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**Inverse source problem for equation of mixed parabolic-hyperbolic type: one-dimensional case**

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**Abstract.** In this paper, we state an inverse problem for a model equation of a mixed parabolic-hyperbolic type with discontinuous coefficients and with a nonlocal condition in a rectangular domain in the one-dimensional case and prove a theorem on the uniqueness of a solution to this problems.

**Keywords:** : classification of the equations, state of the problem, correct stated problem, direct problem, inverse problem, mixed type equation, eigen value, eigen function

**Mathematics Subject Classification (2010):**

# 1 Introduction

The direct and inverse problem for an equation of a mixed parabolic-hyperbolic type with a non-local condition was studied in the work of Sabitov K.B. and Safin E.M. [25]. A criterion for the uniqueness and existence of a solution to the inverse problem for determining the unknown right-hand side was established. Also in [2] other formulations of the inverse problem were considered. In this paper, we consider a problem with discontinuous coefficients.

In mathematical physics, problems of the following type are usually considered: a differential equation is given and additional conditions that must be satisfied by the solution of the differential equation. These additional conditions distinguish one solution from the entire set of solutions of the differential equation. There is a classification for the equations of mathematical physics. Moreover, for each class of differential equations there are typical problem statements. A characteristic feature of these problems is their correctness. Correct problems in mathematical physics are called direct problems. To solve the direct problem, a given set of functions is associated with a new function - the solution of a boundary value problem. In this way, a certain operator is constructed, which is defined on the data of the direct problem.

Let us now imagine that some of the functions that are usually given in the direct problem are unknown (and it is precisely their finding that is of main interest), and instead of them some additional information about the solution of the direct problem is given. Similar problems are called inverse problems of mathematical physics.

The theory of boundary value problems for equations of mixed type is one of the central sections of the theory of partial differential equations and occurs in solving many important problems of an applied nature.

Direct and inverse problems for equations of mixed type have been studied relatively less than problems for equations of a particular type.

To date, the most complete results have been obtained on the study of direct problems for equations of mixed type [6]-[22], [4]-[5], but there are practically few works related to the search for a solution to inverse problems for an equation of mixed type [25]-[2], [18]-[10].

Generally speaking, direct and inverse problems for equations of mixed type are not as well known as similar problems for classical equations. The problem considered in the article is relevant to practice. For example, the problem considered in this work is related to modeling the process of gas movement in a closed channel with porous walls, and the gas movement in the channel is described by the wave equation, and outside it, by the diffusion equation [8], [6]. Non-classical problems, problems of mixed type were considered in the works of Trikomi F.[6],[30], Fiker G. [7], Vragov V. N.[32], Nakhusheva A.M. [1], [20] and many other researchers (see, for example, [3]-[19]).

Numerical solutions of the inverse problem were considered in the works [11]-[16].

# 2 Formulation of the problem

In a rectangular region $G := \{(x,t) : 0 < x < l; -\alpha < t < \beta\}$, here $\alpha$ and $\beta$ — are given positive numbers, consider equations of mixed parabolic-hyperbolic type::

$$c^2\theta(-t)u_{tt} + \theta(t)u_t = \widehat{D}u_{xx} + f(t)g(x), \tag{2.1}$$

where, $\widehat{D} = \begin{cases} D, & \text{if } t > 0 \\ 1, & \text{if } t < 0 \end{cases}$ ,$\theta(t)$ is Heaviside function; $D$ is diffusion coefficient and $c$ is the speed of the gas are given constants.

For this equation, we can consider the following inverse problem:

**Problem**. Find in the domain $G$ the solution of equation (2.1) and the unknown function $g(x)$ on the right side satisfying the conditions:

$$u(x,t) \in C^{0,1}_{x,t}\left(\overline{G}\right) \cap C^{2,1}_{x,t}\left(G_+ \cup \{t=\beta\}\right) \cap C^{2,2}_{x,t}(G_- \cup \{t=-\alpha\}), \quad (2.2)$$

$$g(x) \in C[0;l]. \quad (2.3)$$

Boundary conditions:

$$u(0,t) = u(l,t) = 0, \quad -\alpha \le t \le \beta, \quad (2.4)$$

we assume that the nonlocal condition takes place:

$$u(x,\beta) - u(x,-\alpha) = \varphi(x), \quad 0 \le x \le l, \quad (2.5)$$

bonding conditions at $t=0$:

$$\lim_{t\to+0} u(x,t) = \lim_{t\to-0} u(x,t), \ \lim_{t\to+0} \frac{\partial u(x,t)}{\partial t} = \lim_{t\to-0} \frac{\partial u(x,t)}{\partial t}, x \in [0,l], \quad (2.6)$$

if the following additional information about the solution of the equation is known:

$$u(x,\beta) = \psi(x), 0 \le x \le l, \quad (2.7)$$

where $f(t)$, $\varphi(x)$ и $\psi(x)$ — given sufficiently smooth functions, $\varphi(0) = \varphi(l) = 0$, $\psi(0) = \psi(l) = 0$, $G_+ = G \cap \{t > 0\}$, $G_- = G \cup \{t < 0\}$.

