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Asymptotical Behavior of Trajectories of Non-Volterra Quadratic Stochastic Operators

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Abstract—In the paper, we consider a family of discrete-time dynamical systems generated by non-Volterra stochastic operators. Namely non-Volterra stochastic operators depending on the two parameters $a, b \in [-1, 1]$. It is described the set of fixed points and their types and the set of periodic points. We proved that for any parameters any trajectory of a non-Volterra QSO from this family converges to either period point with period two or a fixed point, that is we showed that such operators have a property of being ergodic.

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1. INTRODUCTION

Let $E = \{1, \dots, m\}$ be a finite set and the set of all probability distributions on E

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_k \geq 0,\ \forall\, k \in E,\ \sum_{k=1}^{m} x_k = 1 \right\}$$

be the $(m-1)$-dimensional simplex.

Let us consider a continuous mapping $\Psi\colon S^{m-1} \to S^{m-1}$. The mapping Ψ is called *stochastic* operator, that is, if $\Psi(S^{m-1}) \subset S^{m-1}$, then Ψ is a stochastic operator.

The trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ of Ψ for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = \Psi\left(\mathbf{x}^{(n)}\right) = \Psi^{n+1}\left(\mathbf{x}^{(0)}\right), \quad n = 0, 1, 2, \dots$$

By $\omega_\Psi\left(\mathbf{x}^{(0)}\right)$ we denote ω-limit set points of the trajectory $\{\mathbf{x}^{(n)}\}_{n\geq 0}$.

A mapping Ψ is called *regular*, if there is the limit $\lim\limits_{n\to\infty} \Psi^n(\mathbf{x})$ for any initial value $\mathbf{x} \in S^{m-1}$. A mapping Ψ is said to be *ergodic*, if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Psi^k(\mathbf{x})$$

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exists for any $\mathbf{x} \in S^{m-1}$.

A *quadratic stochastic operator* (QSO) is a mapping $V : S^{m-1} \to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$V : x'_k = \sum_{i,j \in E} p_{ij,k} x_i x_j, \quad k \in E, \tag{1}$$

and the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad i, j, k \in E. \tag{2}$$

Such operators frequently arise in many models of mathematical genetics, namely theory of heredity (see e.g. [1–4]).

The main problem in mathematical biology consists in the study of the asymptotical behaviour of the trajectories for a given QSO. In other words, the main task is the description of the ω-limit set $\omega_V\left(\mathbf{x}^{(0)}\right)$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO V. This problem is an open problem even in two-dimensional case. This is a motivation considering the quadratic stochastic operator (5).

A QSO is called a *Volterra* operator, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k = 1, \ldots, m$. The asymptotic behaviour of trajectories Volterra QSOs was analysed in [5, 6] using the theory of Lyapunov functions and tournaments.

On the basis of numerical calculations, Ulam conjectured that any QSO is ergodic [7]. But in 1978 in [8], Zakharevich considered the following QSO on S^2

$$x'_1 = x_1^2 + 2x_1x_2, \quad x'_2 = x_2^2 + 2x_2x_3, \quad x'_3 = x_3^2 + 2x_1x_3 \tag{3}$$

and showed that it is a non-ergodic transformation, that is he proved that Ulam's conjecture is false in general. Later in [9] established a sufficient condition for a QSO defined on S^2 to be a non-ergodic transformation, that is Zakharevich's result was generalized to a class of Volterra QSOs defined on S^2. In [10], we have shown the correlation between non-ergodicity of Volterra QSOs and rock-paper-scissors games, so the QSO (3) can be reinterpreted in terms game theory as a rock-paper-scissors game. In [11] the random dynamics of Volterra QSOs is studied. In [12–19] some classes of non-Volterra QSOs were studied. For a recent review on the theory of quadratic stochastic operators see [20].

