Superposition of linear and quadratic operators arising in statistical mechanics

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ABSTRACT

A significant portion of scientific and applied research worldwide focuses on the study of nonlinear dynamic systems. Understanding the behavior of such systems based on their initial conditions is crucial for predicting future developments. The theory of dynamical systems plays a vital role in deepening our comprehension of intricate phenomena across various domains, including biology, physics, economics, and healthcare. The current research addresses the superposition of a linear operator and a Volterra quadratic stochastic operator arising in statistical mechanics, which is established on a two-dimensional simplex space. The paper determines the stationary states of the combined transformation and classified their nature. Furthermore, it is demonstrated that nearly every trajectory under this transformation tends to approach the third corner of the two-dimensional simplex.

Keywords: linear operator, quadratic stochastic operator, Volterra operator, trajectory, simplex, statistical mechanics.

1. INTRODUCTION

Quadratic stochastic operators are widely utilized in various mathematical genetics models (see, for example, [1,2]), particularly within the scope of heredity studies (see also [3,4]). The foundational concept of a quadratic stochastic operator was introduced by S. Bernstein in 1924. These operators naturally emerge in genetic models in the following manner: Let us consider a population of organisms that forms a reproductively isolated group. Each member of this population is assumed to belong to a specific genetic type (or species). The classification of species is organized in such a way that the genetic types of the two parents uniquely determine the probability distribution over the possible types of their offspring in the first generation. This probability, referred to as the inheritance coefficient, specifies the likelihood of each genetic type appearing among the descendants. Evidently, that for all and that

$$\sum_{k=1}^{m} p_{ij,k} = 1, \quad i,j,k = 1,\ldots,m.$$

Let us denote the relative proportions of each genetic type in the current generation by a probability distribution x. Under the assumption of panmixia—that is, random mating within the population—the pairing of individuals of types i and $\boldsymbol{j}$ occurs with a probability determined by the state x. Specifically, for a given distribution xxx, the likelihood of selecting a parent pair of types i and j is given by $x_i\, x_j$. Therefore, the overall likelihood of obtaining a particular genetic type in the first-generation offspring is determined by the combined probabilities of all possible parent pairings that can produce that type

$$x_k' = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad k = 1,\ldots,m.$$

This relationship gives rise to an evolutionary quadratic operator. Consequently, the process of population evolution can be analyzed through the framework of a dynamical system governed by a quadratic stochastic operator (see [5,6]).

In the study conducted by D.B. Eshmamatova, Sh.J. Seytov, and N.B. Narziyev (see [7]), the authors examined dynamical systems arising from the superposition of Volterra operators with distinct characteristics, defined over a two-dimensional

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simplex. Their findings reveal that the long-term behavior of orbits generated by the superposition operator significantly differs from those produced by the original individual operators.

Notably, they demonstrated that even when the initial operators are non-ergodic and possess different directions of evolution, their superposition can result in a system exhibiting regular (predictable) dynamics.

The structure of the paper is as follows. Section 2 provides a review of fundamental definitions and established results related to Volterra and non-Volterra quadratic stochastic operators (QSOs). In Section 3, we examine the superposition of a linear operator and a Volterra quadratic stochastic operators which defined on the two-dimensional simplex. Fixed points are found and their types are studied, as well as the limit behavior of the trajectory.

2. MATERIALS AND METHODS

Quadratic operators serve as a key factor in the study of various models in biology, physics, chemistry, statistical mechanics.

Let

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_m) \in R^m : x_i \geq 0,\ \sum_{i=1}^{m} x_i = 1 \right\}$$

be the $(m-1)$-dimensional simplex. A map V of S^{m-1} into itself is called a quadratic stochastic operator (QSO) if

$$(V\mathbf{x})_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j \tag{1}$$

for any $\mathbf{x} \in S^{m-1}$ and for all $k = 1, \ldots, m$, where

$$p_{ij,k} \geq 0,\ p_{ij,k} = p_{ji,k},\ \sum_{k=1}^{m} p_{ij,k} = 1,\ \forall i, j, k = 1, \ldots, m. \tag{2}$$

Let $\{\mathbf{x}^{(n)} \in S^{m-1} : n = 0, 1, 2, \ldots\}$ represent the trajectory (orbit) originating from the initial state $\mathbf{x} \in S^{m-1}$, where each subsequent point is generated by the recurrence relation $\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right)$ for all $n = 0, 1, 2, \ldots$ and $\mathbf{x}^{(0)} = \mathbf{x}$.

