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Inverse Problem for an Equation of Mixed Parabolic–Hyperbolic Type with a Bessel Operator

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Abstract—For an equation of the mixed parabolic–hyperbolic type with a Bessel operator, we study the inverse problem associated with the search for the unknown right-hand side. By separation of variables, the problem is reduced to solving ordinary differential equations for the coefficients of the Fourier–Bessel series expansions of the unknown functions in orthonormal Bessel functions of the first kind and zero order. A criterion for the uniqueness and existence of a solution of the problem is established.

Keywords: *inverse problem, Fourier–Bessel series, eigenvalue, eigenfunction, uniqueness, existence*

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Direct and inverse problems for mixed type equations are not studied so well as similar problems for classical equations. Nevertheless, such problems are relevant from the viewpoint of applications. For example, the electromagnetic field strength in a homogeneous medium satisfies the wave equation in the case of low conductivity of the medium, but in the case of relatively high conductivity, when the displacement currents can be neglected compared with conduction currents, the electromagnetic field strength satisfies the heat equation [1, pp. 443–447 of the Russian edition]. Another example is given by the following phenomenon in gas dynamics: when modeling gas motion processes in a closed channel with porous walls, the gas motion is described by the wave equation in the channel and by the diffusion equation outside the channel [2, 3]. There are many such examples.

Direct problems for mixed parabolic–hyperbolic type equations were studied in [4–8]. Inverse problems of determining the right-hand side or the initial function in initial–boundary value problems for mixed parabolic–hyperbolic type equations in a rectangular domain were considered in the papers [9–12], where criteria for the uniqueness and existence of solutions of inverse problems were established based on the spectral method. In the present paper, we study direct and inverse problems related to finding a solution of an initial–boundary value problem for mixed parabolic–hyperbolic type equations and an unknown right-hand side of the equation in a cylindrical domain. When studying the problem in question, we need the Bessel function and the conditions for the convergence of the Fourier–Bessel series [13].

Various inverse problems of determining the coefficients and right-hand sides of certain types of second-order partial differential equations, i.e., parabolic, hyperbolic, and elliptic equations, have been studied in many papers (see, e.g., the monographs [14–19] and the extensive bibliography therein). Inverse problems of reconstructing the kernel in hyperbolic integro-differential equations were studied in [20, 21]. Numerical methods for finding the coefficients of equations can be found in the papers [22, 23] (see also the bibliography therein).

1. STATEMENT OF THE PROBLEM

In the domain $G := \{(\rho,t) \mid 0 < \rho < 1,\ -\alpha < t < \beta\}$, consider the differential equation of the mixed parabolic–hyperbolic type

$$\theta(t)d\frac{\partial u}{\partial t} + \theta(-t)c^2\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \rho^2} + f(\rho), \quad (\rho,t) \in G, \tag{1}$$

with the boundary conditions

$$\left[\rho\frac{\partial}{\partial\rho}u(\rho,t)\right]_{\rho=0} = 0, \quad u|_{\rho=1} = 0, \quad -\alpha \le t \le \beta, \tag{2}$$

the matching conditions

$$u(\rho,+0) = u(\rho,-0), \quad u_t(\rho,+0) = u_t(\rho,-0), \quad 0 \le \rho \le 1, \tag{3}$$

at $t = 0$, and the initial condition

$$u(\rho,-\alpha) = \varphi(\rho), \quad 0 \le \rho \le 1. \tag{4}$$

Here $\theta(t)$ is the Heaviside theta function, and α, β, d, and c are given positive numbers. We assume that the functions $f(\rho)$ and $\varphi(\rho)$ are sufficiently smooth.

Note that the earlier-described problems in a cylindrical domain in the case of axial symmetry can be reduced to equations of the form (1).

Relations (1)–(4) are a direct problem; i.e., if the functions $f(\rho)$ and $\varphi(\rho)$ and the constants d, c, α, and β are known, then the solution $u(\rho,t)$ can be found from Eqs. (1)–(4).

Denote $G_+ = G \cap \{t > 0\}$ and $G_- = G \cap \{t < 0\}$.

Definition 1. By a solution of the direct problem (1)–(4) we mean a function $u(\rho,t)$ in the class $C^{2,1}_{\rho,t}(G_+ \cup \{t = \beta\}) \cap C^2(G_- \cup \{t = -\alpha\})$ that is a solution of Eq. (1) in the domain G and satisfies conditions (1)–(4).

