|x_1^0 - \varphi_1^0| < \delta, \qquad |x_2^0 - \varphi_2^0| < \delta, \qquad |y_1^0 - \psi_1^0| < \delta, \qquad |y_2^0 - \psi_2^0| < \delta.$$

Then from (2.15) we find

$$|C_1 - \overline{C}_1| = |x_1^0 - \varphi_1^0 - (y_1^0 - \psi_1^0))| < 2\delta. \tag{2.16}$$

By similar reasoning we establish that

$$|C_2 - \overline{C}_2| < 2\delta, \qquad |C_3 - \overline{C}_3| < \alpha\delta, \qquad |C_4 - \overline{C}_4| < \alpha\delta, \tag{2.17}$$

where α is a known finite constant, $\overline{C}_1$, $\overline{C}_2$, $\overline{C}_3$, and $\overline{C}_4$ are constants depending on φ_1^0, φ_2^0, ψ_1^0, and ψ_2^0 according to relations (2.15), respectively.

Let us show that $\forall \varepsilon > 0$ as soon as (2.16) and (2.17) are satisfied,

$$|x_i(t) - \varphi_i(t)| < \varepsilon, \quad |y_i(t) - \psi_i(t)| < \varepsilon, \qquad i = 1, 2. \tag{2.18}$$

For brevity, we will demonstrate the proof of inequalities (2.18) using the example

$$|y_2(t) - \psi_2(t)| < \varepsilon.$$

So, from (2.14) we find

$$|y_2(t) - \psi_2(t)| = \left| \frac{-C_2 e^{-t}}{1 - C_2 C_3 \exp(C_2 e^{-t})} - \frac{-\overline{C}_2 e^{-t}}{1 - \overline{C}_2 \overline{C}_3 \exp(\overline{C}_2 e^{-t})} \right|$$
$$= \left| \frac{(C_2 - \overline{C}_2) + C_2 \overline{C}_2 \big((C_3 - \overline{C}_3) \exp(\overline{C}_2 e^{-t}) + C_3(\exp(C_2 e^{-t}) - \exp(\overline{C}_2 e^{-t}))\big)}{(1 - C_2 C_3 \exp(C_2 e^{-t}))(1 - \overline{C}_2 \overline{C}_3 \exp(\overline{C}_2 e^{-t}))} \right| e^{-t}.$$

Due to the continuity and boundedness of the function $\exp(C_2 e^{-t})$, taking into account (2.16) and (2.17), we obtain

$$|y_2(t) - \psi_2(t)| < \beta\delta,$$

where β is a known finite constant. If $\delta = \varepsilon/\gamma$, $\gamma = \max(2, \alpha, \beta)$, then

$$|y_2(t) - \psi_2(t)| < \varepsilon.$$

The remaining inequalities (2.18), which are true for $t \in [0, +\infty)$, are proved by similar reasoning. It follows from the above that the solutions to the system are Lyapunov stable [19, p. 318].

2. Let us study the equilibria $M_2^1(0,1,0,1)$. The Jacobian has the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where $det|A| \neq 0$. The matrix A has the eigenvalues $\lambda_{1,2} = 1$ and $\lambda_{3,4} = -1$; this implies that system (2.4) is unstable at the equilibrium M_2^1 [22, p. 72].

3. Consider system (2.4) on S^1. Based on this, considering (2.12) and (2.13), we find that $C_1 = -C_2$. Let us modify C_4 in (2.15). So,

$$C_4 = \frac{x_2^0 y_1^0 - y_1^0 y_2^0 + y_1^0 y_2^0 - x_1^0 y_2^0}{x_2^0 - y_2^0} = \frac{(x_2^0 - y_2^0) y_1^0 + (y_1^0 - x_1^0) y_2^0}{x_2^0 - y_2^0}.$$

Taking into account the first and second relations in (2.15), as well as $C_1 = -C_2$, we find that $C_4 = 1$ and $C = 1$. This implies that

$$\lim_{t \to +\infty} |x_i(t) - \varphi_i(t)| = 0, \qquad \lim_{t \to +\infty} |y_i(t) - \psi_i(t)| = 0, \ i = 1, 2. \tag{2.19}$$