Definition 1. A vector $\mathbf{x} \in S^{m-1}$ is referred to as a fixed point of a quadratic stochastic operator V if it satisfies the condition $V(\mathbf{x}) = \mathbf{x}$.

Definition 2. A quadratic stochastic operator V is said to exhibit *regularity* if, for every starting point $\mathbf{x} \in S^{m-1}$, the sequence $V^n(x)$ converges as $n \to \infty$; that is, the limit

$$\lim_{n\to\infty} V\left(\mathbf{x}^{(n)}\right)$$

exists.

It is important to observe that any limit point of an orbit corresponds to a stationary point of the quadratic stochastic operator. Consequently, these fixed points characterize the asymptotic, or long-term, behavior of trajectories originating from any initial state. Understanding the convergence of orbits and the nature of fixed points is crucial in numerous practical applications (see, for example, [8,9]). From a biological perspective, the regularity of a QSO implies that the population dynamics stabilize over time, indicating a steady future for the system.

Definition 3. A continuous function $\varphi :\ int\, S^{m-1} \to \mathbb{R}$ is said to be a Lyapunov function for the operator V if the limit

$$\lim_{n\to\infty} \varphi\left(V^n(\mathbf{x})\right)$$

exists for every $\mathbf{x} \in S^{m-1}$.

Let $D_{\mathbf{x}}V\left(\mathbf{x}^*\right)$ represent the Jacobian matrix of the operator V evaluated at the fixed point $\mathbf{x}^*$.

Definition 4 (see [10]). A fixed point $\mathbf{x}^*$ is said to be hyperbolic if the Jacobian matrix $D_{\mathbf{x}}V\left(\mathbf{x}^*\right)$ possesses no eigenvalues lying on the complex unit circle.

Definition 5 (see [10]). A hyperbolic fixed point $\mathbf{x}^*$ is classified as follows:

- Attracting if every eigenvalue of the matrix $D_{\mathbf{x}}V\left(\mathbf{x}^*\right)$ has modulus strictly less than one;
- Repelling if all eigenvalues of $D_{\mathbf{x}}V\left(\mathbf{x}^*\right)$ have modulus strictly greater than one;
- Saddle if the eigenvalues include both moduli less than and greater than one.

3. RESULTS AND DISCUSSION

Within this analysis we will consider the superposition operator $B = A \circ V$, which is a superposition of a linear operator $A: S^2 \to S^2$ and a quadratic stochastic operator $V: S^2 \to S^2$, where $V: x_1 = x_1^2,\ x_2 = x_2^2 + 2x_1x_2,\ x_3 = x_3^2 + 2x_1x_2 + 2x_2x_3$ and $A = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & & \\ 1-\alpha-\beta & 1-\gamma & 1 \end{pmatrix}$ with $0 \le \alpha,\ \beta,\ \alpha+\beta,\ \gamma \le 1$. Under these conditions, the operator B takes the form:

$$B = A\left(V(x)\right) = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 1-\alpha-\beta & 1-\gamma & 1 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 + 2x_1x_2 \\ x_3^2 + 2x_1x_2 + 2x_2x_3 \end{pmatrix} \Rightarrow$$

$$B: \begin{cases} x_1' = \alpha x_1^2, \\ x_2' = \beta x_1^2 + \gamma x_2^2 + 2\gamma x_1 x_2 \\ x_3' = \left(1-\alpha-\beta\right)x_1^2 + \left(1-\gamma\right)x_2^2 + 2\left(1-\gamma\right)x_1x_2 + x_3^2 + 2\left(x_1 + x_2\right)x_3. \end{cases} \tag{3}$$

It is evident to verify that if $\alpha = 1,\ \gamma = 1$ then the matrix A is the identity matrix and, in this case, we have $B = V$. This case is not interesting.