Inverse problem. Determine the function $f(\rho)$ if the following additional information is known about the solution of the direct problem (1)–(4):

$$u(\rho,\beta) = \psi(\rho), \quad 0 \le \rho \le 1, \tag{5}$$

where $\psi(\rho)$ is a given sufficiently smooth function.

Definition 2. A solution of the inverse problem (1)–(5) is a pair of functions $u(\rho,t)$ and $f(\rho)$ in the classes $C^{2,1}_{\rho,t}(G_+ \cup \{t = \beta\}) \cap C^2(G_- \cup \{t = -\alpha\})$ and $C[0,1]$, respectively, satisfying relations (1)–(5).

The next two sections deal with constructing a solution of the direct problem by the spectral method and proving existence and uniqueness theorems for this solution.

2. ANALYSIS OF THE DIRECT PROBLEM

Seeking particular solutions of Eq. (1) with $f(\rho) = 0$ in the form $u(\rho,t) = R(\rho)T(t)$ according to the Fourier method, we obtain the following relations:

$$dT^{'}(t)R(\rho) = \frac{1}{\rho}T(t)R^{'}(\rho) + T(t)R^{''}(\rho), \quad t > 0,$$

$$c^2T^{''}(t)R(\rho) = \frac{1}{\rho}T(t)R^{'}(\rho) + T(t)R^{''}(\rho), \quad t < 0.$$

Separating the variables, we have

$$d\frac{T^{'}(t)}{T(t)} = \frac{1}{\rho}\frac{R^{'}(\rho)}{R(\rho)} + \frac{R^{''}(\rho)}{R(\rho)} = -\lambda^2, \quad t > 0,$$

$$c^2\frac{T^{''}(t)}{T(t)} = \frac{1}{\rho}\frac{R^{'}(\rho)}{R(\rho)} + \frac{R^{''}(\rho)}{R(\rho)} = -\lambda^2, \quad t < 0,$$

where λ is an arbitrary real parameter. Hence for the function $R(\rho)$ we obtain the problem for the equation

$$R^{''}(\rho) + \frac{1}{\rho}R^{'}(\rho) + \lambda^2 R(\rho) = 0 \tag{6}$$

with the boundary conditions

$$\lim_{\rho\to 0}\left(\rho R^{'}(\rho)\right) = 0, \quad R(1) = 0, \tag{7}$$

which is a self-adjoint problem.

The following Bessel functions of the first kind and zero order are solutions of Eq. (6):

$$R_k(\rho) = J_0(\lambda_k\rho), \quad k = 1, 2, 3, \ldots$$

They are also eigenfunctions. We find the eigenvalues using the boundary conditions (7) and the positive roots of the equation $J_0(\lambda_k) = 0$. They are as follows:

$$\lambda_k = k\pi - \frac{\pi}{4} = (4k-1)\frac{\pi}{4}.$$

Now let us expand the unknown function and the right-hand side of the equation in Bessel Fourier series in the eigenfunctions $J_0(\lambda_k\rho)$; i.e.,

$$u(\rho, t) = \sum_{k=1}^{\infty} u_k(t)J_0(\lambda_k\rho), \tag{8}$$

$$f(\rho) = \sum_{k=1}^{\infty} f_k J_0(\lambda_k\rho), \tag{9}$$

where

$$u_k(t) = \frac{2}{J_1^2(\lambda_k)}\int_0^1 \rho u(\rho, t)J_0(\lambda_k\rho)\,d\rho,$$

$$f_k = \frac{2}{J_1^2(\lambda_k)}\int_0^1 \rho f(\rho)J_0(\lambda_k\rho)\,d\rho.$$

Substituting (8) and (9) into (1), we obtain

$$\begin{aligned} du_k'(t) &= -\lambda_k^2 u_k(t) + f_k, \quad t > 0, \\ c^2 u_k''(t) &= -\lambda_k^2 u_k(t) + f_k, \quad t < 0. \end{aligned} \tag{10}$$

One can readily establish that these differential equations have the general solutions

$$\begin{aligned} u_k(t) &= c_k e^{-\lambda_k^2 t/d} + f_k/\lambda_k^2, \quad t > 0, \\ u_k(t) &= a_k\cos\left(\frac{\lambda_k}{c}t\right) + b_k\sin\left(\frac{\lambda_k}{c}t\right) + f_k/\lambda_k^2, \quad t < 0, \end{aligned} \tag{11}$$

where a_k, b_k, and c_k are arbitrary constants.