From (2.19) we obtain the asymptotic stability o9f the solution at the equilibrium $M_1^1(1,0,1,0)$. Moreover,

$$\lim_{t \to +\infty} X(t) = (1,0,1,0),$$

as was to be proved. □

Now consider systems (2.4) on S^1. After some transformations we find (due to $\dot{x}_1(t) = -\dot{x}_2(t)$ and $\dot{y}_1(t) = -\dot{y}_2(t)$, the first and second, as well as the third and fourth equations, respectively, in system (2.4) are equivalent),

$$\begin{cases} \dot{x}_1(t) = (1 - x_1(t)) y_1(t) \\ \dot{y}_1(t) = x_1(t)(1 - y_1(t)). \end{cases} \tag{2.20}$$

We readily find the equilibria $M_1^1(1,1)$ and $M_2^1(0,0)$. The roots of the characteristic equation (2.20) of the first approximation at the equilibrium $M_1^1(1,1)$ are negative ($\lambda_{1,2} = -1$). By Lyapunov's theorem 1 [22, p. 72], the equilibria $M_1^1(1,1)$ are asymptotically stable, and the roots of the characteristic equation (2.20) at $M_2^1(0,0)$ are equal to 1 (multiple). According to the same theorem [22, p. 72], the equilibria are unstable.

Note that the system has the following first two solutions: $x_1(t) = 1$, $y_1(t) = 1$ and $x_1(t) = 0$, $y_1(t) = 0$. Moreover, setting $C_1 = -C_2$ and $C_4 = 1$ in (2.14), we formally obtain its solution from (2.14),

$$\begin{cases} x_1(t) = \dfrac{1 + C_2 e^{-t} - C_2 C_3 \exp(C_2 e^{-t})}{1 - C_2 C_3 \exp(C_2 e^{-t})} - C_2 e^{-t} \\ x_2(t) = \dfrac{-C_2 e^{-t}}{1 - C_2 C_3 \exp(C_2 e^{-t})} + C_2 e^{-t} \\ y_1(t) = \dfrac{1 + C_2 e^{-t} - C_2 C_3 \exp(C_2 e^{-t})}{1 - C_2 C_3 \exp(C_2 e^{-t})} \\ y_2(t) = \dfrac{-C_2 e^{-t}}{1 - C_2 C_3 \exp(C_2 e^{-t})}. \end{cases} \tag{2.21}$$

We compare analytical and numerical solutions to system (2.4) using graphs. The numerical solutions (2.4) were found in the mathematical editor MathCAD. With the following initial values (taking into account (2.5)) we construct graphs of the numerical solutions to (2.4) (see Fig. 1), where $C_1 = (C_1, C_2, C_3, C_4)^\top$,

$$C^0 = \begin{pmatrix} 0 & 0.11 & 0.22 & 0.32 & 0.44 & 0.38 & 0.67 & 0.71 & 0.25 & 0.77 \\ 1 & 0.89 & 0.78 & 0.68 & 0.56 & 0.62 & 0.33 & 0.29 & 0.75 & 0.23 \\ 0.45 & 0.14 & 0.33 & 0.24 & 0.15 & 0.5 & 0.59 & 0.28 & 0.9 & 0.46 \\ 0.55 & 0.86 & 0.67 & 0.76 & 0.85 & 0.5 & 0.41 & 0.78 & 0.1 & 0.54 \end{pmatrix} \tag{2.22}$$