Theorem 1. For the given operator B, the following assertions are valid:

a) if $\alpha = 1,\ 0 \le \gamma < 1$ then the vertices $e_1 = \left(1,0,0\right)$ and $e_3 = \left(0,0,1\right)$ are fixed points and they are attracting and saddle points, respectively;

b) if $\gamma = 1,\ 0 \le \alpha < 1$ then the vertices $e_2 = \left(0,1,0\right)$ and $e_3 = \left(0,0,1\right)$ are fixed points and they are attracting and saddle points, respectively;

c) if $0 \le \alpha < 1,\ 0 \le \gamma < 1$ then the vertex $e_3 = \left(0,0,1\right)$ is a unique fixed point and it has the attracting type;

d) in all cases there are no periodic points.

Proof. Let us determine the stationary states of the given operator B. It is established that such a point satisfies the equation $B(x) = x$. characterizing fixed points of the operator. This equation can be rewritten in the following form

$$\begin{cases} x_1 = \alpha x_1^2, \\ x_2 = \beta x_1^2 + \gamma x_2^2 + 2\gamma x_1 x_2 \\ x_3 = (1-\alpha-\beta) x_1^2 + (1-\gamma) x_2^2 + 2(1-\gamma) x_1 x_2 + x_3^2 + 2(x_1 + x_2) x_3. \end{cases} \quad (4)$$

a) Let $\alpha = 1$ and $0 \le \gamma < 1$. Then it holds $\beta = 0$ and (4) has the form:

$$\begin{cases} x_1 = x_1^2, \\ x_2 = \gamma x_2^2 + 2\gamma x_1 x_2 \\ x_3 = (1-\gamma) x_2^2 + 2(1-\gamma) x_1 x_2 + x_3^2 + 2(x_1 + x_2) x_3. \end{cases} \quad (5)$$

From the first relation in system (5), we obtain

$$x_1 = x_1^2 \;\Rightarrow\; x_1(1-x_1) = 0 \;\Rightarrow\; x_{1(1)}^* = 0 \text{ and } x_{1(2)}^* = 1.$$

If we take the solution $x_{1(2)}^* = 1$ then we obtain that the vertex $e_1 = (1,0,0)$. If we take the solution $x_{1(1)}^* = 0$ then in the case $0 < \gamma < 1$ from the 2-nd equation of (5) we have $x_2 = \gamma x_2^2 \Rightarrow x_2(1-\gamma x_2) = 0 \Rightarrow x_{2(1)}^* = 0$ and $x_{2(2)}^* = \frac{1}{\gamma} > 1$. If $\gamma = 0$ then from the 2-nd equation of (5) we have $x_{2(1)}^* = 0$. Consequently, it follows that $x_{3(1)}^* = 1$. Therefore, we get that the vertex $e_3 = (0,0,1)$. Hence, we have that in the case $\alpha = 1$ and $0 \le \gamma < 1$ the vertices $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$ are fixed points of the operator V.

b) Let $\gamma = 1$ and $0 \le \alpha < 1$. Then (4) has the form:

$$\begin{cases} x_1 = \alpha x_1^2, \\ x_2 = \beta x_1^2 + x_2^2 + 2x_1 x_2 \\ x_3 = (1-\alpha-\beta) x_1^2 + 2(1-\gamma) x_1 x_2 + x_3^2 + 2(x_1 + x_2) x_3. \end{cases} \quad (6)$$

In the condition $0 < \alpha < 1$ (resp. $\alpha = 0$) from the 1-st equation of (6) we have

$$x_1 = \alpha x_1^2 \text{ (resp. } x_1 = 0) \Rightarrow x_1(1-\alpha x_1) = 0 \Rightarrow x_{1(1)}^* = 0 \text{ and } x_{1(2)}^* = \frac{1}{\alpha} > 1 \text{ (resp. } x_1^* = 0).$$

Hence, we have that $x_1^* = 0$.