In the present paper we consider a family of discrete-time dynamical systems generated by non-Volterra QSOs depending on the parameters $a, b \in [-1, 1]$. We proved that if $a = b = -1$, then for the corresponding QSO there are infinitely many invariant sets. Moreover, this operator has two fixed points and infinitely many two-periodic points. It is shown that almost all trajectories of the QSO converge to a two-periodic orbit (Theorems 2, 3). If $a = b = 1$, then for the corresponding QSO there are two invariant sets and infinitely many fixed points. Moreover, any trajectory of such non-Volterra QSO converges to a fixed point (Theorems 2, 4). There are two fixed points and any trajectory of the operator (5) converges to a fixed point when $-1 < a, b < 1$ (Theorems 2, 7). These facts show that depending on parameters the discrete-time dynamical systems generated by non-Volterra QSOs may exhibit varied dynamics. Therefore, each non-Volterra QSO provides an interesting example in the theory of nonlinear dynamical systems and such operators require investigation even in the small dimensional cases.

The paper is organized as follows. In Section 2 we recall the necessary definitions, give the form of a non-Volterra stochastic operator and describe the set of fixed points. In Section 3 we prove that for any parameters a non-Volterra QSO from this family is an ergodic operator.

2. A NON-VOLTERRA QSO AND ITS FIXED POINTS

Let us recall necessary definitions. Let V be a quadratic stochastic operator. A point $\mathbf{x} \in S^{m-1}$ is called a *periodic* point of V if there exists an n such that $V^n(\mathbf{x}) = \mathbf{x}$. The smallest positive integer n satisfying $V^n(\mathbf{x}) = \mathbf{x}$ is called the *prime period* or *least period* of the point $\mathbf{x}$. A period-one point is called a *fixed* point of V.

Denote the set of all fixed points by Fix (V) and the set of all periodic points of (not necessarily prime) period n by $\mathrm{Per}_n\ (V)$. Evidently that the set of all iterates of a periodic point form a periodic trajectory (orbit).

Let $DV(\mathbf{x}^*) = (\partial V_i/\partial x_j(\mathbf{x}^*))_{i,j=1}^m$ be the Jacobi matrix of operator V at the point $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is called *hyperbolic*, if its Jacobi matrix $DV(\mathbf{x}^*)$ has no eigenvalues 1 in absolute value. A hyperbolic fixed point $\mathbf{x}^*$ is called *attracting* (respectively, *repelling*), if all the eigenvalues of the Jacobi matrix $DV(\mathbf{x}^*)$ are less (respectively, greater) than 1 in absolute value; it is called a *saddle*, if some of the eigenvalues of $DV(\mathbf{x}^*)$ are less than 1 in absolute value and other eigenvalues are greater than 1 in absolute value (see [21]).

A QSO V given by (1) on the S^2 has the form

$$V: \begin{cases} x_1' = p_{11,1}x_1^2 + p_{22,1}x_2^2 + p_{33,1}x_3^2 + 2p_{12,1}x_1x_2 + 2p_{13,1}x_1x_3 + 2p_{23,1}x_2x_3, \\ x_2' = p_{11,2}x_1^2 + p_{22,2}x_2^2 + p_{33,2}x_3^2 + 2p_{12,2}x_1x_2 + 2p_{13,2}x_1x_3 + 2p_{23,2}x_2x_3, \\ x_3' = p_{11,3}x_1^2 + p_{22,3}x_2^2 + p_{33,3}x_3^2 + 2p_{12,3}x_1x_2 + 2p_{13,3}x_1x_3 + 2p_{23,3}x_2x_3. \end{cases} \tag{4}$$

In [5] the QSO (4) studied in the case $p_{11,1} = p_{22,2} = p_{33,3} = 1$, $p_{22,1} = p_{33,1} = p_{23,1} = p_{11,2} = p_{33,2} = p_{13,2} = p_{11,3} = p_{22,3} = p_{12,3} = 0$ and other coefficients of the QSO given by (4) are non-negative.

In [19] the QSO (4) studied in the case $p_{23,1} = p_{13,2} = p_{12,3} = 1$, $p_{11,1} = p_{22,2} = p_{33,3} = p_{12,1} = p_{12,2} = p_{13,1} = p_{13,3} = p_{23,2} = p_{23,3} = 0$, and other coefficients of the QSO given by (4) are non-negative.

In [22] the QSO (4) studied in the case $p_{23,1} = p_{13,2} = p_{12,3} = 1$, $p_{11,1} = p_{22,2} = p_{12,1} = p_{12,2} = p_{13,1} = p_{13,3} = p_{23,2} = p_{23,3} = 0$, and other coefficients of the QSO given by (4) are non-negative.