To find the coefficients a_k, b_k, and c_k, we use the matching conditions

$$u_k(0+0) = u_k(0-0),$$
$$u'_k(0+0) = u'_k(0-0)$$

and obtain $a_k = c_k$ and $b_k = -\dfrac{\lambda_k c}{d} c_k$. From the initial condition (3), we have

$$a_k \cos\left(\frac{\lambda_k}{c}\alpha\right) - b_k \sin\left(\frac{\lambda_k}{c}\alpha\right) + f_k/\lambda_k^2 = \varphi_k,$$

where the φ_k are the Fourier–Bessel coefficients of the series

$$\varphi(\rho) = \sum_{k=1}^{\infty} \varphi_k J_0(\lambda_k \rho), \tag{12}$$

Substituting the values a_k and b_k expressed via c_k and solving the resulting system for c_k, we find

$$c_k = \frac{\varphi_k - f_k/\lambda_k^2}{\cos\left(\dfrac{\lambda_k}{c}\alpha\right) + \dfrac{\lambda_k c}{d}\sin\left(\dfrac{\lambda_k}{c}\alpha\right)}. \tag{13}$$

We introduce the notation

$$\delta_\alpha(k) = \cos\left(\frac{\lambda_k}{c}\alpha\right) + \frac{\lambda_k c}{d}\sin\left(\frac{\lambda_k}{c}\alpha\right). \tag{14}$$

3. EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE DIRECT PROBLEM

Substituting the eigenvalues, we find the values of α for which the expression (14) takes nonzero values. To this end, we rewrite (14) as follows:

$$\delta_\alpha(k) = \sqrt{1 + \frac{(4k-1)^2\pi^2c^2}{16d^2}}\sin\left(\frac{4k-1}{4c}\alpha\pi + \gamma_k\right), \tag{15}$$

where

$$\gamma_k = \arcsin\left(\frac{1}{\sqrt{1 + \dfrac{(4k-1)^2\pi^2c^2}{16d}}}\right).$$

Let us find the values of α for which $\delta_\alpha(k) = 0$; here we have $\alpha = \frac{4c}{(4k-1)\pi}(\pi n - \gamma_k)$.

Now let us find the values of α for which

$$\left|\delta_\alpha(k)\right| \geq C_0 > 0. \tag{16}$$

To this end, we estimate $\delta_\alpha(k)$ in absolute value. Owing to the equality $\alpha = 4cp$, $p \in \mathbb{N}$, we obtain $|\delta_\alpha(k)| = 1 \geq C_0 > 0$.

Let $\alpha = 4cn/m$, $n, m \in \mathbb{N}$, $(n, m) = 1$; then from (15) for the values $\alpha = 4cp$, $p \in \mathbb{N}$, or $\alpha = 4cn/m$, $n, m \in \mathbb{N}$, where n and m are coprime numbers, i.e., GCF $(n, m) = 1$, $\beta > 0$; for (14) and (15), we have

$$\left|\delta_\alpha(k)\right| = \left|\sqrt{1 + \frac{(4k-1)^2\pi^2c^2}{16d^2}}\sin\left(\frac{4k-1}{4c}\alpha\pi + \gamma_k\right)\right|$$

$$= \left| \sqrt{1 + \frac{(4k-1)^2\pi^2 c^2}{16d^2}} \sin\left(\frac{4k-1}{m} n\pi + \gamma_k \right) \right|.$$

Let us divide $(4k-1)n$ by m with remainder, $(4k-1)n = sm + r$, $s, r \in \mathbb{N} \cup 0$, $0 \le r < m$. Then the expression for $|A_{\alpha\beta}(k)|$ has the form

$$\left|\delta_\alpha(k)\right| = \left| \sqrt{1 + \frac{(4k-1)^2\pi^2 c^2}{16d^2}} \sin\left(\frac{r}{m} n\pi + \gamma_k \right) \right|.$$

Since $0 \le r\pi/m < \pi$, as $\gamma_k \to 0$, it follows that (16) holds for sufficiently large k and an arbitrary $\beta > 0$.

Thus, we have obtained the following uniqueness criterion.

Theorem 1. *If there exists a solution of problem* (1)–(4), *then it is unique for the values* $\alpha = 4cp$, $p \in \mathbb{N}$, *or* $\alpha = 4cn/m$, $n, m \in \mathbb{N}$, $\mathrm{GCF}\,(n, m) = 1$.