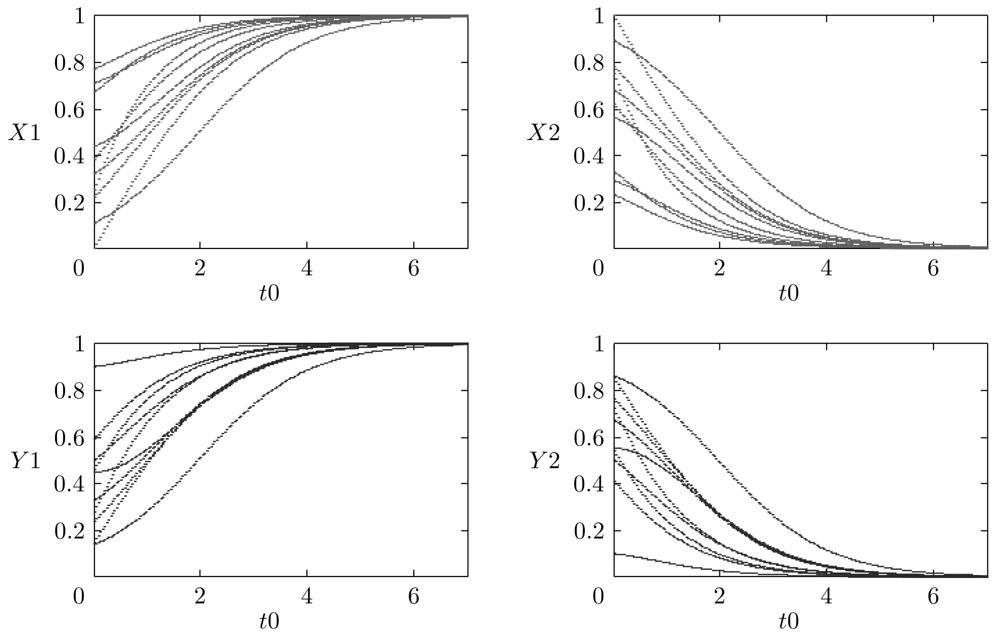

Fig. 1.

Here $t0$ stands for time t, $X1$, for the variable $x_1(t)$, $X2$, for the variable $x_2(t)$, $Y1$, for the variable $y_1(t)$, and $Y2$, for the variable $y_2(t)$.

The graphs of the variables $(x_1(t), x_2(t), t)$ and $(y_1(t), y_2(t), t)$ on the space are given in Fig. 2, where $Z1 = (x_1(t), x_2(t), t)^\top$ and $Z2 = (y_1(t), y_2(t), t)^\top$.

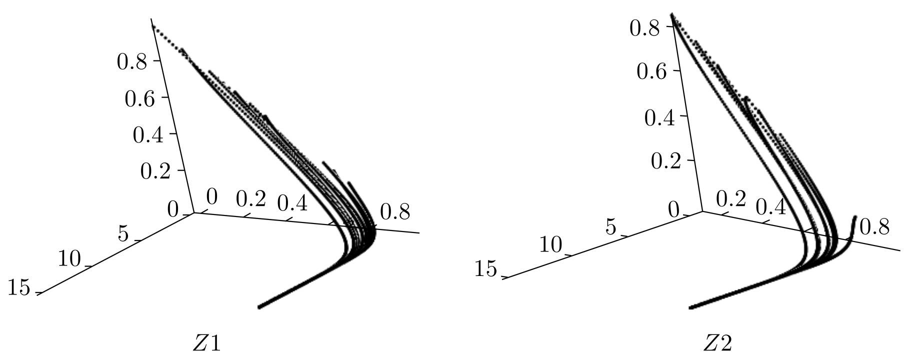

Fig. 2.

In MathCAD, we compared analytical and numerical solutions to system (2.14). In the course of the research, it was found that the difference between analytical and numerical solutions to (2.4) does not exceed 0.001, and for $t \geq 7$ they coincide and tend to the equilibrium $M_1^1(1, 0, 1, 0)$ (see Fig. 3). Here $C_1 = 0.22$, $C_2 = 0.78$, $C_3 = 0.33$, and $C_4 = 0.67$.