We consider the following family of non-Volterra QSOs defined on S^2

$$V: \begin{cases} x_1' = (1+a)x_1x_3 + (1-b)x_2x_3, \\ x_2' = (1-a)x_1x_3 + (1+b)x_2x_3, \\ x_3' = x_3^2 + (x_1+x_2)^2, \end{cases} \tag{5}$$

where $a, b \in [-1, 1]$.

Note the QSO (5) does not coincide with the QSOs which are investigated in [5, 19, 22]. Since in general case the main problem is an open problem even in two-dimensional case. It plays as motivation of considering the QSO (5). Let $\Gamma_{ij} = \{\mathbf{x} \in S^2 :\ x_k = 0;\ k \notin \{i,j\}\}$, $i \neq j$, $i,j \in \{1,2,3\}$ be the faces of S^2 and let $\mathbf{m}_1 = (0, 1/2, 1/2)$, $\mathbf{m}_2 = (1/2, 0, 1/2)$, $\mathbf{m}_3 = (1/2, 1/2, 0)$ be their centers, respectively. Denote by $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ the vertices of the S^2 and also denote

$$M = \left\{\mathbf{x} \in S^2 : x_1 + x_2 = \frac{1}{2},\ x_3 = \frac{1}{2}\right\}, \quad \mathbf{x}_{a,b} = \left(\frac{1-b}{2(2-a-b)}, \frac{1-a}{2(2-a-b)}, \frac{1}{2}\right).$$

First, we consider the following function

$$f(x) = 2x^2 - 2x + 1, \quad x \in [0,1]. \tag{6}$$

Denote $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n \text{ times}}$, the n-fold composition of $f(x)$ with itself.

Theorem 1 [16]. *For the function f defined by (6) the followings are true:*

$i)$ the points $x = 1/2$ and $x = 1$ are fixed points;

$ii)$ there is no periodic orbit of $f(x)$ with period $n \geq 2$;

$iii)$ if $0 < x < 1$, then $\lim_{n\to\infty} f^n(x) = 1/2$ and $f(0) = f(1) = 1$.

Theorem 2. *For the operator V the following statements are true:*

$i)$ $\mathrm{Fix}\,(V) = \begin{cases} \{\mathbf{e}_3\} \cup M, & \text{if } \ a = 1 \ \text{ or } \ b = 1, \\ \{\mathbf{e}_3, \mathbf{x}_{a,b}\}, & \text{if } \ -1 \leq a, b < 1; \end{cases}$

ii) The vertex $\mathbf{e}_3$ *is repelling (respectively, non-hyperbolic, saddle) point when* $|a+b|>1$ *(respectively,* $|a+b|=1$, $|a+b|<1$*).*

iii) If $a+b<2$*, then the fixed points from the set* M *are attracting points and they are non-hyperbolic points when* $a+b=2$*;*

iv) If $-1<a,b<1$*, then the fixed point* $\mathbf{x}_{a,b}$ *is an attracting point and it is a non-hyperbolic point when* $a=b=-1$.

Proof. $i)$ A fixed point of the operator V is a solution of the equation $V(\mathbf{x})=\mathbf{x}$, that is

$$\begin{cases} x_1=(1+a)x_1x_3+(1-b)x_2x_3, \\ x_2=(1-a)x_1x_3+(1+b)x_2x_3, \\ x_3=x_3^2+(x_1+x_2)^2, \end{cases} \tag{7}$$

where $a,b\in[-1,1]$. Using the $x_1+x_2+x_3=1$, we can write the third equation of the system (7) in the form

$$x_3^2-2x_3+1=x_3.$$

Due to Theorem 1 the last equation has the solutions $x_3^*=1$ and $x_3^*=1/2$. If we take $x_3^*=1$, then it follows that the vertex $\mathbf{e}_3$ is a fixed point of the operator (5) for any parameters. Let $x_3^*=1/2$. Then, the simple analysis of the following

$$\begin{cases} x_1=\dfrac{1+a}{2}x_1+\dfrac{1-b}{2}x_2, \\ x_2=\dfrac{1-a}{2}x_1+\dfrac{1+b}{2}x_2, \end{cases}$$

where $a,b\in[-1,1]$, system of linear equation gives that the set of fixed points has form either

$$\mathrm{Fix}\,(V)=\{\mathbf{e}_3\}\cup M \quad \text{or} \quad \mathrm{Fix}\,(V)=\{\mathbf{e}_3,\mathbf{x}_{a,b}\}.$$

