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Markov Processes and **Related Fields**

Quadratic Stochastic Processes of Type $(\sigma|\mu)$

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Abstract. We construct quadratic stochastic processes (QSP) (also known as Markov processes of cubic matrices) in continuous and discrete times. These are dynamical systems given by (a fixed type, called σ) stochastic cubic matrices satisfying an analogue of Kolmogorov – Chapman equation (KCE) with respect to a fixed multiplications (called μ) between cubic matrices. The existence of a stochastic (at each time) solution to the KCE provides the existence of a QSP called a QSP of type $(\sigma|\mu)$.

In this paper, our aim is to construct and study trajectories of QSPs for specially chosen notions of stochastic cubic matrices and a wide class of multiplications of such matrices (known as Maksimov's multiplications).

Keywords: quadratic stochastic dynamics; cubic matrix; Kolmogorov – Chapman equation; time

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1. Introduction

The Kolmogorov – Chapman equation (KCE) gives the fundamental relationship between the probability transitions (kernels). Namely, it is known that (see e.g. [18]) if each element of a family of matrices satisfying the KCE is stochastic, then it generates a Markov process. In this paper following [2] we study Markov process of cubic matrices, which is a two-parametric family of cubic stochastic matrices (we fix a notion of stochastic matrix and fix a multiplication rule of cubic matrices) satisfying the KCE. The main question of this study is to describe dynamics of the process given by cubic matrices. This question is very important in the theory of dynamical systems to know future evolution of the system.

Let us give necessary definitions and facts.

1.1. Maksimov's cubic stochastic matrices

Denote $I = \{1, 2, \dots, m\}$. Let $\mathfrak{C}$ be the set of all m^3-dimensional cubic matrices over the field of real numbers [9]. Denote by E_{ijk}, $i, j, k \in I$ the basis cubic matrices in $\mathfrak{C}$, i.e., E_{ijk} is a m^3 cubic matrix whose (i, j, k)th entry is equal to 1 and all other entries are equal to 0.

Following [11] define the following multiplications for basis matrices E_{ijk}:

$$E_{ijk} *_0 E_{lnr} = \delta_{kl}\delta_{jn}E_{ijr}, \tag{1.1}$$

where δ_{kl} is the Kronecker symbol.

Then for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathfrak{C}$ the matrix $A *_0 B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{k=1}^{m} a_{ijk}b_{kjr}. \tag{1.2}$$

The following results of this section are proven in [11] (see also [16] for detailed proofs)

Proposition 1.1. *The algebra of cubic matrices* $(\mathfrak{C}, *_0)$ *is a direct sum of algebras of square matrices.*

Define multiplication:

$$E_{ijk} *_a E_{lnr} = \delta_{kl}E_{ia(j,n)r}, \tag{1.3}$$

where $a\colon I \times I \to I$, $(j, n) \mapsto a(j, n) \in I$, is an arbitrary associative binary operation.

Note that (1.1) is *not* a particular case of (1.3).

Denote by $\mathcal{O}_m$ the set of all associative binary operations on I.

The general formula for the multiplication is the extension of (1.3) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathfrak{C}$ the matrix $A *_a B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{l,n\colon a(l,n)=j} \sum_{k} a_{ilk}b_{knr}.$$

Note that $c_{ijr} = 0$ for j such that $\{l, n : a(l, n) = j\} = \emptyset$.

Lemma 1.1. *The multiplication* (1.3) *is associative for each associative* $a \in \mathcal{O}_m$.

If the equation $a(x,u)=v$ (resp. $a(u,x)=v$) is uniquely solvable for any $u,v\in I$ then the operation a on I has right (resp. left) unique solvability.

Lemma 1.2. *If the operation a on I has right or left unique solvability, then*

$$\sum_{d\in I}\sum_{\substack{j,m:\\ a(j,m)=d}}\gamma_{j,m}=\sum_{j\in I}\sum_{m\in I}\gamma_{j,m}.$$

1.2. Stochasticity

Define several kinds of cubic stochastic matrices (see [11,12]): a cubic matrix $P=(p_{ijk})_{i,j,k=1}^m$ is called

(1, 2)-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{i,j=1}^m p_{ijk}=1,\quad \text{for all } k.$$

(1, 3)-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{i,k=1}^m p_{ijk}=1,\quad \text{for all } j.$$

(2, 3)-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{j,k=1}^m p_{ijk}=1,\quad \text{for all } i.$$

3-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{k=1}^m p_{ijk}=1,\quad \text{for all } i,j.$$

The last one can be also given with respect to first and second index.

Maksimov [11] also defined a twice stochastic matrix: a (2,3)-stochastic cubic matrix is called *twice stochastic* if

$$\sum_{i=1}^m p_{ijk}=\frac{1}{m},\qquad \text{for all}\ \ j,k.$$

Proposition 1.2. $(2,3)$-*stochastic* (*and twice*) *stochastic cubic matrices form a convex semigroup*[1] *with respect to multiplication* (1.3).

Remark 1.1. One also can show that $(1,2)$-stochastic cubic matrices form a convex semigroup. But the collection of $(1,3)$-stochastic matrices does not form a semigroup with respect to multiplication (1.3).

[1] A semigroup is an algebraic structure consisting of a set together with an associative binary operation.

1.3. Quadratic stochastic processes

Denote by $\mathcal{S}$ the set of all possible kinds of stochasticity and denote by $\mathbb{M}$ the set of all possible multiplication rules of cubic matrices.

Let parameters $s \geq 0$, $t \geq 0$, are considered as time.

Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and by $\mathcal{M}^{[s,t]} = \left(P^{[s,t]}_{ijk}\right)_{i,j,k=1}^{m}$ a cubic matrix with two parameters.

Definition 1.1 ([7]). *A family $\{\mathcal{M}^{[s,t]} : s, t \in \mathbb{R}_+\}$ is called a Markov process of cubic matrices (or a quadratic stochastic process (QSP)) of type $(\sigma|\mu)$ if for each time s and t the cubic matrix $\mathcal{M}^{[s,t]}$ is stochastic in sense $\sigma \in \mathcal{S}$ and satisfies the Kolmogorov – Chapman equation (for cubic matrices):*

$$\mathcal{M}^{[s,t]} = \mathcal{M}^{[s,\tau]} *_\mu \mathcal{M}^{[\tau,t]}, \qquad \text{for all } \ 0 \leq s < \tau < t \tag{1.4}$$

with respect to the multiplication $\mu \in \mathbb{M}$.

QSPs arise naturally in the study of biological and physical systems with interactions (see [6]).