Now let us prove the existence of a solution. Substituting (13) into (11), we find

$$u_k(t) = \frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d} t} + f_k/\lambda_k^2, \qquad t > 0,$$

$$u_k(t) = \frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} \left(\cos\left(\frac{\lambda_k}{c} t \right) - c\frac{\lambda_k}{d} \sin\left(\frac{\lambda_k}{c} t \right) \right) + f_k/\lambda_k^2, \quad t < 0.$$

Taking into account these relations, from (8) and (9) we obtain a formal solution of the posed problem in the form of the series

$$u(\rho, t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d} t} + f_k/\lambda_k^2 \right] J_0(\lambda_k \rho), \quad t > 0, \tag{17}$$

$$u(\rho, t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} \left(\cos\left(\frac{\lambda_k}{c} t \right) - c\frac{\lambda_k}{d} \sin\left(\frac{\lambda_k}{c} t \right) \right) + f_k/\lambda_k^2 \right] J_0(\lambda_k \rho), \quad t < 0. \tag{18}$$

To prove the existence of a solution, we need to show that the series (17), (18) and the series obtained by differentiating the function $u(\rho, t)$ two times with respect to ρ and once with respect to t in the domain G_+ and twice with respect to ρ and with respect to t in the domain G_- converge uniformly.

To estimate the coefficients of the Fourier–Bessel series of the function $u(x, t)$ and the series obtained by its differentiation, we calculate the general terms of these series,

$$u_k(\rho, t) = \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d} t} + f_k/\lambda_k^2 \right] J_0(\lambda_k \rho), \quad t > 0, \tag{19}$$

$$u_k(\rho, t) = \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} \left(\cos\left(\frac{\lambda_k}{c} t \right) - c\frac{\lambda_k}{d} \sin\left(\frac{\lambda_k}{c} t \right) \right) + f_k/\lambda_k^2 \right] J_0(\lambda_k \rho), \quad t < 0, \tag{20}$$

$$\frac{\partial u_k(\rho, t)}{\partial t} = -\frac{1}{d\delta_\alpha(k)} (f_k - \lambda_k^2 \varphi_k) J_0(\lambda_k \rho), \quad t > 0, \tag{21}$$

$$\frac{\partial^2 u_k(\rho, t)}{\partial t^2} = \frac{1}{\delta_\alpha(k)} \left[-\frac{\lambda_k^2}{c^2} \cos\left(\frac{\lambda k}{c} t \right) + \frac{\lambda_k^3}{cd} \sin\left(\frac{\lambda k}{c} t \right) \right] (\varphi_k - f_k/\lambda_k^2) J_0(\lambda_k \rho), \quad t < 0, \tag{22}$$

$$\frac{\partial^2 u_k(\rho,t)}{\partial\rho^2} = \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d}t} + f_k/\lambda_k^2\right]\lambda_k^2 J_0^{''}(\lambda_k\rho), \quad t>0, \tag{23}$$

$$\frac{\partial^2 u_k(\rho,t)}{\partial\rho^2} = \left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)}\left(\cos\left(\frac{\lambda_k}{c}t\right) - c\frac{\lambda_k}{d}\sin\left(\frac{\lambda_k}{c}t\right)\right) + f_k/\lambda_k^2\right]\lambda_k^2 J_0^{''}(\lambda_k\rho), \quad t<0. \tag{24}$$

Let the functions $\varphi(\rho)$ and $f(\rho)$ satisfy the assumptions of the theorem in ([13, pp. 289–291 of the Russian edition]) with some $s \ge 1$ (the number s will be defined later). Then for the Fourier–Bessel coefficients of these functions one has the estimates [13, p. 282 of the Russian edition]

$$|\varphi_k| \le \frac{M_1}{\lambda_k^{2s-1/2}},$$
$$|f_k| \le \frac{M_2}{\lambda_k^{2s-1/2}}.$$

In order to estimate the expressions in (19)–(24), we need estimates for the functions $J_0(z)$ and $J_0''(z)$, $z \in [0,+\infty)$. For the Bessel function $J_\nu(z)$, one has the well-known integral representation

$$J_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\varphi + i\nu\sin\varphi}\,d\varphi,$$

which implies the estimate $|J_\nu(z)| \le 1$, $z \in \mathbb{R}$. Taking into account this estimate and the relations [13]

$$J_0'(z) = -J_1(z),$$
$$J_0''(z) = (1/2)\big(J_2(z) - J_0(z)\big),$$

we find the estimates $|J_0(z)| \le 1$ and $\left|J_0''(z)\right| \le 1$, $z \in [0,+\infty)$, which will be used below.