To check the types of the fixed points, we rewrite the operator V in the following form

$$\begin{aligned} x_1'&=(1-x_1-x_2)\left((1+a)x_1+(1-b)x_2\right), \\ x_2'&=(1-x_1-x_2)\left((1-a)x_1+(1+b)x_2\right), \end{aligned} \tag{8}$$

where $(x_1,x_2)\in\tilde{S}=\{(x_1,x_2):x_1,x_2\geq 0, x_1+x_2\in[0;1]\}$ and x_1,x_2 are the first two coordinates of a point lying in the simplex S^2.

$ii)$ It is easy to verify that the Jacobian of the operator (8) at the vertex $\mathbf{e}_3$ has the eigenvalues $\mu_1=2,\mu_2=a+b$. Since $a,b\in[-1,1]$ it follows that the vertex $\mathbf{e}_3$ is a non-hyperbolic (repelling, saddle) point when $|a+b|=1$ ($|a+b|>1$, $|a+b|<1$).

$iii)$ $\mathbf{x}\in M$ is a fixed point for the (8) and it has the eigenvalues $\mu_1=0$ and $\mu_2=(a+b)/2$. Therefore, if $a+b<2$ (resp. $a+b=2$), then any point from this set an attracting (a non-hyperbolic) point.

$iv)$ After simple algebra one has that the Jacobian of the operator (8) at the point $\mathbf{x}_{ab}$ has the eigenvalues $\mu_1=0$ and $\mu_2=(a+b)/2$. Therefore, if $a,b\in(-1,1)$ it follows that the fixed point $\mathbf{x}_{ab}$ is an attracting point and it is a non-hyperbolic point when $a=b=-1$.

The theorem is proved. □

3. THE SET OF LIMIT POINTS OF A TRAJECTORY

Let us describe the set of ω-limit points of a trajectory. The problem of describing the $\omega_V\left(\mathbf{x}^{(0)}\right)$ of a trajectory is of great importance in the theory of dynamical systems. Since $\omega_V\left(\mathbf{x}^{(0)}\right)$ of a trajectory is a subset of S^2, and since S^2 is compact, it follows that $\omega_V\left(\mathbf{x}^{(0)}\right)\neq\emptyset$. We note that if $\omega_V\left(\mathbf{x}^{(0)}\right)$ consists of a single point, then the trajectory converges to this point, as it is a fixed point of the operator V given by (5).

We will consider all possible cases.

Theorem 3. *Let* $a=b=-1$*. Then, for the QSO* V *given by (5) the following statements are true.*

i) *the vertex* $\mathbf{e}_3$ *and* $\mathbf{x}_{-1,-1} = (1/4, 1/4, 1/2)$ *are fixed points;*

ii) $\mathrm{Per}_2(V) = \left\{\mathbf{x} \in S^2 : x_1 + x_2 = 1/2, x_3 = 1/2\right\}$;

iii) *the sets* $M_\nu = \left\{\mathbf{x} \in S^2 : x_1 = \nu x_2 \vee x_2 = \nu x_1\right\}$ *for* $\nu > 0$*, and* $M_0 = \left\{\mathbf{x} \in S^2 : x_1 x_2 = 0\right\}$ *for* $\nu = 0$*, are invariant sets of* V;

iv) *if* $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_3\}$*, then* $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{e}_3\}$;

v) $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{m}_1, \mathbf{m}_2\}$ *for any initial* $\mathbf{x}^{(0)} \in M_0$;

vi) *for any* $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{12}$ *there exists* M_ν*,* $\nu \in (0, +\infty)$ *such that* $\mathbf{x}^{(0)} \in M_\nu$ *and* $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\tilde{\mathbf{x}}, \hat{\mathbf{x}}\}$*, where*

$$\tilde{\mathbf{x}} = \left(\frac{\nu}{2(1+\nu)}, \frac{1}{2(1+\nu)}, \frac{1}{2}\right) \quad \text{and} \quad \hat{\mathbf{x}} = \left(\frac{1}{2(1+\nu)}, \frac{\nu}{2(1+\nu)}, \frac{1}{2}\right).$$