We note that this definition of QSP gives an alternative of [12, Definition 3.1.1] (which was initially introduced in [3], [17]). In [13], to each QSP (in sense of [12, Definition 3.1.1]) two kind of marginal processes are associated. Weak ergodicity of such QSP is studied in terms of the marginal processes. See pages 271-272 of [7] for a detailed comparison between our definition of QSP and [12, Definition 3.1.1].

Following [2] assume there are m different types of particles, denoted by $I = \{1, 2, \dots, m\}$ the set of all types and our main aim is to study the asymptotic behavior of the variables $\xi_i^{(t)}$ = number of particles of type i at the time t. The initial state is taken to be fixed and described by $\xi_i^{(0)}$, the numbers of particles of type $i \in I$ in the initial (zero) time. These numbers are assumed finite. Denote by

$$x_i^{(t)} = P^{(t)}(i) = \frac{\xi_i^{(t)}}{\sum_{j=1}^m \xi_j^{(t)}},$$

the fraction of particles of type i at the time t. Thus

$$x^{(t)} = (x_1^{(t)}, \dots, x_m^{(t)}) \in S^{m-1} = \left\{ x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}$$

is a distribution of the system (i.e. the vector describing fractions of all types of particles) at the moment t.

Let $x^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)}) \in S^{m-1}$ be an initial distribution on I. For arbitrary moments of time $s \geq 0$ and $t > 0$, with $s < t$, the matrix $\mathcal{M}^{[s,t]}$ gives the transition probabilities from the distribution $x^{(s)}$ to the distribution $x^{(t)}$. To use the matrix $\mathcal{M}^{[s,t]} = \left(P_{ijk}^{[s,t]}\right)$ we assume that a particle of type $i \in I$ and a particle of type $j \in I$ have interaction at time s, as an interaction process, then with probability $P_{ijk}^{[s,t]}$ a particle of type k appears at time t. The equation (1.4) gives the time-dependent evolution law of the interacting process (dynamical system).

Since we should have $x^{(t)} \in S^{m-1}$, one can consider the following models:

- Consider $P_{ijk}^{[s,t]}$ as the conditional probability $P^{[s,t]}(k|i,j)$ that ith and jth particles (physics) or species (biology) interbred successfully at time s, then they produce an individual k at time t.

  Assume the "parents" ij are independent for any moment of time s, that is
$$P^{(s)}(i,j) = P^{(s)}(i)P^{(s)}(j) = x_i^{(s)}x_j^{(s)},$$
  and we assume that the matrix $\left(P_{ijk}^{[s,t]}\right)$ is 3-stochastic, then the probability distribution $x^{(t)}$ can be found by the formula of the total probability as
$$\begin{aligned} x_k^{(t)} &= \sum_{i,j=1}^{m} P^{(s)}(i,j)P^{[s,t]}(k|i,j) \\ &= \sum_{i,j=1}^{m} P_{ijk}^{[s,t]} x_i^{(s)} x_j^{(s)}, \quad k = 1, \dots, m, \quad 0 \le s < t. \end{aligned} \tag{1.5}$$
  For 1−stochastic and 2−stochastic it can be defined similarly, by replacing the corresponding indices.

- Consider now a physical (biological, chemical) system where there are m types of "particles" or molecules, the set of types is denoted by $I = \{1, \dots, m\}$, and each particle may split to two new ones having types from I. Consider $P_{ijk}^{[s,t]}$ as the conditional probability $P^{[s,t]}(i,j|k)$ that a particle of type k starts splitting at time s and finishes splitting at time t and the result is two particles with ith and jth types.

  Assume $\left(P_{ijk}^{[s,t]}\right)$ is (1,2)-stochastic then $x^{(t)}$ can be defined by
$$x_k^{(t)} = \frac{1}{2}\sum_{i,j=1}^{m}\left(P_{kij}^{[s,t]} + P_{ikj}^{[s,t]}\right)x_j^{(s)}, \qquad k = 1, \dots, m, \quad 0 \le s < t. \tag{1.6}$$
  For (1,3)-stochastic and (2,3)-stochastic cases one can define $x^{(t)}$ similarly by replacing the indices.

Thus finding $P_{ijk}^{[s,t]}$ from the equation (1.4) (at a fixed (σ, μ)) and studying the time-dependent behavior of $P_{ijk}^{[s,t]}$ we can describe the time-dependent evolution of $x^{(t)}$.

1.4. The main problem

To construct QSPs of type $(\sigma|\mu)$, i.e. to solve (1.4). To study the dynamics of such system when $t-s\to+\infty$. In this paper our aim is to construct and study QSPs, for the Maksimov's multiplication corresponding to arbitrary operation a on I which has right (or left) unique solvability.

2. Construction of QSPs

The equation (1.4) has the following form

$$P_{ijr}^{[s,t]}=\sum_{l,n:\,a(l,n)=j}\sum_{k=1}^{m}P_{ilk}^{[s,\tau]}P_{knr}^{[\tau,t]},\quad \forall i,j,r\in I. \tag{2.1}$$

We have to fix a stochasticity of cubic matrices first and solve (2.1) in class of such matrices.

Consider Maksimov's multiplication corresponding to arbitrary operation a on I which has right (resp. left) unique solvability. Denote

$$q_{ir}^{[s,t]}=\sum_{j=1}^{m}P_{ijr}^{[s,t]},\quad \mathbb{Q}^{[s,t]}=\left(q_{ir}^{[s,t]}\right)_{i,r=1}^{m}. \tag{2.2}$$

Then using the solvability condition and Lemma 1.2 we reduce equation (1.4) (i.e. (2.1)) to the following

$$q_{ir}^{[s,t]}=\sum_{k=1}^{m}q_{ik}^{[s,\tau]}q_{kr}^{[\tau,t]},\quad \forall i,r\in I,\quad \text{i.e.,}\quad \mathbb{Q}^{[s,t]}=\mathbb{Q}^{[s,\tau]}\mathbb{Q}^{[\tau,t]}. \tag{2.3}$$

Thus the Kolmogorov – Chapman equation for cubic matrices reduced to the Kolmogorov – Chapman equation for square matrices. Summarizing we have

Proposition 2.1. *Any solution of equation* (1.4) *for a multiplication* μ, *corresponding to operation a satisfying the condition of Lemma* 1.2, *can be given by a solution of the system* (2.3) *with a matrix* $\mathbb{Q}^{[s,t]}=\left(q_{ir}^{[s,t]}\right)_{i,r=1}^{m}$ *which satisfies* (2.2).