Let us estimate the function $u_k(\rho,t)$,

$$\begin{aligned}\big|u_k(\rho,t)\big| &= \left|\left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d}t} + f_k/\lambda_k^2\right] J_0(\lambda_k\rho)\right| = \left|\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d}t} + f_k/\lambda_k^2\right|\big|J_0(\lambda_k\rho)\big| \\ &\le \left|\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)} e^{-\frac{\lambda_k^2}{d}t} + f_k/\lambda_k^2\right| \le |\varphi_k|\left|\frac{e^{-\frac{\lambda_k^2}{d}t}}{\delta_\alpha(k)}\right| + |f_k|\left|\frac{e^{-\frac{\lambda_k^2}{d}t}}{\lambda_k^2\delta_\alpha(k)}\right| + \big|f_k/\lambda_k^2\big| \\ &\le \frac{M_1}{\lambda_k^{2s-(1/2)}} + \frac{M_2}{\lambda_k^{2s-(1/2)+2}} \le \frac{N_1}{\lambda_k^{2s-(1/2)}}, \quad t>0;\end{aligned}$$

in this case, $s = 2$ for the function $\varphi(x)$ and $s = 0$ for the functions $f(x)$.

One has the estimate

$$\begin{aligned}\big|u_k(\rho,t)\big| &= \left|\left[\frac{\varphi_k - f_k/\lambda_k^2}{\delta_\alpha(k)}\left(\cos\left(\frac{\lambda_k}{c}t\right) - c\frac{\lambda_k}{d}\sin\left(\frac{\lambda_k}{c}t\right)\right) + f_k/\lambda_k^2\right] J_0(\lambda_k\rho)\right| \\ &\le M_1\frac{\lambda_k}{\lambda_k^{2s-(1/2)}} + \frac{M_2}{\lambda_k^{2s-(1/2)+1}} \le N_2\frac{\lambda_k}{\lambda_k^{2s-(1/2)}}, \quad t<0;\end{aligned}$$

here $s = 3$ for the function $\varphi(x)$ and $s = 1$ for the function $f(x)$; M_1, M_2, N_1, and N_2 are positive constants.

Further, in a similar way we carry out obvious estimates for the expressions (21)–(24) and find

$$\left|\frac{\partial u_k(\rho,t)}{\partial t}\right| \le N_3\frac{\lambda_k^2}{\lambda_k^{2s-(1/2)}},\quad t>0,\qquad \left|\frac{\partial^2 u_k(\rho,t)}{\partial t^2}\right| \le N_4\frac{\lambda_k^3}{\lambda_k^{2s-(1/2)}},\quad t<0,$$
$$\left|\frac{\partial^2 u_k(\rho,t)}{\partial \rho^2}\right| \le N_5\frac{\lambda_k^2}{\lambda_k^{2s-(1/2)}},\quad t>0,\qquad \left|\frac{\partial^2 u_k(\rho,t)}{\partial \rho^2}\right| \le N_6\frac{\lambda_k^3}{\lambda_k^{2s-(1/2)}},\quad t<0.$$

Here N_3, N_4, N_5, and N_6 are positive constants.

From these inequalities, we obtain

$$\max\left\{\max_{(\rho,t)\in G_+}\left|\frac{\partial u_k(\rho,t)}{\partial t}\right|,\ \max_{(\rho,t)\in G_-}\left|\frac{\partial^2 u_k(\rho,t)}{\partial^2 t}\right|,\ \max_{(\rho,t)\in G_+}\left|\frac{\partial^2 u_k(\rho,t)}{\partial^2 \rho}\right|,\ \max_{(\rho,t)\in G_-}\left|\frac{\partial^2 u_k(\rho,t)}{\partial^2 \rho}\right|\right\} \le N\frac{\lambda_k^3}{\lambda_k^{2s-(1/2)}}, \tag{25}$$

where N is a positive constant.

It follows that if $s=3$ for the function $\varphi(x)$ and $s=2$ for the function $f(x)$, then, according to the theorem in [13, p. 282 of the Russian edition], the series (17), (18), the series obtained by differentiating $u(\rho,t)$ twice with respect to ρ and once with respect to t in the domain G_+, and the series obtained by differentiating the function $u(\rho,t)$ two times with respect to ρ and with respect to t in the domain G_- converge uniformly.

Thus, we have proved the following assertion.

