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Bobokhon Mamurov

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On Convergence of Trajectories of a Non-Volterra Quadratic Stochastic Operator

Bobokhon Mamurov[a)]

Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan.

[a)]*bmamurov.51@mail.ru*

Abstract. In the present paper we consider a non-Volterra quadratic stochastic operator defined on the two-dimensional simplex. We showed that the center of the simplex is a unique fixed point of this operator and it has attracting type. We constructed a Lyapunov function and using it we showed that for any initial point the trajectory of this operator approaches to the center of the simplex.

INTRODUCTION

The notion of quadratic stochastic operator was introduced by S. Bernstein in [1]. Such quadratic operators arise in many models of mathematical genetics, namely, in the theory of heredity (see e.g. [2, 3, 4, 5, 6, 7, 8]).

A quadratic stochastic operator may arise in mathematical genetics as follows. Consider a biological (ecological) population, i.e., a community of organisms closed with respect to reproduction. Suppose that each individual of the population belongs only to one of the species (genetic type) $1,\dots,m$. The scale of species is such that the species of the parents i and j, unambiguously, determine the probability of every species k for the first generation of direct descendants. Denote this probability, called the inheritance coefficient, by $p_{ij,k}$. Evidently that $p_{ij,k}\geq 0$ for all i,j,k and that

$$\sum_{k=1}^{m} p_{ij,k}=1, \quad i,j,k=1,\dots,m.$$

Let $(x_1,x_2,\dots,x_m)$ be the relative frequencies of the genetic types within the whole population in the present generation, which is a probability distribution. In the case of panmixia (random interbreeding) the parent pairs i and j arise for a fixed state $\mathbf{x}=(x_1,x_2,\dots,x_m)$ with probability x_ix_j. Hence, the total probability of the species k in the first generation of direct descendants is defined by

$$x'_k=\sum_{i,j=1}^{m} p_{ij,k}x_ix_j, \quad k=1,\dots,m.$$

The association $\mathbf{x}\mapsto\mathbf{x}'$ defines an evolutionary quadratic operator. Thus evolution of a population can be studied as a dynamical system of a quadratic stochastic operator [5].

A QSO is called a *Volterra* operator if $p_{ij,k}=0$ for any $k\notin\{i,j\}$, $i,j,k=1,\dots,m$. The asymptotic behaviour of trajectories Volterra QSOs was analysed in [2] using the theory of Lyapunov functions and tournaments. However, in the non-Volterra case, many questions remain open and there seems to be no general theory available [3, 4, 7, 8, 9, 10, 11].

In [7] the conception of $F-$ quadratic stochastic operators was introduced, and it was proved that for any parameters such quadratic operator has a unique fixed point. Also it was showed that any trajectory of F- quadratic stochastic operator converges to the unique fixed point exponentially rapidly.

In [8] the notion of strictly non-Volterra quadratic stochastic operators was introduced, and it was proved that an arbitrary strictly non-Volterra quadratic stochastic operator on the two-dimensional simplex has a unique fixed point, which is not attracting.

In [11] symmetric strictly non-Volterra quadratic stochastic operators defined on the three-dimensional simplex were studied. It was proved that this operator has a unique fixed point and some classes of such operators have infinitely many periodic points. Also it was showed that therein trajectories which are asymptotically cyclic with period two.

In [10] quadratic stochastic operators corresponding to graphs were studied, and it was proved that for any parameters the quadratic operator has a unique fixed point and any trajectory of such operators converge to the unique fixed point.

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The paper is organised as follows. In Section 2 we recall definitions and well known results from the theory of Volterra and non-Volterra QSOs . In Section 3 we consider a non-Volterra QSO and show that this QSO has a unique fixed point. Moreover, we prove that this operator has the property being regular.

PRELIMINARIES

Let

$$S^{m-1}=\left\{\mathbf{x}=(x_1,x_2,\ldots,x_m)\in R^m : x_i\geq 0,\ \sum_{i=1}^{m}x_i=1\right\}$$

be the $(m-1)$-dimensional simplex. A map V of S^{m-1} into itself is called a *quadratic stochastic operator* (QSO) if

$$(V\mathbf{x})_k=\sum_{i,j=1}^{m}p_{ij,k}x_ix_j \tag{1}$$

for any $\mathbf{x}\in S^{m-1}$ and for all $k=1,\ldots,m$, where

$$p_{ij,k}\geq 0,\quad p_{ij,k}=p_{ji,k},\qquad \sum_{k=1}^{m}p_{ij,k}=1,\ \ \forall i,j,k=1,\ldots,m. \tag{2}$$

Assume $\{\mathbf{x}^{(n)}\in S^{m-1}: n=0,1,2,\ldots\}$ is the trajectory of the initial point $\mathbf{x}\in S^{m-1}$, where $\mathbf{x}^{(n+1)}=V(\mathbf{x}^{(n)})$ for all $n=0,1,2,\ldots$, with $\mathbf{x}^{(0)}=\mathbf{x}$.