Recall that a square matrix $\mathbb{Q}=(q_{ij})_{i,j=1}^{m}$ is called *right stochastic* if

$$q_{ij}\geq 0,\quad \forall i,j=1,\dots,m;\qquad \sum_{j=1}^{m}q_{ij}=1,\quad \forall i=1,\dots,m.$$

Similarly one can define a *left stochastic* matrix being a non-negative real square matrix, with each column summing to 1 and a *doubly stochastic* matrix being a square matrix of non-negative real numbers with each row and column summing to 1.

A family of stochastic matrices $\{\mathbb{Q}^{[s,t]} : s,t \geq 0\}$ is called a *Markov process* if it satisfies the Kolmogorov – Chapman equation (2.3).

The full set of solutions to (2.3) is not known yet. But there is a very wide class of its solutions see [1, 2, 7, 8, 14, 15]. One of these known solutions is the following (non-stochastic, time-homogeneous) matrix:

$$\begin{pmatrix} q_{11}^{[s,t]} & q_{12}^{[s,t]} \\ q_{21}^{[s,t]} & q_{22}^{[s,t]} \end{pmatrix} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}. \tag{2.4}$$

In [15] to construct chains of some algebras, for $m = 2$, a wide class of solutions of (2.3) is presented, many of them are non-stochastic matrices, in general. Let us give a list of families of (left, right, doubly) stochastic square matrices (see [15]), which satisfy the equation (2.3), i.e. they generate interesting Markov processes:

$$\mathbb{Q}_1^{[s,t]} = \begin{pmatrix} g(s) & g(s) \\ 1-g(s) & 1-g(s) \end{pmatrix}, \text{ where } g(s) \in [0,1] \text{ is an arbitrary function;}$$

$$\mathbb{Q}_2^{[s,t]} = \frac{1}{2}\begin{pmatrix} 1+\frac{\Psi(t)}{\Psi(s)} & 1-\frac{\Psi(t)}{\Psi(s)} \\ 1-\frac{\Psi(t)}{\Psi(s)} & 1+\frac{\Psi(t)}{\Psi(s)} \end{pmatrix},$$

where $\Psi(t) > 0$ is an arbitrary decreasing function of $t \geq 0$;

$$\mathbb{Q}_3^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } s \leq t < b, \\ \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } t \geq b, \end{cases} \quad \text{where } b > 0;$$

$$\mathbb{Q}_4^{[s,t]} = \begin{pmatrix} 1 & 0 \\ 1-\frac{\psi(t)}{\psi(s)} & \frac{\psi(t)}{\psi(s)} \end{pmatrix},$$

where $\psi(t) > 0$ is a decreasing function of $t \geq 0$;

$$\mathbb{Q}_5^{[s,t]} = \begin{pmatrix} f(t) & 1-f(t) \\ f(t) & 1-f(t) \end{pmatrix}, \text{ where } f(t) \in [0,1] \text{ is an arbitrary function;}$$

$$\mathbb{Q}_6^{[s,t]}(\lambda,\mu) = \begin{pmatrix} 1-\frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) & \frac{\lambda-2\mu}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) \\ \frac{\lambda}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) & 1-\frac{\lambda}{2(\lambda-\mu)}\left(1-\frac{\theta(t)}{\theta(s)}\right) \end{pmatrix},$$

where λ, μ are real parameters such that $0 < 2\mu < \lambda$ and $\theta(t) > 0$ is an arbitrary decreasing function;

$$\mathbb{Q}_7^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } \ s \le t < a, \\ \begin{pmatrix} g(t) & 1-g(t) \\ g(t) & 1-g(t) \end{pmatrix}, & \text{if } \ t \ge a, \end{cases}$$

where $g(t) \in [0,1]$ is an arbitrary function.

We note that the matrices $\mathbb{Q}_i^{[s,t]}$, $i = 1, \dots, 7$, generate interesting usual Markov processes: some of them independent on time, some depend only on t, but many of them non-homogenously depend on both s, t. Depending on the statistical models of real-world processes one can choose parameter functions (i.e. g, Ψ, ψ, f, θ) and be able then to control the evolution (with respect to time) of such Markov processes.

The following lemma gives a connection between stochastic matrices.

Lemma 2.1. *The matrix* $\mathcal{M}^{[s,t]} = \left(P_{ijk}^{[s,t]}\right)$, *with* $P_{ijk}^{[s,t]} \ge 0$, *is*

- *(1,2)-stochastic (resp. (2,3)-stochastic) if and only if the corresponding by* (2.2) *matrix* $\mathbb{Q}^{[s,t]}$ *is left (resp. right) stochastic.*
- *(1,3)-stochastic if and only if the corresponding matrix* $\mathbb{Q}^{[s,t]}$ *satisfies* $\sum_{i,r=1}^m q_{ir}^{[s,t]} \equiv m$.
- *1-stochastic (resp. 3-stochastic) if and only if the corresponding matrix* $\mathbb{Q}^{[s,t]}$ *satisfies*
$$\sum_{i=1}^m q_{ir}^{[s,t]} \equiv m \quad (\textit{resp.} \ \sum_{r=1}^m q_{ir}^{[s,t]} \equiv m).$$
- *2-stochastic iff* $q_{ir}^{[s,t]} \equiv 1$.

Proof. It is consequence of the equality (2.2). □

Proposition 2.2. *If* $m > 1$ *and the operation* a *on* I *has right or left unique solvability, then equation* (1.4) *does not have any solution in class of* i*-stochastic (for any* $i = 1, 2, 3$*) cubic matrices.*

Proof. We prove in case $i = 1$ (the cases $i = 2, 3$ are similar). Assume there is a solution $\mathcal{M}^{[s,t]} = \left(P_{ijk}^{[s,t]}\right)$, which is 1-stochastic, i.e.,

$$P_{ijk}^{[s,t]} \ge 0,$$
$$\sum_{i=1}^m P_{ijk}^{[s,t]} = 1,$$

for all j, k, $0 \le s < t$. Then by Lemma 2.1 for corresponding square matrix $\mathbb{Q}^{[s,t]}$ we should have $\sum_{i=1}^m q_{ir}^{[s,t]} \equiv m$. Moreover, by Proposition 2.1 the matrix $\mathbb{Q}^{[s,t]}$ should satisfy (2.3), which is impossible, because

$$m = \sum_{i=1}^m q_{ir}^{[s,t]} = \sum_{i=1}^m \sum_{k=1}^m q_{ik}^{[s,\tau]} q_{kr}^{[\tau,t]} = \sum_{k=1}^m m q_{kr}^{[\tau,t]} = m^2 > m.$$

$\square$

Remark 2.1. In case $m = 1$ the equation (1.4) becomes the following functional equation

$$P^{[s,t]} = P^{[s,\tau]} P^{[\tau,t]}, \tag{2.5}$$

where unknown function is $P^{[s,t]} = P_{111}^{[s,t]}$.