Theorem 2. *Let $\varphi(\rho)\in C^6[0,1]$, let $f(\rho)\in C^4[0,1]$, and let, in addition, condition (16) and the following equalities be satisfied*:

$$\varphi^{(i)}(0)=0,\quad i=0,1,\dots,5,\qquad f^{(i)}(0)=0,\quad i=0,1,\dots,3,$$
$$\varphi^{(i)}(1)=0,\quad i=0,1,\dots,4,\qquad f^{(i)}(1)=0,\quad i=0,1,2.$$

Then there exists a unique solution of problem (1)–(4). This solution is defined by formulas (17), (18), where $\varphi^{(i)}$ and $f^{(i)}$ are the ith derivatives of the functions φ and f, while φ_k and f_k are the Fourier–Bessel coefficients of φ and f, respectively.

4. EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE INVERSE PROBLEM

Let us proceed to the study of the inverse problem. Using the additional condition (8), we expand the function ψ in the Fourier–Bessel series

$$\psi(\rho)=\sum_{k=1}^{\infty}\psi_k J_0(\lambda_k\rho), \tag{26}$$

where the ψ_k are the Fourier–Bessel coefficients. As a result, we obtain

$$c_k e^{\frac{-\lambda_k^2}{d}\beta}+f_k/\lambda_k^2=\psi_k,$$

Substituting the value of c_k into this equality, we find

$$f_k=\lambda_k^2\left(\frac{\varphi_k e^{-\lambda_k^2\beta/d}-\psi_k\delta_\alpha(k)}{e^{-\lambda_k^2\beta/d}-\delta_\alpha(k)}\right). \tag{27}$$

Set

$$A_{\alpha\beta}(k)=e^{-\lambda_k^2\beta/d}-\cos\left(\frac{\lambda_k}{c}\alpha\right)-\frac{\lambda_k c}{d}\sin\left(\frac{\lambda_k}{c}\alpha\right). \tag{28}$$

It is easily seen that the relation

$$A_{\alpha\beta}(k) \neq 0 \tag{29}$$

holds if α satisfies the assumptions of Theorem 1 with an arbitrary $\beta > 0$. Then, just as Theorem 1, we have the following criterion for the uniqueness of the solution of the inverse problem.

Theorem 3. *If there exists a solution of the inverse problem* (1)–(5), *then it is unique for the values* α *satisfying the assumptions of Theorem* 1 *and for any* $\beta > 0$.

Let us prove the existence of a solution of the inverse problem. By analogy with estimate (28), we obtain an estimate for the function $f(\rho)$,

$$f(\rho) = \sum_{k=1}^{\infty} \lambda_k^2 \left(\psi_k - \frac{\psi_k - \varphi_k}{A_{\alpha\beta}(k)}\right) e^{-\lambda_k^2 \beta/d} J_0(\lambda_k \rho), \tag{30}$$

where the functions $\varphi(\rho)$ and $\psi(\rho)$ satisfy the assumptions of the theorem in [13, p. 282 of the Russian edition] with some $s \geq 1$. Then for the Fourier–Bessel coefficients of these functions we have the estimates

$$|\varphi_k| \leq \frac{M_1}{\lambda_k^{2s-(1/2)}},$$
$$|\psi_k| \leq \frac{M_3}{\lambda_k^{2s-(1/2)}},$$

and the estimate for $f(\rho)$ has the form

$$\big|f(\rho)\big| \leq N \frac{\lambda_k^2}{\lambda_k^{2s-(1/2)}}.$$

It follows from this estimate and the estimate (25) that if $s = 3$, then, according to the theorem in [13, p. 282 of the Russian edition], the series in (17), (18), and (30), the series obtained by differentiating the function $u(\rho, t)$ two times with respect to ρ and once with respect to t in the domain G_+, and the series obtained by differentiating the function $u(\rho, t)$ twice with respect to ρ and with respect to t in the domain G_- converge uniformly.

Thus, we have proved the main result of the present paper.

Theorem 4. *Assume that the function* $\varphi(\rho)$ *satisfies the conditions in Theorem* 2, *conditions* (29) *are satisfied, and for the function* $\psi(\rho) \in C^6[0,1]$ *one has the equalities*

$$\psi^{(i)}(0) = 0, \quad i = 0, 1, \ldots, 5,$$
$$\psi^{(i)}(1) = 0, \quad j = 0, 1, \ldots, 4.$$

Then there exists a unique solution of problem (1)–(5). *This solution is defined by formulas* (17), (18), *and* (30), *where the* $\psi^{(i)}$ *are the ith derivatives of the function* ψ *and the* ψ_k *are the Fourier–Bessel coefficients of the function* ψ.

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