Definition 1. A point $\mathbf{x}\in S^{m-1}$ is called a fixed point of a QSO V if $V(\mathbf{x})=\mathbf{x}$.

Definition 2. A QSO V is called regular if for any initial point $\mathbf{x}\in S^{m-1}$, the limit

$$\lim_{n\to\infty}V(\mathbf{x}^{(n)})$$

exists.

Note that the limit point is a fixed point of a QSO. Thus, the fixed points of a QSO describe limit or long run behavior of the trajectories for any initial point. The limit behavior of trajectories and fixed points play an important role in many applied problems (see e.g. [2, 5, 6, 7]). The biological treatment of the regularity of a QSO is rather clear: in the long run the distribution of species in the next generation coincides with the distribution of species in the previous one, i.e., it is stable.

Definition 3. A continuous function φ: int $S^{m-1}\to R$ for an operator V if the limit $\lim_{n\to\infty}\varphi(V^n(\mathbf{x}))$ exists and finite for all $\mathbf{x}\in S^{m-1}$.

A Lyapunov function is very helpful to describe an upper estimate of the set of limit points. However there is no general recipe on how to find such Lyapunov functions.

Let $D_{\mathbf{x}}V(\mathbf{x}^*)=(\partial V_i/\partial x_j)(\mathbf{x}^*)$ be a Jacobian of V at the point $\mathbf{x}^*$.

Definition 4. [12] A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ has no eigenvalues on the unit circle.

Definition 5. [12] A hyperbolic fixed point $\mathbf{x}^*$ is called:

i) attracting if all the eigenvalues of the Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ are less than 1 in absolute value;

ii) repelling if all the eigenvalues of the Jacobian $D_{\mathbf{x}}V(\mathbf{x}^*)$ are greater than 1 in absolute value;

iii) a saddle otherwise.

MAIN RESULT

Consider the following two strictly non-Volterra QSOs on the two-dimensional simplex

$$V:\begin{cases} x_1' = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_1x_2, \\ x_2' = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_2x_3, \\ x_3' = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_3x_1. \end{cases} \tag{3}$$

Lemma 1. The center $\mathbf{x}$ is a unique and attracting point of the QSO (3).
Proof. The equation $V(\mathbf{x}) = \mathbf{x}$ has the form

$$\begin{cases} x_1 = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_1x_2, \\ x_2 = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_2x_3, \\ x_3 = \frac{1}{3}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + 2x_3x_1. \end{cases} \tag{4}$$

As before, a solution of the system (4) is a fixed point. It is known that the QSO (3) is a continuous operator and that the simplex over a finite set is compact and convex, so that by the Brouwer fixed-point theorem there is always at least one fixed point. We rewrite the system (4) in the form

$$\begin{cases} x_1 - x_2 = 2x_2(x_1 - x_3), \\ x_2 - x_3 = 2x_3(x_2 - x_1), \\ x_2 - x_1 = 2x_1(x_3 - x_2). \end{cases} \tag{5}$$

Let $\mathbf{x}^* \in S^2$ be a solution of the system (5). Assume that $x_1^* \geq x_2^*$. The rest is similar to this case. Since $x_1^* \geq x_2^*$ from the first equation of the system (5) we get $x_1^* \geq x_3^*$. Using it from the second equation of the system (5) we have $x_3^* \geq x_2^*$. Using it from the last equation we obtain that $x_2^* \geq x_1^*$, that is $x_1^* \geq x_2^* \;\Rightarrow\; x_1^* \geq x_3^* \;\Rightarrow\; x_3^* \geq x_2^* \;\Rightarrow\; x_2^* \geq x_1^* \;\Rightarrow$

$$x_1^* \geq x_3^* \geq x_2^* \geq x_1^* \;\Rightarrow x_1^* = x_2^* = x_3^* = \frac{1}{3}.$$

Therefore it follows that the center $\mathbf{c} = (1/3, 1/3, 1/3)$ is a unique fixed point.

To find the type of the unique fixed point, using $x_3 = 1 - x_1 - x_2$, we rewrite the quadratic operator (3) in the form:

$$\begin{cases} x_1' = -\frac{4}{3}x_1^2 + \frac{2}{3}x_2^2 - \frac{4}{3}x_1x_2 + \frac{4}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}, \\ x_2' = \frac{2}{3}x_1^2 - \frac{4}{3}x_2^2 - \frac{4}{3}x_1x_2 - \frac{2}{3}x_1 + \frac{4}{3}x_2 + \frac{1}{3}. \end{cases} \tag{6}$$

where $(x_1, x_2) \in \{(x_1, x_2) : x_1, x_2 \geq 0, 0 \leq x_1 + x_2 \leq 1\}$, and x_1, x_2 are the first two coordinates of the points of the simplex S^2.