This equation is known as Cantor's second equation which has a very rich family of solutions:

(a) $P^{[s,t]} \equiv 0$;

(b) $P^{[s,t]} = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$;

(c)

$$P^{[s,t]} = \begin{cases} 1, & \text{if } \ s \le t < c, \\ 0, & \text{if } \ t \ge c. \end{cases} \qquad \text{where } \ c > 0.$$

Remark 2.2.

1) According to Proposition 2.2 we do not have i-stochastic solutions.

2) As it was mentioned above multiplication of two $(1,3)$-stochastic matrices may be non $(1,3)$-stochastic. Therefore it is not clear existence of a $(1,3)$-stochastic solution to (1.4) (i.e. (2.1)). Below we shall construct some examples of such solution.

3) By above mentioned results one can see that $(1,2)$-stochasticity and $(2,3)$-stochasticity play a symmetric role. Therefore below we find only $(1,2)$-stochastic solutions of (2.1).

Condition 1. For definiteness let us take $I = \{0, 1, 2, \dots, m-1\}$ as a group with respect to operation a, defined by $a(i,j) = (i+j)(\operatorname{mod} m)$. Then it is easy to see that a is uniquely solvable.

Under this condition the elements of the matrix $\mathcal{M}^{[s,t]}$ can be renumbered as $\mathcal{M}^{[s,t]} = \left(P_{ijk}^{[s,t]}\right)_{i,j,k=0}^{m-1}$.

For convenience of the writing of this cubic matrix we introduce square matrix

$$\mathcal{M}_i^{[s,t]} = \left(P_{ijk}^{[s,t]}\right)_{j,k=0}^{m-1}, \quad i = 0, 1, \dots, m-1.$$

Then the cubic matrix can be written as

$$\mathcal{M}^{[s,t]} = \left(\mathcal{M}_0^{[s,t]}|\mathcal{M}_1^{[s,t]}|\dots|\mathcal{M}_{m-1}^{[s,t]}\right).$$

The equation (2.1) can be written as

$$P_{ijr}^{[s,t]} = \sum_{k=0}^{m-1}\left(\sum_{l=0}^{j} P_{ilk}^{[s,\tau]}P_{k(j-l)r}^{[\tau,t]} + \sum_{l=1}^{m-j-1} P_{i(j+l)k}^{[s,\tau]}P_{k(m-l)r}^{[\tau,t]}\right), \ \forall i,j,r \in I. \quad (2.6)$$

This is a non-linear system of functional equations with m^3 unknown two-variable functions $P_{ijr}^{[s,t]}$. Trivial solution is $P_{ijr}^{[s,t]} = 1/m^2$, $\forall i,j,r \in I$, $0 \le s < t$. The analysis of the system (2.6) is difficult. Therefore below we shall mainly consider the case $m = 2$.

2.1. $(1,3)$-stochastic solutions

Now we construct QSPs of type $(13|a)$, where 13 means $(1,3)$-stochasticity and a means that we are considering multiplication (1.3).

For simplicity let us consider the case $m = 2$. Write a cubic matrix $\mathcal{M}^{[s,t]}$ in the following convenient form:

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} P_{000}^{[s,t]} & P_{001}^{[s,t]} & P_{100}^{[s,t]} & P_{101}^{[s,t]} \\ P_{010}^{[s,t]} & P_{011}^{[s,t]} & P_{110}^{[s,t]} & P_{111}^{[s,t]} \end{array}\right). \quad (2.7)$$

This matrix generates QSP of type $(13|a)$ iff

$$\begin{aligned} &P_{000}^{[s,t]} + P_{001}^{[s,t]} + P_{100}^{[s,t]} + P_{101}^{[s,t]} = 1, \\ &P_{010}^{[s,t]} + P_{011}^{[s,t]} + P_{110}^{[s,t]} + P_{111}^{[s,t]} = 1. \\ &P_{i0j}^{[s,t]} + P_{i1j}^{[s,t]} = q_{ij}^{[s,t]}, \quad i,j = 0,1. \end{aligned} \quad (2.8)$$

In addition to these conditions by Condition 1 the equation (1.4) becomes

$$\begin{cases} P_{i0j}^{[s,t]} = \sum_{k=0}^{1}\left(P_{i0k}^{[s,\tau]}P_{k0j}^{[\tau,t]} + P_{i1k}^{[s,\tau]}P_{k1j}^{[\tau,t]}\right) \\ P_{i1j}^{[s,t]} = \sum_{k=0}^{1}\left(P_{i0k}^{[s,\tau]}P_{k1j}^{[\tau,t]} + P_{i1k}^{[s,\tau]}P_{k0j}^{[\tau,t]}\right), \quad i,j = 0,1. \end{cases} \quad (2.9)$$

In general it is difficult to solve the system (2.9). Let us solve it in class of functions satisfying

$$P_{000}^{[s,t]} = P_{001}^{[s,t]} = P_{100}^{[s,t]} \equiv f(s,t).$$

Assume also that matrix $\mathbb{Q}^{[s,t]}$ is left *and* right stochastic.

Using these assumptions and (2.8) from (2.9) we get

$$\begin{aligned} f(s,t) &= P_{000}^{[s,t]} \\ &= P_{000}^{[s,\tau]}P_{000}^{[\tau,t]} + P_{010}^{[s,\tau]}P_{010}^{[\tau,t]} + P_{001}^{[s,\tau]}P_{100}^{[\tau,t]} + P_{011}^{[s,\tau]}P_{110}^{[\tau,t]} \\ &= 4f(s,\tau)f(\tau,t) - (q_{00}^{[s,\tau]} + q_{01}^{[s,\tau]})f(\tau,t) \\ &\quad - q_{00}^{[\tau,t]} + q_{10}^{[\tau,t]})f(s,\tau) + q_{00}^{[s,\tau]}q_{00}^{[\tau,t]} + q_{01}^{[s,\tau]}q_{10}^{[\tau,t]}. \end{aligned}$$

Since the matrix $\mathbb{Q}^{[s,t]}$ satisfies (2.3) we have

$$q_{00}^{[s,\tau]}q_{00}^{[\tau,t]} + q_{01}^{[s,\tau]}q_{10}^{[\tau,t]} = q_{00}^{[s,t]}$$

and since the matrix is left and right stochastic we get

$$f(s,t) = 4f(s,\tau)f(\tau,t) - f(\tau,t) - f(s,\tau) + q_{00}^{[s,t]}. \tag{2.10}$$

Note that $f(s,t) \leq 1/3$.