Proof. Let $a = b = -1$, then the QSO (5) has the form

$$V : \begin{cases} x_1' = 2x_2x_3, \\ x_2' = 2x_1x_3, \\ x_3' = x_3^2 + (x_1 + x_2)^2 \end{cases} \tag{9}$$

and this operator was studied in [23]. Consequently the proofs of assertions of Theorem 3 follow from the results of [23]. □

Theorem 4. *Let* $a = b = 1$*. For the QSO* V *given by (5) the following statements are true:*

i) *the faces* Γ_{13} *and* Γ_{23} *are invariant sets;*

ii) $$\omega_V\left(\mathbf{x}^{(0)}\right) = \begin{cases} \{\mathbf{e}_3\}, & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \\ \{\mathbf{m}_2\}, & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{13} \setminus \{\mathbf{e}_1, \mathbf{e}_3\}, \\ \{\mathbf{m}_1\}, & \text{if} \quad \mathbf{x}^{(0)} \in \Gamma_{23} \setminus \{\mathbf{e}_2, \mathbf{e}_3\}; \end{cases}$$

iii) *if* $\mathbf{x}^{(0)} \in \mathrm{int}\, S^2 = \left\{\mathbf{x} \in S^2 : x_1x_2x_3 > 0\right\}$*, then*

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \left(\frac{x_1^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \frac{x_2^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \frac{1}{2}\right).$$

Proof. Assume that $a = 1,\ b = 1$. In this case the QSO (5) has the form

$$V : \begin{cases} x_1' = 2x_1x_3, \\ x_2' = 2x_2x_3, \\ x_3' = x_3^2 + (x_1 + x_2)^2. \end{cases} \tag{10}$$

i) Obviously.

ii) a) Let $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an initial point. Then, it is clear that $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$.

b) Let $\mathbf{x}^{(0)} \in \Gamma_{13} \setminus \{\mathbf{e}_1, \mathbf{e}_3\}$. Then, from (10) one has that

$$x_1^{(n)} > 0, \quad \text{and} \quad x_2^{(n)} = 0 \quad \text{for any} \quad n = 0, 1, 2, \ldots$$

and using

$$\min_{x_3\in[0,1]} x_3' = \min_{x_3\in[0,1]} \left(2x_3^2 - 2x_3 + 1\right) = \frac{1}{2}, \tag{11}$$

one has that

$$x_1' \geq x_1 \quad \Rightarrow \quad x_1^{(n+1)} \geq x_1^{(n)}, \quad n = 0,1,2,\ldots \quad \Rightarrow \quad \lim_{n\to\infty} x_1^{(n)} = x_1^*.$$

Consequently, it follows that $\lim_{n\to\infty} x_3^{(n)} = 1 - x_1^*$, and we have

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^* = \left(x_1^*, 0, 1 - x_1^*\right).$$

Since the limit point should be a fixed point we have $\mathbf{x}^* = \mathbf{m}_2$.

c) Similarly in the case $\mathbf{x}^{(0)} \in \Gamma_{23} \setminus \{\mathbf{e}_2, \mathbf{e}_3\}$ we obtain $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{m}_1$.

iii) Let $\mathbf{x}^{(0)} \in \operatorname{int} S^2$. Then, due to Theorem 1 we have

$$\lim_{n\to\infty} x_3^{(n)} = x_3^* = \frac{1}{2}.$$

Using (11), we have

$$x_1' \geq x_1 \quad \Rightarrow \quad x_1^{(n+1)} \geq x_1^{(n)}, \quad n = 0,1,2,\ldots \quad \Rightarrow \quad \lim_{n\to\infty} x_1^{(n)} = x_1^*,$$

$$x_2' \geq x_2 \quad \Rightarrow \quad x_2^{(n+1)} \geq x_2^{(n)}, \quad n = 0,1,2,\ldots \quad \Rightarrow \quad \lim_{n\to\infty} x_2^{(n)} = x_2^*.$$

Moreover, from (10) it follows that

$$\frac{x_1^{(n)}}{x_2^{(n)}} = \frac{x_1^{(0)}}{x_2^{(0)}}, \quad n = 0,1,2,\ldots \quad \Rightarrow \quad \frac{x_1^*}{x_2^*} = \frac{x_1^{(0)}}{x_2^{(0)}}.$$