One has that the partial derivations has the form

$$\frac{\partial x_1'}{\partial x_1} = -\frac{8}{3}x_1 - \frac{4}{3}x_2 + \frac{4}{3}, \quad \frac{\partial x_1'}{\partial x_2} = \frac{4}{3}x_2 - \frac{4}{3}x_1 - \frac{2}{3},$$

$$\frac{\partial x_2'}{\partial x_1} = -\frac{4}{3}x_1 - \frac{4}{3}x_2 - \frac{2}{3}, \quad \frac{\partial x_2'}{\partial x_2} = \frac{8}{3}x_2 - \frac{4}{3}x_1 + \frac{4}{3}$$

and their values at the center $\mathbf{c}$ are equal

$$\frac{\partial x_1'}{\partial x_1}(\mathbf{c}) = 0, \quad \frac{\partial x_1'}{\partial x_2}(\mathbf{c}) = -\frac{1}{3}, \quad \frac{\partial x_2'}{\partial x_1}(\mathbf{c}) = -\frac{2}{3}, \quad \frac{\partial x_2'}{\partial x_2}(\mathbf{c}) = 0.$$

Thus, the Jacobian matrix of the operator (6) at the center $\mathbf{c}$ has the form:

$$D_{\mathbf{x}}V(\mathbf{c}) = \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{3} & 0 \end{pmatrix}$$

Now let's find the eigenvalues of the matrix $D_{\mathbf{x}}V(\mathbf{c})$:

$$\det \begin{pmatrix} -\mu & -\frac{1}{3} \\ -\frac{2}{3} & -\mu \end{pmatrix} = 0 \;\Rightarrow\; \mu_{1,2} = \pm\frac{\sqrt{2}}{3} \;\Rightarrow\; |\mu_{1,2}| \leq 1.$$

This shows that the center $\mathbf{c} = (1/3, 1/3, 1/3)$ is an attracting point.

The proof of lemma completed.

Lemma 2. The function $\varphi(\mathbf{x}) = |x_1 - x_2| \cdot |x_2 - x_3| \cdot |x_3 - x_1|$ is a Lyapunov function for the operator (3).

Proof. For a $\mathbf{x} \in S^2$ from (3) one has

$$\varphi(V(\mathbf{x})) = \left|x_1' - x_2'\right| \cdot \left|x_2' - x_3'\right| \cdot \left|x_3' - x_1'\right|$$

$$= |2x_1x_2 - 2x_2x_3| \cdot |2x_2x_3 - 2x_1x_3| \cdot |2x_1x_3 - 2x_1x_2|$$

$$= 8\,|x_2| \cdot |x_1 - x_3| \cdot |x_3| \cdot |x_2 - x_1| \cdot |x_1| \cdot |x_3 - x_2|$$

$$= 8x_1 \cdot x_2 \cdot x_3 \cdot \varphi(\mathbf{x}) \leq 8\left(\frac{x_1 + x_2 + x_3}{3}\right)^3 \cdot \varphi(\mathbf{x}) = \left(\frac{2}{3}\right)^3 \cdot \varphi(\mathbf{x}) \leq \varphi(\mathbf{x}).$$

Thus the function $\varphi(\mathbf{x})$ is a decreasing Lyapunov function for the operator (3).

The proof of the lemma completed.

Lemma 3. $\lim_{n\to\infty} V^n\left(\mathbf{x}^{(0)}\right) = \mathbf{c}$ for any initial point $\mathbf{x}^{(0)} \in S^2$.

Proof. By Lemma 2 the function $\varphi(\mathbf{x})$ is a decreasing Lyapunov function. So it follows that

$$\varphi(V(\mathbf{x})) \leq \varphi(\mathbf{x}).$$

So from the form of QSO (3) for the $\mathbf{x}^{(n)}$, $n = 0, 1, 2, \ldots$ one has

$$\varphi\left(\mathbf{x}^{(n+1)}\right) \leq \left(\frac{2}{3}\right)^3 \cdot \varphi\left(\mathbf{x}^{(n)}\right) \leq \cdots \leq \left(\frac{2}{3}\right)^{(3n+1)} \cdot \varphi\left(\mathbf{x}^{(0)}\right).$$

Therefore it follows that

$$\lim_{n\to\infty} \varphi\left(\mathbf{x}^{(n)}\right) = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{c}$$

where we have used that $\varphi\left(\mathbf{x}^{(0)}\right)$ is a bounded value.

The proof of the lemma completed.

Theorem. a) The QSO (3) has a unique fixed point $\mathbf{c} = (1/3, 1/3, 1/3)$;
b) The fixed point $\mathbf{c}$ is an attracting point;
c) For any $\mathbf{x}^{(0)} \in S^2$, the trajectory $\{\mathbf{x}^{(n)}\}$ tends to the fixed point $\mathbf{c}$;
d) The QSO (3) is a regular transformation.

Proof. Collecting together all three Lemmas we complete the proof of Theorem.

CONCLUSION

In this work, we considered a non-Volterra quadratic stochastic operator defined on the two-dimensional simplex. We showed that the center of the simplex is a unique fixed point of this operator and it has attracting type. Moreover we proved that for an initial point the trajectory of this quadratic stochastic operator converges to the center of the two-dimensional simplex, that is, we showed that such operator has the property being regular.

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