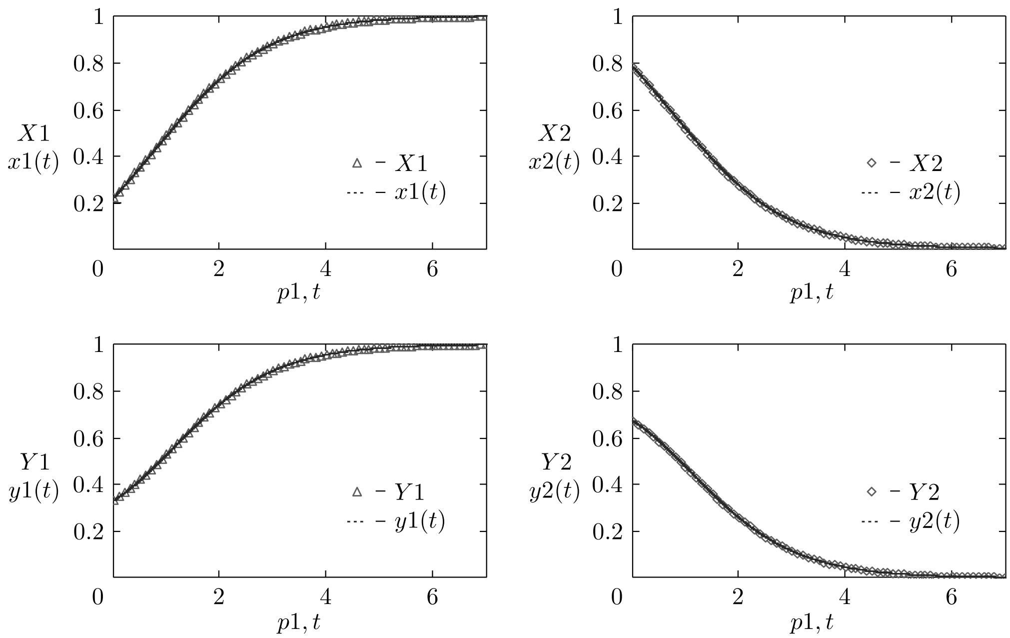

Fig. 3.

Here in the graphs, by $x1(t), x2(t)$ and $y1(t), y2(t)$ we have denoted the values of the analytical solutions to (2.14), $p1$ and t is the change in time, respectively.

It has been established that not all extreme operators of the continuous-time VQSOBP admit solutions in quadratures. For example, the continuous analog of the fourteenth extreme operator $\widetilde{W}_{14}$ of the continuous-time VQSOBP [6] has the form

$$\begin{cases} \dot{x}_1(t) = x_1(t)y_1(t) - x_1(t) \\ \dot{x}_2(t) = x_2(t)y_1(t) + y_2(t) - x_2(t) \\ \dot{y}_1(t) = x_1(t) - y_1(t) \\ \dot{y}_2(t) = x_2(t) - y_2(t). \end{cases} \tag{2.23}$$

After some transformations from the third equation in (2.23) we obtain $\dot{y}_1(t) + y_1(t) = C_3 e^{(y_1-1)^2/2}$. It was not possible to find analytical solutions to the differential equation using modern mathematical methods.

Let us analyze system (2.23). Let $(x_1(t), x_1(t)) \in \Omega_1$ and $(y_1(t)y_1(t)) \in \Omega_2$. Let us find the equilibria of the system. The system has infinitely many equilibria $M_1^{14}(1,0,1,0)$ and $M_2^{14}(0, C_5, 0, C_5)$, $C_5 = \text{const}$ and $C_5 \in [0,1]$.

We study the systems at the equilibria $M_1^{14}(1,0,1,0)$. The matrix of the linearized system A is nonsingular and has the eigenvalues

$$\lambda_{1,2} = \frac{-1-\sqrt{5}}{2} < 0, \qquad \lambda_{3,4} = \frac{-1+\sqrt{5}}{2} > 0;$$

this implies that system (2.23) is unstable at the equilibrium M_1^{14} [22, p. 72]. At the equilibrium $M_2^{14}(0, C_5, 0, C_5)$ the characteristic equation has a zero root. Therefore, it is impossible to investigate the stability of system (2.23) using the equations of the first approximation.

Let us consider the function $V(x_1(t), x_2(t), y_1(t), y_2(t)) = x_1(t) + x_2(t) + y_1(t) + y_2(t)$, which is a Lyapunov function [23, p. 44]. We calculate the derivative of the function $V(x_1(t), x_2(t), y_1(t), y_2(t))$ by virtue of system (2.23) and after transferring the origin of the system to the point $M_2^{14}(0, C_5; 0, C_5)$ (for convenience, we again denote the variables by $(x_1(t), x_2(t), y_1(t), y_2(t))$ we find

$$\dot{V}(t) = (x_1(t) + x_2(t) - C_5)y_2(t) < 0$$

if $x_1(t) + x_2(t) < C_5$. This means that the zero solution to the system is asymptotically Lyapunov stable as $t \to +\infty$ in the subset Ω_1.