Consequently, we obtain

$$x_1^* = \frac{x_1^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \quad x_2^* = \frac{x_2^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \quad x_3^* = \frac{1}{2}.$$

Thus, for any $\mathbf{x}^{(0)} \in \operatorname{int} S^2$ we obtain

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \left(\frac{x_1^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \frac{x_2^{(0)}}{2\left(x_1^{(0)} + x_2^{(0)}\right)}, \frac{1}{2}\right).$$

The proof of the theorem is complete. □

Theorem 5. *For the operator V given by (5) the following statements are true:*

i) *if $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$;*

ii) *if $a = 1$, $-1 \leq b < 1$ and $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{m}_2\}$;*

iii) *if $b = 1$, $-1 \leq a < 1$ and $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{m}_1\}$.*

Proof. Assume that $a = 1$, $-1 \leq b < 1$. In this case the QSO (5) has the form

$$V: \begin{cases} x_1' = 2x_1x_3 + (1-b)x_2x_3, \\ x_2' = (1+b)x_2x_3, \\ x_3' = x_3^2 + (x_1 + x_2)^2. \end{cases} \tag{12}$$

i) Let $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an initial point. Then, it is clear that $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$.

ii) Let $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then, by Theorem 1 we have $\lim_{n\to\infty} x_3^{(n)} = 1/2$, that is for any small $\varepsilon > 0$ there is a natural number n_0 such that for any $n > n_0$ it holds

$$\left|x_3^{(n)} - \frac{1}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{2} - \varepsilon < x_3^{(n)} < \frac{1}{2} + \varepsilon.$$

Using the last inequalities, from the second equation of (7) one has

$$\frac{1+b}{2} x_2^{(n)} - \varepsilon (1+b)\left(x_2^{(n)}\right) < x_2^{(n+1)} < \frac{1+b}{2} x_2^{(n)} + \varepsilon (1+b)\left(x_2^{(n)}\right). \tag{13}$$

Since $-1 \leq b < 1$, it follows that $0 \leq (1+b) < 2$ and, using it for all $x_2^{(0)} \in [0,1]$, we have

$$\lim_{n\to\infty} x_2^{(n)} = \left(\frac{1+b}{2}\right)^n x_2^{(0)} = 0, \tag{14}$$

$$\lim_{n\to\infty} x_1^{(n)} = \frac{1}{2} - \lim_{n\to\infty} x_2^{(n)} = \frac{1}{2}. \tag{15}$$

Therefore, using the relations (13), (14) and the last (15) we obtain

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{m}_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right).$$

iii) The case $b = 1$, $-1 \leq a < 1$ can be considered in a similar manner. □

Theorem 6. *For the operator V given by (5) the following statements are true:*

i) *if* $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, *then* $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$;

ii) *if* $a = -1$, $-1 < b < 1$ *and* $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, *then* $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{x}_{-1,b}\}$;

iii) *if* $b = -1$, $-1 < a < 1$ *and* $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, *then* $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{x}_{a,-1}\}$.

Proof. Assume that $a = -1$, $-1 < b < 1$. In this case the QSO (5) has the form

$$V: \begin{cases} x_1' = (1-b)x_2x_3, \\ x_2' = 2x_1x_3 + (1+b)x_2x_3, \\ x_3' = x_3^2 + (x_1+x_2)^2. \end{cases} \tag{16}$$

i) Let $\mathbf{x}^{(0)} \in \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an initial point. Then, it is clear that $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$.

ii) Let $\mathbf{x}^{(0)} \notin \Gamma_{12} \cup \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then, by Theorem 1 we have $\lim_{n\to\infty} x_3^{(n)} = 1/2$, that is for any small $\varepsilon > 0$ there is a natural number n_0 such that for any $n > n_0$ it holds

$$\left|x_3^{(n)} - \frac{1}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{2} - \varepsilon < x_3^{(n)} < \frac{1}{2} + \varepsilon.$$