Take the matrix $\mathbb{Q}^{[s,t]} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Thus $q_{00}^{[s,t]} \equiv 1/2$ then denoting $h(s,t) = 4f(s,t) - 1$ the equation (2.10) can be written as

$$h(s,t) = h(s,\tau)h(\tau,t). \tag{2.11}$$

As Cantor's second equation this equation has solutions:

(a) $h(s,t) \equiv 0$;

(b) $h(s,t) = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$;

(c)

$$h(s,t) = \begin{cases} 1, & \text{if } \ s \leq t < c, \\ 0, & \text{if } \ t \geq c. \end{cases} \quad \text{where } \ c > 0.$$

Now for each solutions we give corresponding QSP:

(a′) The functional equation (2.10) has solutions $f(s,t) \equiv 1/4$. Thus the cubic matrices (independent on time)

$$\mathcal{M}_1^{[s,t]} = \left(\begin{array}{cc|cc} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array}\right). \tag{2.12}$$

generate a QSP of type $(13|a)$.

(b′) The functional equation (2.10) has solutions $f(s,t) = \frac{1}{4}\left(\frac{\Phi(t)}{\Phi(s)} + 1\right)$. Thus the cubic matrices

$$\mathcal{M}_2^{[s,t]} = \left(\begin{array}{cc|cc} f(s,t) & f(s,t) & f(s,t) & 1-3f(s,t) \\ \frac{1}{2} - f(s,t) & \frac{1}{2} - f(s,t) & \frac{1}{2} - f(s,t) & 3f(s,t) - \frac{1}{2} \end{array}\right) \tag{2.13}$$

generate a QSP of type $(13|a)$ iff

$$\frac{1}{6} \leq f(s,t) = \frac{1}{4}\Big(\frac{\Phi(t)}{\Phi(s)} + 1\Big) \leq \frac{1}{3},$$

i.e.

$$-\frac{1}{3} \leq \frac{\Phi(t)}{\Phi(s)} \leq \frac{1}{3}. \tag{2.14}$$

Note that the condition (2.14) can be satisfied for a function Φ when time is discrete, i.e., $t \in \mathbb{N}$. Then for example we can take $\Phi(n) = 3^{-n}$.

(c$'$) In case (c) we have

$$f(s,t) = \begin{cases} 1/2, & \text{if } \ s \leq t < c \\ 1/4, & \text{if } \ t \geq c. \end{cases}$$

But this does not satisfy condition $f(s,t) \leq 1/3$.

Summarizing we have

Proposition 2.3. *The matrices $\mathcal{M}_1^{[s,t]}$ defined in* (2.12) *generate a QSP of type* $(13|a)$. *The matrices $\mathcal{M}_2^{[n,m]}$, $n, m \in \mathbb{N}$, $n < m$ defined in* (2.13) *generate a discrete-time QSP of type* $(13|a)$.

2.2. $(1,2)$-stochastic solutions

Now we construct QSPs of type $(12|a)$, where 12 means $(1,2)$-stochasticity and a means that we are considering multiplication (1.3).

The matrix (2.7) generates QSP of type $(12|a)$ iff

$$\begin{aligned} &P_{000}^{[s,t]} + P_{010}^{[s,t]} + P_{100}^{[s,t]} + P_{110}^{[s,t]} = 1, \\ &P_{001}^{[s,t]} + P_{011}^{[s,t]} + P_{101}^{[s,t]} + P_{111}^{[s,t]} = 1, \\ &P_{i0j}^{[s,t]} + P_{i1j}^{[s,t]} = q_{ij}^{[s,t]}, \quad i,j = 0,1. \end{aligned} \tag{2.15}$$

Note that for any left stochastic matrix $\mathbb{Q}^{[s,t]}$ the conditions (2.15) are satisfied. Therefore it suffices to solve the equation (2.9). Assuming

$$P_{000}^{[s,t]} = P_{001}^{[s,t]} = P_{100}^{[s,t]} \equiv g(s,t)$$

we get

$$\begin{aligned} g(s,t) &= P_{000}^{[s,t]} \\ &= P_{000}^{[s,\tau]} P_{000}^{[\tau,t]} + P_{010}^{[s,\tau]} P_{010}^{[\tau,t]} + P_{001}^{[s,\tau]} P_{100}^{[\tau,t]} + P_{011}^{[s,\tau]} P_{110}^{[\tau,t]} \\ &= 4g(s,\tau)g(\tau,t) - (q_{00}^{[s,\tau]} + q_{01}^{[s,\tau]})g(\tau,t) \\ &\quad - (q_{00}^{[\tau,t]} + q_{10}^{[\tau,t]})g(s,\tau) + q_{00}^{[s,\tau]} q_{00}^{[\tau,t]} + q_{01}^{[s,\tau]} q_{10}^{[\tau,t]}. \end{aligned} \tag{2.16}$$

Take the matrix $\mathbb{Q}^{[s,t]} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. In this case the equation (2.16) has solutions as (a′)–(c′).

Now for each solutions we give corresponding QSP:

(a″) The functional equation (2.16) has solutions $g(s,t) \equiv 1/4$. Thus $(1,2)$-stochastic cubic matrix has the form

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} 1/4 & 1/4 & 1/4 & F(s,t) \\ 1/4 & 1/4 & 1/4 & 1/2 - F(s,t) \end{array}\right). \tag{2.17}$$

where $F(s,t)$ should satisfy the equation

$$\begin{aligned} F(s,t) = P_{101}^{[s,t]} &= P_{100}^{[s,\tau]} P_{001}^{[\tau,t]} + P_{110}^{[s,\tau]} P_{011}^{[\tau,t]} + P_{101}^{[s,\tau]} P_{101}^{[\tau,t]} + P_{111}^{[s,\tau]} P_{111}^{[\tau,t]} \\ &= 2F(s,\tau)F(\tau,t) - \frac{1}{2}F(s,\tau) - \frac{1}{2}F(\tau,t) + \frac{3}{8}. \end{aligned}$$