If we consider the problem in S^1, then the equilibrium $M_2^{14}(0, C_5, 0, C_5)$ coincides with $M_2^{14}(0, 1, 0, 1)$ and is asymptotically stable. The second equilibrium of system (2.23) $M_1^{14}(1, 0, 1, 0)$ is unstable in S^1.

Note that numerical methods usually do not provide a good qualitative understanding of the behavior of the system. Therefore, at least as part of the analysis of dynamic models, we use graphical methods. Because of the simplicity of graphical methods, along with their geometric nature, they allow introducing the fundamental modeling concepts used to study dynamical systems. However, here, it is mainly a matter of writing computer programs correctly, without involving many time-consuming numerical calculations.

The numerical solutions to system (2.23) on S^1 considering the condition (2.5) wi the initial data (2.21) have the following form ($C = 1$) (see Fig. 4).

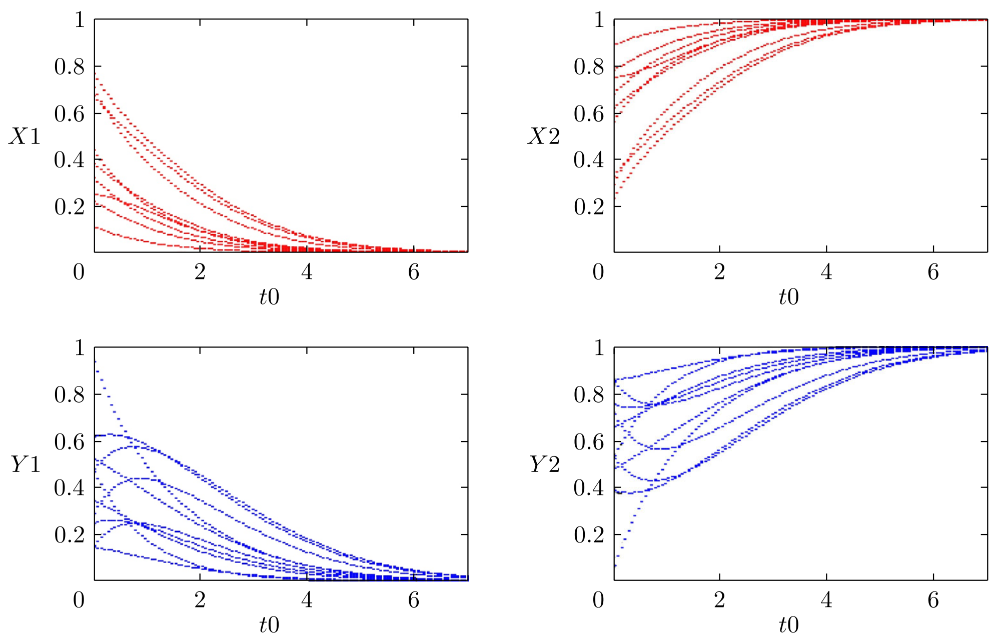

Fig. 4.

When studying the operator $\widetilde{W}_{14}$, it was found that, unlike the numerical and analytical solutions of the operator $\widetilde{W}_1$, the solutions of the operator $\widetilde{W}_{14}$ tend to the equilibrium $M_1^2(0, 1, 0, 1)$ starting only with $t = 14$ (graphs were constructed and solutions were compared more than 100 times with various initial conditions). This is explained by the design of the dynamical systems (2.4) and (2.23).

The remaining continuous analogs of extreme operators have been completely studied in S^1 and the results are presented in table form.

Analysis of the data presented in Table 1 shows that only six operators have asymptotically stable equilibria in S^1. The operators $\widetilde{W}_3$, $\widetilde{W}_5$, $\widetilde{W}_6$, $\widetilde{W}_7$, $\widetilde{W}_8$, $\widetilde{W}_{11}$, and $\widetilde{W}_{15}$ have constant individual components and have a stable equilibrium point. This may correspond to the fact that the individual is sick or old.