Relations (2.1)-(2.6) are a direct problem, i.e., if the functions $\varphi(x)$, $g(x)$ and the constants $c$ and $D$, are known, then the solution $u(x,t)$ can be found from the relations (2.1)-(2.6).

# 3 Solutions of the inverse problem for $f(t) = 1$

To solve this problem, we use the method of separation of variables. This method is used in constructing solutions to the so-called mixed problems for a wide class of partial differential equations.

Simple reasoning shows that $\omega_k = \frac{\pi k}{l}$, $k \in Z$ are eigenvalues, a $X_k(x) = \sin\frac{\pi k}{l}x$ are eigenfunctions. Hence, the functions $u(x,t)$ , $f(t)$, $g(x)$, $\varphi(x)$ и $\psi(x)$ are expanded in the Fourier series as follows:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin(\omega_k x), \quad (3.1)$$

$$g(x)=\sum_{k=1}^{\infty} g_k \sin(\omega_k x), \tag{3.2}$$

$$\varphi(x)=\sum_{k=1}^{\infty} \varphi_k \sin(\omega_k x), \tag{3.3}$$

$$\psi(x)=\sum_{k=1}^{\infty} \psi_k \sin(\omega_k x). \tag{3.4}$$

Substituting(3.1)-(3.2) in the equation(2.1) we get the following equations:

$$u_k'(t)+D\omega_k^2 u_k(t)=g_k f(t), \tag{3.5}$$

$$u_k''(t)+\frac{1}{c^2}\omega_k^2 u_k(t)=\frac{1}{c^2}g_k f(t). \tag{3.6}$$

From conditions (2.5) and (2.7), we find the following relations:

$$\begin{cases} u_k(\beta)-u_k(-\alpha)=\varphi_k, \\ u_k(\beta)=\psi_k. \end{cases} \tag{3.7}$$

Here,

$$\varphi_k=\frac{2}{l}\int_0^l \varphi(x)\sin(\omega_k x)dx,$$
$$\psi_k=\frac{2}{l}\int_0^l \psi(x)\sin(\omega_k x)dx.$$

Let's first consider the case, $f(t)=1$. Equations (3.5) and (3.6) ) take the following form:

$$u_k'(t)+D\omega_k^2 u_k(t)=g_k, \tag{3.8}$$

$$u_k''(t)+\frac{1}{c^2}\omega_k^2 u_k(t)=\frac{1}{c^2}g_k. \tag{3.9}$$

Solving equations (3.8) and (3.9), we obtain the following solutions:

$$u_k(t)=C_k e^{-D\omega_k^2 t}+\frac{g_k}{D\omega_k^2}, \quad t>0 \tag{3.10}$$

$$u_k(t)=A_k\cos\left(\frac{\omega_k}{c}t\right)+B_k\sin\left(\frac{\omega_k}{c}t\right)+\frac{g_k}{\omega_k^2}, \quad t<0. \tag{3.11}$$

From conditions (2.6), we find the gluing conditions for $u_k$

$$\begin{cases} u_k(0-0) = u_k(0+0), \\ u_k'(0-0) = u_k'(0+0). \end{cases} \tag{3.12}$$

Using conditions (3.7) and (3.12) we get the following linear system of equations:

$$\begin{cases} A_k + \frac{g_k}{\omega_k^2} = C_k + \frac{g_k}{D\omega_k^2}, \\ \frac{1}{c}\omega_k B_k = -D\omega_k^2 C_k, \\ C_k e^{-D\omega_k^2\beta} + \frac{g_k}{D\omega_k^2} - \left(A_k \cos\left(\frac{\omega_k}{c}\alpha\right) - B_k \sin\left(\frac{\omega_k}{c}\alpha\right) + \frac{g_k}{\omega_k^2}\right) = \varphi_k \end{cases}$$

Having obtained a system of linear equations for determining unknown coefficients, we find them:

$$\begin{cases} C_k = \frac{\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\left(1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right)}{e^{-d\omega_k^2\beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right)} \\ A_k = C_k + \frac{g_k}{\omega_k^2}\frac{1-D}{D}, \\ B_k = -Dc\omega_k C_k \end{cases}$$

$$\text{i.e.} \begin{cases} A_k = \frac{\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\left(1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right)}{e^{-d\omega_k^2\beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right)} + \frac{g_k}{\omega_k^2}\frac{1-D}{D}, \\ B_k = -Dc\cdot\omega_k\cdot\frac{\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\left(1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right)}{e^{-d\omega_k^2\beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right)}, \\ C_k = \frac{\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\left(1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right)}{e^{-d\omega_k^2\beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right)}. \end{cases}$$