Using the latter from the 2-nd equation of (6) one has

$$x_2 = x_2^2 \;\Rightarrow\; x_2(1-x_2) = 0 \;\Rightarrow\; x_{2(1)}^* = 0 \text{ and } x_{2(2)}^* = 1.$$

Hence, we conclude that $x_{3(1)}^* = 1$ and $x_{3(2)}^* = 0$. Consequently, we obtain that in the case $\alpha = 1$ and $0 \le \gamma < 1$ the vertices $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ serve as stationary states of the operator V.

c) Let $0 \le \alpha < 1$ and $0 \le \gamma < 1$. Then in the case $0 < \alpha < 1$ (resp. $\alpha = 0$) from the 1-st equation of (4) we have $x_1 = \alpha x_1^2$ (resp. $x_1 = 0$) $\Rightarrow x_1(1-\alpha x_1) = 0 \Rightarrow x_{1(1)}^* = 0$ and $x_{1(2)}^* = \frac{1}{\alpha} > 1$ (resp. $x_1^* = 0$). Hence, we have that $x_1^* = 0$.

In the case $0 < \gamma < 1$ (resp. $\gamma = 0$) using latter the from the 2-nd equation of (4) one has $x_2^* = 0$ (resp. $x_2 = 0$) $\Rightarrow$ $x_2(1-\gamma x_2) = 0 \Rightarrow x_{2(1)}^* = 0$ and $x_{2(2)}^* = \frac{1}{\gamma} > 1$ (resp. $x_2^* = 0$). Therefore, we have that $x_2^* = 0$. Using $x_1^* = 0$ and $x_2^* = 0$

we get $x_2^* = 1$ and consequently we obtain that in the case $0 \le \alpha < 1$ and $0 \le \gamma < 1$ the vertex $e_3 = (0,0,1)$ is a single stationary point of the operator V.

To determine the types of the fixed points we rewrite operator (3) as follows

$$\tilde{B}: \begin{cases} x_1' = \alpha x_1^2, \\ x_2' = \beta x_1^2 + \gamma x_2^2 + 2x_1 x_2, \end{cases} \tag{7}$$

where x_1 and x_2 are the 1-st and 2-nd coordinates of the point $x \in S^2$.

From (7) the one has

$$\frac{\partial x_1'}{\partial x_1} = 2\alpha x_1, \frac{\partial x_1'}{\partial x_2} = 0, \frac{\partial x_2'}{\partial x_1} = 2\beta x_1 + 2\gamma x_2, \frac{\partial x_2'}{\partial x_2} = 2\gamma x_2 + 2\gamma x_1.$$

At the fixed point $e_1 = (1,0,0)$ we have

$$D_B(e_1) = \begin{pmatrix} 2\alpha & 0 \\ 2\beta & 2\gamma \end{pmatrix} \Rightarrow \det\begin{pmatrix} 2\alpha - \mu & 0 \\ 2\beta & 2\gamma - \mu \end{pmatrix} = 0 \Rightarrow \mu_1 = 2\alpha, \ \mu_2 = 2\gamma.$$

At the fixed point $e_2 = (0,1,0)$ we have

$$D_B(e_2) = \begin{pmatrix} 0 & 0 \\ 2\beta & 2\gamma \end{pmatrix} \Rightarrow \det\begin{pmatrix} -\mu & 0 \\ 2\beta & 2\gamma - \mu \end{pmatrix} = 0 \Rightarrow \mu_1 = 0, \ \mu_2 = 2\gamma.$$

Similarly, at the fixed point $e_3 = (0,0,1)$ we have

$$D_B(e_3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \det\begin{pmatrix} -\mu & 0 \\ 0 & -\mu \end{pmatrix} = 0 \Rightarrow \mu_1 = \mu_2 = 0.$$