Denoting $G(s,t) = 4F(s,t) - 1$ one can rewrite this equation in the following form

$$G(s,t) = \frac{1}{2}G(s,\tau)G(\tau,t).$$

The last equation has solutions $G(s,t) \equiv 0$, $G(s,t) = \frac{2\psi(t)}{\psi(s)}$ (for any $\psi \neq 0$) and

$$G(s,t) = \begin{cases} 2, & \text{if } 0 \le s < t \le a \\ 0, & \text{if } t > a. \end{cases}$$

To these solutions correspond (by (2.17)) the following QSPs of type $(12|a)$:

$$\mathcal{M}_3^{[s,t]} = \left(\begin{array}{cc|cc} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{array}\right). \tag{2.18}$$

$$\mathcal{M}_4^{[s,t]} = \left(\begin{array}{cc|cc} 1/4 & 1/4 & 1/4 & \frac{1}{4} + \frac{\psi(t)}{2\psi(s)} \\ 1/4 & 1/4 & 1/4 & \frac{1}{4} - \frac{\psi(t)}{2\psi(s)} \end{array}\right), \tag{2.19}$$

where ψ is such that $-1/2 \le \psi(t)/\psi(s) \le 1/2$. This condition can be satisfied for a function ψ when time is discrete, i.e., $t \in \mathbb{N}$. Then for example we can take $\psi(n) = 2^{-n}$.

The case $G(s,t) = 2$ does not define a QSP, because in this case $F(s,t) > 1/2$.

(b″) For the case $g(s,t) = \frac{1}{4}\left(\frac{\varphi(t)}{\varphi(s)} + 1\right)$ (where $\varphi \neq 0$ is an arbitrary function) in (2.16) the $(1,2)$-stochastic cubic matrix has the form

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} g(s,t) & g(s,t) & g(s,t) & L(s,t) \\ \frac{1}{2} - g(s,t) & \frac{1}{2} - g(s,t) & \frac{1}{2} - g(s,t) & \frac{1}{2} - L(s,t) \end{array}\right), \tag{2.20}$$

where

$$g(s,t)=\frac{1}{4}\left(\frac{\varphi(t)}{\varphi(s)}+1\right)\quad\text{with}\quad -1\leq\frac{\varphi(t)}{\varphi(s)}\leq 1.$$

In particular, this condition is satisfied for a positive and decreasing function φ. Here the function $L(s,t)$ should satisfy the following equation

$$L(s,t)=2L(s,\tau)L(\tau,t)-\frac{1}{2}(L(s,\tau)+L(\tau,t))+\frac{1}{4}+\frac{1}{2}g(s,t). \tag{2.21}$$

Note that the equation (2.21) has solution $L(s,t)=g(s,t)=\frac{1}{4}\left(\frac{\varphi(t)}{\varphi(s)}+1\right)$. We do not know any other solution of (2.21).

Thus

$$\mathcal{M}_5^{[s,t]}=\left(\begin{array}{cc|cc} g(s,t) & g(s,t) & g(s,t) & g(s,t)\\ \frac{1}{2}-g(s,t) & \frac{1}{2}-g(s,t) & \frac{1}{2}-g(s,t) & \frac{1}{2}-g(s,t)\end{array}\right), \tag{2.22}$$

where $g(s,t)=\frac{1}{4}\left(\frac{\varphi(t)}{\varphi(s)}+1\right)$ with $-1\leq\frac{\varphi(t)}{\varphi(s)}\leq 1$ generates a QSP of type $(12|a)$.

(c$''$) In this case

$$g(s,t)=\begin{cases}1/2, & \text{if } s\leq t<c\\ 1/4, & \text{if } t\geq c.\end{cases}$$

Then corresponding matrix is

$$\mathcal{M}_6^{[s,t]}=\begin{cases}\left(\begin{array}{cc|cc}1/2 & 1/2 & 1/2 & 1/2\\ 0 & 0 & 0 & 0\end{array}\right), & \text{if } s\leq t<c\\[2ex] \left(\begin{array}{cc|cc}1/4 & 1/4 & 1/4 & 1/4\\ 1/4 & 1/4 & 1/4 & 1/4\end{array}\right), & \text{if } t\geq c.\end{cases} \tag{2.23}$$

Summarizing we have

Proposition 2.4. *The matrices $\mathcal{M}_i^{[s,t]}$, $i=3,5,6$ defined above generate QSPs of type $(12|a)$. The matrices $\mathcal{M}_4^{[n,m]}$, $n,m\in\mathbb{N}$, $n<m$ generate a discrete-time QSP of type $(12|a)$.*

Take now the matrix $\mathbb{Q}^{[s,t]}=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$ then

$$g(s,t)=4g(s,\tau)g(\tau,t)-g(s,\tau). \tag{2.24}$$

It is easy to see that this equation has solution $g(s,t)\equiv 1/2$ (we do not know any other solution). But this solution does not define a QSP, because from $P_{000}^{[s,t]}+P_{010}^{[s,t]}=q_{00}^{[s,t]}=0$ it follows that $P_{000}^{[s,t]}=0\neq 1/2$.

3. An example when Condition 1 is not satisfied

In this section we consider an operation a on $I = \{1, 2, \dots, m\}$ which is not uniquely solvable. Consider binary operation $a(i,j) = \max\{i,j\}$. It is not uniquely solvable, in general. Indeed, for $m \geq 2$, the equation $\max\{x, m\} = m$ has many solutions: $x = 1, 2, \dots, m$.

Let σ be a fixed stochasticity of cubic matrices then the QSP corresponding to max operation is denoted as type $(\sigma|\max)$. Here we give some examples of such QSP.

For simplicity we take $m = 2$ and solve the equation (1.4) for matrix $\mathcal{M}^{[s,t]} = \left(a_{ijk}^{[s,t]}\right)_{i,j,k=1}^{2}$.