Using the last inequalities and $x_1 + x_2 + x_3 = 1$, from the first equation of (16) one has

$$(1-b)\left(X_1^{(n)} - \varepsilon\left(x_1^{(n)} + 2 + \varepsilon\right)\right) < x_1^{(n+1)} < (1-b)\left(X_1^{(n)} + \varepsilon\left(x_1^{(n)} + 2 - \varepsilon\right)\right), \tag{17}$$

where we have used the following notation

$$X_1^{(n)} = -\frac{1}{2}x_1^{(n)} + \frac{1}{4}.$$

Since $-1 < b < 1$ it follows that $0 < 1-b < 2$ and using it for all $x_1^{(0)} \in [0,1]$ we have

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty}\left((-1)^n\left(\frac{1-b}{2}\right)^n x_1^{(0)} + \frac{1-b}{4}\sum_{k=0}^{n-1}(-1)^k\left(\frac{1-b}{2}\right)^k\right) = \frac{1-b}{2(3-b)}, \quad (18)$$

$$\lim_{n\to\infty} x_2^{(n)} = \frac{1}{2} - \lim_{n\to\infty} x_1^{(n)} = \frac{1}{3-b}. \quad (19)$$

Therefore, using the relations (17), (18) and the last (19) we obtain

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}_{-1,b} = \left(\frac{1-b}{2(3-b)}, \frac{1}{3-b}, \frac{1}{2}\right).$$

iii) Similarly in the case $b=-1$, $-1<a<1$ can be proved

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}_{a,-1} = \left(\frac{1}{3-a}, \frac{1-a}{2(3-a)}, \frac{1}{2}\right).$$

□

Theorem 7. *Let $-1<a<1$ and $-1<b<1$. Then, for the operator V given by (5) the following statements are true:*

i) *if $\mathbf{x}^{(0)} \in \Gamma_{12}\cup\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, then $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$;*

ii) *if $\mathbf{x}^{(0)} \notin \Gamma_{12}\cup\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, then $\omega_V\left(\mathbf{x}^{(0)}\right) = \{\mathbf{x}_{a,b}\}$, where*

$$\mathbf{x}_{a,b} = \left(\frac{1-b}{2(2-a-b)}, \frac{1-a}{2(2-a-b)}, \frac{1}{2}\right).$$

Proof. i) Let $\mathbf{x}^{(0)} \in \Gamma_{12}\cup\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be an initial point. Then, it is clear that $V\left(\mathbf{x}^{(0)}\right) = \mathbf{e}_3$.

ii) Let $\mathbf{x}^{(0)} \notin \Gamma_{12}\cup\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Then, by Theorem 1 we have $\lim\limits_{n\to\infty} x_3^{(n)} = 1/2$, that is for any small $\varepsilon > 0$ there is a natural number n_0 such that for any $n > n_0$ it holds

$$\left|x_3^{(n)} - \frac{1}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{2} - \varepsilon < x_3^{(n)} < \frac{1}{2} + \varepsilon.$$

Using the last inequalities, from the first equation of (7) one has

$$g(x_1^{(n)}) - \varepsilon\,\widetilde{g}\left(x_1^{(n)}\right) < x_1^{(n+1)} < g(x_1^{(n)}) + \varepsilon\,\widehat{g}\left(x_1^{(n)}\right), \quad (20)$$

where $g(x) = \frac{a+b}{2}x + \frac{1-b}{4}$, $\widetilde{g}(x) = (a+b)x - (1-b)(1-\varepsilon)$, $\widehat{g}(x) = (a+b)x + (1-b)(1+\varepsilon)$.

Since $-1<a<1$, $-1<b<1$ it follows that $|a+b|<2$ and using it for all $x \in [0,1]$ we have

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} g^n(x) = \frac{1-b}{2(2-a-b)}, \quad (21)$$

$$\lim_{n\to\infty} x_2^{(n)} = \frac{1}{2} - \lim_{n\to\infty} x_1^{(n)} = \frac{1-a}{2(2-a-b)}. \quad (22)$$

Therefore, using the relations (20), (21) and the last (22), we obtain

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}_{a,b} = \left(\frac{1-b}{2(2-a-b)}, \frac{1-a}{2(2-a-b)}, \frac{1}{2}\right).$$

The proof of the theorem is complete. □

If an operator is regular, then it satisfies the ergodic hypothesis, and by Theorems 4–7 a non-Volterra QSO (5) is a regular transformation. Hence, from the Theorems 3–7 we have the following corollary.

Corollary 1. *Any non-Volterra QSO (5) is an ergodic transformation.*

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