3. CONCLUSIONS

The study of the continuous-time analog of the VQSOBP [6] gives some advantage, since it was established on the basis of numerical solutions that, starting from $t = 7$, the numerical solutions to (2.4) coincide with the analytical solutions to (2.14). Starting from $t = 14$, the numerical solutions to (2.23) tend to the equilibrium.

Table 1

Operators	Equilibria and stability		Found solutions
$\widetilde{W}_2$	$M_1^2(1,0,1,0)$	$M_2^2(0,1,0,1)$	numerical
	stable	unstable	
$\widetilde{W}_3$	$M_1^3(1,0,C_3,1-C_3)$, $C_3 \in [0,1]$	$M_2^3(C_1,1-C_1,0,1)$, $C_1 \in [0,1]$	analytical and numerical
	stable	unstable	
$\widetilde{W}_4$	$M_1^4(1,0,C_3,1-C_3)$, $C_3 \in [0,1]$	$M_2^4(C_1,1-C_1,0,1)$, $C_1 \in [0,1]$	analytical and numerical
	stable	asymptotically stable	
$\widetilde{W}_5$	$M_1^5(1,0,C_3,1-C_3)$, $C_3 \in [0,1]$	$M_2^5(C_1,1-C_1,1,0)$, $C_1 \in [0,1]$	analytical and numerical
	stable	unstable	
$\widetilde{W}_6$	$M_1^6(C_1,1-C_1,C_1,1-C_1)$, $C_1 \in [0,1]$	–	analytical and numerical
	stable	–	
$\widetilde{W}_7$	$M_1^7(C_1,1-C_1,C_3,1-C_3)$, $C_1, C_3 \in [0,1]$	–	analytical and numerical
	stable	–	
$\widetilde{W}_8$	$M_1^8(C_1,1-C_1,0,1)$, $C_1 \in [0,1]$	$M_2^8(1,0,C_3,1-C_3)$, $C_3 \in [0,1]$	analytical and numerical
	stable	–	
$\widetilde{W}_9$	$M_1^9(0,1,0,1)$	$M_2^9(1,0,1,0)$	numerical
	stable	asymptotically stable	
$\widetilde{W}_{10}$	$M_1^{10}(C_1,1-C_1,C_1,1-C_1)$, $C_1 \in [0,1]$	–	analytical and numerical
	stable	–	
$\widetilde{W}_{11}$	$M_1^{11}(C_1,1-C_1,C_1,1-C_1)$, $C_1 \in [0,1]$	–	analytical and numerical
	stable	–	
$\widetilde{W}_{12}$	$M_1^{12}(1,0,1,0)$	$M_2^{12}(0,1,0,1)$	numerical
	unstable	asymptotically stable	
$\widetilde{W}_{13}$	$M_1^{13}(0,1,C_3,1-C_3)$, $C_3 \in [0,1]$	$M_2^{13}(C_1,1-C_1,1,0)$, $C_1 \in [0,1]$	analytical and numerical
	unstable	stable	
$\widetilde{W}_{15}$	$M_1^{15}(0,1,C_3,1-C_3)$, $C_3 \in [0,1]$	$M_2^{15}(C_1,1-C_1,1,0)$, $C_1 \in [0,1]$	analytical and numerical
	stable	stable	
$\widetilde{W}_{16}$	$M_1^{16}(1,0,1,0)$	$M_2^{16}(0,1,0,1)$	analytical and numerical
	asymptotically stable	unstable	

Scientists often assume that in the biological models under study, it is mainly the attractive equilibria that are important, since they are observed in the real system that is subject to perturbations.

The analysis of continuous analogs of the VQSOBP [6] shows that although the stability assessment directly by the roots of the characteristic equation is possible, it is of little use in engineering and scientific practice. In most cases, it is impossible to establish the stability of the VQSOBP due to the degeneracy of the matrix of the linearized first-approximation system. In addition, knowledge of the numerical values of the roots does not provide information on the ways of stabilizing the system if it is unstable or has small stability reserves.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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