In the case of multiplication corresponding to the binary operation $a(i,j) = \max\{i,j\}$ the equation (1.4) is in the following form

$$\begin{cases} a_{111}^{[s,t]} = a_{111}^{[s,\tau]}a_{111}^{[\tau,t]} + a_{112}^{[s,\tau]}a_{211}^{[\tau,t]} \\ a_{112}^{[s,t]} = a_{111}^{[s,\tau]}a_{112}^{[\tau,t]} + a_{112}^{[s,\tau]}a_{212}^{[\tau,t]} \\ a_{211}^{[s,t]} = a_{211}^{[s,\tau]}a_{111}^{[\tau,t]} + a_{212}^{[s,\tau]}a_{211}^{[\tau,t]} \\ a_{212}^{[s,t]} = a_{211}^{[s,\tau]}a_{112}^{[\tau,t]} + a_{212}^{[s,\tau]}a_{212}^{[\tau,t]} \\ a_{121}^{[s,t]} = a_{111}^{[s,\tau]}a_{121}^{[\tau,t]} + a_{112}^{[s,\tau]}a_{221}^{[\tau,t]} + a_{122}^{[s,\tau]}a_{221}^{[\tau,t]} + a_{121}^{[s,\tau]}a_{111}^{[\tau,t]} + a_{121}^{[s,\tau]}a_{121}^{[\tau,t]} + a_{122}^{[s,\tau]}a_{211}^{[\tau,t]} \\ a_{122}^{[s,t]} = a_{111}^{[s,\tau]}a_{122}^{[\tau,t]} + a_{121}^{[s,\tau]}a_{112}^{[\tau,t]} + a_{121}^{[s,\tau]}a_{122}^{[\tau,t]} + a_{112}^{[s,\tau]}a_{222}^{[\tau,t]} + a_{122}^{[s,\tau]}a_{212}^{[\tau,t]} + a_{122}^{[s,\tau]}a_{222}^{[\tau,t]} \\ a_{221}^{[s,t]} = a_{211}^{[s,\tau]}a_{121}^{[\tau,t]} + a_{212}^{[s,\tau]}a_{221}^{[\tau,t]} + a_{221}^{[s,\tau]}a_{111}^{[\tau,t]} + a_{221}^{[s,\tau]}a_{121}^{[\tau,t]} + a_{222}^{[s,\tau]}a_{211}^{[\tau,t]} + a_{222}^{[s,\tau]}a_{221}^{[\tau,t]} \\ a_{222}^{[s,t]} = a_{212}^{[s,\tau]}a_{222}^{[\tau,t]} + a_{221}^{[s,\tau]}a_{112}^{[\tau,t]} + a_{221}^{[s,\tau]}a_{122}^{[\tau,t]} + a_{222}^{[s,\tau]}a_{212}^{[\tau,t]} + a_{222}^{[s,\tau]}a_{222}^{[\tau,t]} + a_{211}^{[s,\tau]}a_{122}^{[\tau,t]} \end{cases} \tag{3.1}$$

Denoting

$$b_{ij}^{[s,t]} = a_{i1j}^{[s,t]} + a_{i2j}^{[s,t]}, \quad B^{[s,t]} = \left(b_{ij}^{[s,t]}\right) \tag{3.2}$$

one can reduce the system (3.1) to the following one

$$\begin{cases} b_{11}^{[s,t]} = b_{11}^{[s,\tau]}b_{11}^{[\tau,t]} + b_{12}^{[s,\tau]}b_{21}^{[\tau,t]} \\ b_{12}^{[s,t]} = b_{11}^{[s,\tau]}b_{12}^{[\tau,t]} + b_{12}^{[s,\tau]}b_{22}^{[\tau,t]} \\ b_{21}^{[s,t]} = b_{21}^{[s,\tau]}b_{11}^{[\tau,t]} + b_{22}^{[s,\tau]}b_{21}^{[\tau,t]} \\ b_{22}^{[s,t]} = b_{21}^{[s,\tau]}b_{12}^{[\tau,t]} + b_{22}^{[s,\tau]}b_{22}^{[\tau,t]}. \end{cases} \tag{3.3}$$

Note that 1–4 equations of the system (3.1) can be solved independently from 5–8 equations. Therefore if we solve system of 1–4 equations of (3.1) and solve system (3.3) then by (3.2) we can find all unknown functions of (3.1). Let us realize this argument.

Denote $c_{ij}^{[s,t]}=a_{i1j}^{[s,t]}$, $C^{[s,t]}=\left(c_{ij}^{[s,t]}\right)$ then 1–4 equations of the system (3.1) is

$$\begin{cases} c_{11}^{[s,t]}=c_{11}^{[s,\tau]}c_{11}^{[\tau,t]}+c_{12}^{[s,\tau]}c_{21}^{[\tau,t]} \\ c_{12}^{[s,t]}=c_{11}^{[s,\tau]}c_{12}^{[\tau,t]}+c_{12}^{[s,\tau]}c_{22}^{[\tau,t]} \\ c_{21}^{[s,t]}=c_{21}^{[s,\tau]}c_{11}^{[\tau,t]}+c_{22}^{[s,\tau]}c_{21}^{[\tau,t]} \\ c_{22}^{[s,t]}=c_{21}^{[s,\tau]}c_{12}^{[\tau,t]}+c_{22}^{[s,\tau]}c_{22}^{[\tau,t]}. \end{cases} \tag{3.4}$$

Both systems of equations (3.3) and (3.4) are Kolmogorov – Chapman equations for square matrices. Using known solutions for these equations, (for example, $\mathbb{Q}_i$, $i=1,2,\dots,7$ introduced in the previous section) one can give concrete solutions of the system (3.1). Namely, if $B^{[s,t]}=\left(b_{ij}^{[s,t]}\right)$ is a solution to (3.3) and $C^{[s,t]}=\left(c_{ij}^{[s,t]}\right)$ is a solution to (3.4) then corresponding solution to the system (3.1) is

$$\mathcal{M}_7^{[s,t]}=\left(\begin{array}{cc|cc} c_{11}^{[s,t]} & c_{12}^{[s,t]} & c_{21}^{[s,t]} & c_{22}^{[s,t]} \\ b_{11}^{[s,t]}-c_{11}^{[s,t]} & b_{12}^{[s,t]}-c_{12}^{[s,t]} & b_{21}^{[s,t]}-c_{21}^{[s,t]} & b_{22}^{[s,t]}-c_{22}^{[s,t]} \end{array}\right). \tag{3.5}$$

Theorem 3.1. *Let $B^{[s,t]}=\left(b_{ij}^{[s,t]}\right)$ be a solution to (3.3) and $C^{[s,t]}=\left(c_{ij}^{[s,t]}\right)$ be a solution to (3.4) with $c_{ij}^{[s,t]}\in[0,1]$ and $b_{ij}^{[s,t]}-c_{ij}^{[s,t]}\in[0,1]$ for any $i,j=1,2$, $0\le s<t$ then the family of matrices $\mathcal{M}_7^{[s,t]}$ given in (3.5) is a QSP of type*