If $\alpha = 1, \ 0 \le \gamma < 1$ then e_1 has the type $\begin{cases} repelling, & if \ 0 \le \gamma < \frac{1}{2}, \\ non-hyperbolic, & if \ \gamma = \frac{1}{2}, \\ saddle, & if \ \frac{1}{2} < \gamma \le 1. \end{cases}$

If $\gamma = 1, \ 0 \le \alpha < 1$ then e_2 has the type $\begin{cases} attracting, & if \ 0 \le \gamma < \frac{1}{2}, \\ non-hyperbolic, & if \ \gamma = \frac{1}{2}, \\ saddle, & if \ \frac{1}{2} < \gamma \le 1. \end{cases}$

If $0 \le \alpha < 1$ and $0 \le \gamma < 1$ then e_3 has the type attracting.

Let $x^{(0)} \in S^2 \setminus Fix(V)$. Then from the 1-st equation of (3) we have

$$x_1' = \alpha x_1^2 \Rightarrow x_1'' = \alpha^3 x_1^4 \Rightarrow x_1^{(n+1)} = \alpha^{2^n - 1}\left(x_1^{(0)}\right)^{2^n}, \ n = 0,1,2,3\dots .$$

Therefore, it follows that there is the limit $\lim_{n\to\infty} x_1^{(n)} = 0$. Also, from the 1-st equation of (3) we obtain

$$x_3' = (1-\alpha-\beta)x_1^2 + (1-\gamma)x_2^2 + 2(1-\gamma)x_1x_2 + x_3^2 + 2(x_1+x_2)x_3 \geq x_3,$$

where, we have used $0 \leq \alpha, \beta, \alpha+\beta, \gamma \leq 1$. Consequently, we get $x_3^{(n+1)} \geq x_3^n$, $n = 0,1,2,3,\ldots$. As a result, the sequence converges to a limit $\lim_{n\to\infty} x_3^{(n)} = x_3^*$. Hence, we have $\lim_{n\to\infty} x_2^{(n)} = 1 - x_3^*$. We claim, that it holds $x_3^* = 1$. Suppose on the contrary, that is, let $x_3^* < 1$. Then one has

$$1 = \lim_{n\to\infty} \frac{1-x_3^{(n+1)}}{1-x_3^{(n)}} \leq \lim_{n\to\infty} \frac{1-x_3^{(n)}\left(1+x_1^{(n)}+x_2^{(n)}\right)}{1-x_3^{(n)}} =$$

$$= 1 - \lim_{n\to\infty} \frac{x_3^{(n)}\left(x_1^{(n)}+x_2^{(n)}\right)}{1-x_3^{(n)}} = 1 - \lim_{n\to\infty} \frac{x_3^{(n)}\left(1-x_3^{(n)}\right)}{1-x_3^{(n)}} = 1 - \lim_{n\to\infty} x_3^{(n)}.$$

From the latter we get $\lim_{n\to\infty} x_3^{(n)} = 0$. The latter contradicts to the fact $\left\{x_3^{(n)}\right\}$ is an increasing sequence. Therefore, it follows that $x_3^* = 1$ and consequently, we have $\lim_{n\to\infty} V\left(x^{(0)}\right) = e_3 = (0,0,1)$.

4. CONCLUSION

In the paper [7], dynamical systems formed by the composition of Volterra operators with distinct features, defined on the two-dimensional simplex, have been investigated. It was demonstrated that the long-term behavior of arbitrary trajectories under the superposed operator significantly differs from the asymptotic properties of the trajectories associated with the individual original operators.. In contrast to this paper, the result of the superposition of a linear operator and a quadratic stochastic Volterra operator is defined on a two-dimensional simplex. The fixed points of the superposition operator are found and their types are determined. In addition, it is proven that almost any orbit for such an operator converges to the third vertex of a two-dimensional simplex. The results obtained in this paper are of great importance in constructing and studying the mathematical models of physical and biological processes.

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