- $(12|\max)$ *iff $B^{[s,t]}$ is left stochastic for any $0\le s<t$.*
- $(13|\max)$ *iff $B^{[s,t]}$ (resp. $C^{[s,t]}$) with non negative elements with sum of all elements equals to 2 (resp. 1).*
- $(23|\max)$ *iff $B^{[s,t]}$ is right stochastic for any $0\le s<t$.*
- $(1|\max)$ *iff $B^{[s,t]}$ (resp. $C^{[s,t]}$) with non negative elements with sum of all elements of each column equals to 2 (resp. left stochastic).*
- $(2|\max)$ *never.*
- $(3|\max)$ *iff $B^{[s,t]}$ (resp. $C^{[s,t]}$) with non negative elements with sum of all elements of each row equals to 2 (resp. right stochastic).*

Proof. All types (expect the type $(2|\max)$) follow from the definitions of the corresponding stochasticity. In the case $(2|\max)$ it is necessary that $b_{ij}^{[s,t]}\equiv 1$, but it is easy to see that such quadratic matrix $B^{[s,t]}$ does not satisfy equation (3.3). □

4. Dynamical systems of QSPs

For QSPs generated by $\mathcal{M}_i^{[s,t]}$, $i = 1, \dots, 7$ using (1.5), (1.6), let us give the time behavior of the distribution $x^{(t)} = (x_0^{(t)}, x_1^{(t)}) \in S^1$. Fix $s \geq 0$ and take a vector $x^{(s)} = (x_0^{(s)}, x_1^{(s)}) \in S^1$.

Case $\mathcal{M}_1^{[s,t]}$ and $\mathcal{M}_3^{[s,t]}$. By formula (1.6) independently on the vector $x^{(s)}$, for *any* $t > s$, we get

$$x_0^{(t)} = \frac{1}{2}, \quad x_1^{(t)} = \frac{1}{2}.$$

Thus the time behavior of $x^{(t)}$ is clear: start process at time s with an arbitrary initial distribution vector $x^{(s)}$ then as soon as the time t turns on the distribution of the system goes to the distribution $(1/2, 1/2)$ and this distribution remains stable during all time $t > s$.

Case $\mathcal{M}_2^{[s,t]}$. By formula (1.6), for fixed $s \geq 0$, given vector $x^{(s)}$ and *any* $t > s$, we get

$$x_0^{(t)} = \left(\frac{1}{2} + \frac{\Phi(t)}{4\Phi(s)}\right) x_0^{(s)} + \left(\frac{1}{2} - \frac{\Phi(t)}{4\Phi(s)}\right) x_1^{(s)},$$

$$x_1^{(t)} = \left(\frac{1}{2} - \frac{\Phi(t)}{4\Phi(s)}\right) x_0^{(s)} + \left(\frac{1}{2} + \frac{\Phi(t)}{4\Phi(s)}\right) x_1^{(s)}.$$

The time behavior of $x^{(t)}$ depends on function Φ (which by our assumption satisfies $-1/3 \leq \Phi(t)/\Phi(s) \leq 1/3$). If for example, Φ is such that

$$\lim_{t-s\to\infty} \frac{\Phi(t)}{4\Phi(s)} = \omega, \quad \text{with} \quad \omega \in \left[-\frac{1}{12}, \frac{1}{12}\right]. \tag{4.1}$$

Then

$$\lim_{t\to\infty} x_0^{(t)} = \left(\frac{1}{2} + \omega\right) x_0^{(s)} + \left(\frac{1}{2} - \omega\right) x_1^{(s)},$$

$$\lim_{t\to\infty} x_1^{(t)} = \left(\frac{1}{2} - \omega\right) x_0^{(s)} + \left(\frac{1}{2} + \omega\right) x_1^{(s)}.$$

In case when the limit (4.1) does not exists then limit of $x^{(t)}$ does not exist too.

Case $\mathcal{M}_4^{[s,t]}$. In this case we have

$$x_0^{(t)} = \frac{1}{2} x_0^{(s)} + \left(\frac{1}{2} + \frac{\psi(t)}{4\psi(s)}\right) x_1^{(s)},$$

$$x_1^{(t)} = \frac{1}{2} x_0^{(s)} + \left(\frac{1}{2} - \frac{\psi(t)}{4\psi(s)}\right) x_1^{(s)}.$$

As previous case, the time behavior of $x^{(t)}$ depends on function ψ (which by our assumption satisfies $-1/2 \leq \psi(t)/\psi(s) \leq 1/2$).

Case $\mathcal{M}_5^{[s,t]}$. In this case independently on the initial state vector $x^{(s)}$ we obtain

$$x_0^{(t)} = \frac{1}{2} + \frac{\varphi(t)}{4\varphi(s)},$$

$$x_1^{(t)} = \frac{1}{2} - \frac{\varphi(t)}{4\varphi(s)}.$$

This is an interesting dynamical system, because at each initial (fixed) time s the system does not depend on the initial state $x^{(s)}$ of the system. The trajectory only depends on the initial time itself and the time behavior of $x^{(t)}$ depends on function φ (which by our assumption satisfies $-1 \leq \varphi(t)/\varphi(s) \leq 1$).

Case $\mathcal{M}_6^{[s,t]}$. In this case independently on the initial state vector $x^{(s)}$ we obtain

$$x_0^{(t)} = 1 - x_1^{(t)} = \begin{cases} \frac{3}{4}, & \text{if } \ 0 \leq s < t < a \\ \frac{1}{2}, & \text{if } \ t \geq a. \end{cases}$$

Thus we get a discontinuous (with respect to time) dynamical system, the trajectory has limit 1/2.

Case $\mathcal{M}_7^{[s,t]}$. Consider QSP of type $(3|\max)$ (other cases can be considered similarly). By (1.5) and Theorem 3.1 we get

$$x_0^{(t)} = c_{11}^{[s,t]}(x_0^{(s)})^2 + \left(b_{11}^{[s,t]} - c_{11}^{[s,t]} + c_{21}^{[s,t]}\right) x_0^{(s)} x_1^{(s)} + \left(b_{21}^{[s,t]} - c_{21}^{[s,t]}\right)(x_1^{(s)})^2,$$

$$x_1^{(t)} = c_{12}^{[s,t]}(x_0^{(s)})^2 + \left(b_{12}^{[s,t]} - c_{12}^{[s,t]} + c_{22}^{[s,t]}\right) x_0^{(s)} x_1^{(s)} + \left(b_{22}^{[s,t]} - c_{22}^{[s,t]}\right)(x_1^{(s)})^2.$$

This is a quadratic continuous time dynamical system. The behavior of $x^{(t)}$ depends on the matrix $\mathcal{M}_7^{[s,t]}$. One can choose this matrix to make the behavior of the dynamical system as rich as needed (see [4], [5], [10], [12] for some examples of quadratic dynamical systems and their applications).

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