Let's introduce the notation:

$$\lambda_k^{-1} = e^{-D\omega_k^2\beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right) \tag{3.13}$$

$$\mu_k = 1 - \cos\left(\frac{\omega_k}{c}\alpha\right).$$

Then

$$\begin{cases} A_k = \lambda_k\left(\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\mu_k\right) + \frac{g_k}{\omega_k^2}\frac{1-D}{D}, \\ B_k = -Dc\lambda_k\cdot\omega_k\cdot\left(\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\mu_k\right), \\ C_k = \lambda_k\left(\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\mu_k\right). \end{cases}$$

Substituting the found coefficients (3.10) and (3.11) we obtain a formal solution, imagining that the given function:

$$u_k(t) = \lambda_k \left( \varphi_k - \frac{g_k}{\omega_k^2} \frac{1-D}{D} \mu_k \right) e^{-D\omega_k^2 t} + \frac{g_k}{D\omega_k^2}, \quad 0 < t < \beta, \tag{3.14}$$

$$u_k(t) = \left( \lambda_k \left( \varphi_k - \frac{g_k}{\omega_k^2} \frac{1-D}{D} \mu_k \right) + \frac{g_k}{\omega_k^2} \frac{1-D}{D} \right) \cos\left( \frac{\omega_k}{c} t \right) -$$

$$-Dc\lambda_k \cdot \omega_k \cdot \left( \varphi_k - \frac{g_k}{\omega_k^2} \frac{1-D}{D} \mu_k \right) \sin\left( \frac{\omega_k}{c} t \right) + \frac{g_k}{\omega_k^2}, \quad -\alpha < t < 0. \tag{3.15}$$

Consider (3.14) i.e. let's rewrite it like this:

$$u_k(t) = \lambda_k \varphi_k e^{-D\omega_k^2 t} + \frac{g_k}{\omega_k^2} \left( \frac{1}{D} - \mu_k \lambda_k \frac{1-D}{D} e^{-D\omega_k^2 t} \right), \quad 0 < t < \beta,$$

let's introduce the notation:

$$d_k(t) = \frac{1}{D} - \mu_k \lambda_k \frac{1-D}{D} e^{-D\omega_k^2 t}.$$

Then (3.14) takes the following form:

$$u_k(t) = \lambda_k \varphi_k e^{-D\omega_k^2 t} + \frac{g_k}{\omega_k^2} d_k(t), \quad 0 < t < \beta. \tag{3.16}$$

Now we will solve the inverse problem, with the given condition (2.7), i.e. if $u_k(\beta) = \psi_k$ find $g(x)$. The uniqueness of the solution (2.1)-(2.7) was proved in [29], for $f(t) = 1$ and for $D = \frac{1}{c^2}$. Consider the discontinuous case of coefficients, i.e. $D \neq \frac{1}{c^2}$.

Using condition (2.7) and found formal solution (3.16), we find $g_k$

$$g_k = \omega_k^2 \left( \frac{\psi_k}{d_k(\beta)} - \frac{\lambda_k \varphi_k e^{-D\omega_k^2 \beta}}{d_k(\beta)} \right). \tag{3.17}$$

From here we obtain the solution of the given inverse problem:

$$u(x,t) = \begin{cases} \sum\limits_{k=1}^{\infty} \left(\lambda_k \varphi_k e^{-D\omega_k^2 t} + \frac{g_k}{\omega_k^2} d_k(t)\right) \sin(\omega_k x), 0 < t < \beta \\ \sum\limits_{k=1}^{\infty} \left( \left(\lambda_k \left(\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\mu_k\right) + \frac{g_k}{\omega_k^2}\frac{1-D}{D}\right) \cos\left(\frac{\omega_k}{c}t\right) - \right. \\ \left. -Dc\lambda_k \cdot \omega_k \cdot \left(\varphi_k - \frac{g_k}{\omega_k^2}\frac{1-D}{D}\mu_k\right) \sin\left(\frac{\omega_k}{c}t\right) + \frac{g_k}{\omega_k^2}\right) \\ \sin(\omega_k x), -\alpha < t < 0 \end{cases} \quad (3.18)$$

and

$$g(x) = \sum_{k=1}^{\infty} g_k \sin(\omega_k x) = \sum_{k=1}^{\infty} \omega_k^2 \left( \frac{\psi_k}{d_k(\beta)} - \frac{\lambda_k \varphi_k e^{-D\omega_k^2 \beta}}{d_k(\beta)} \right) \sin(\omega_k x) \quad (3.19)$$

**Remark 3.1.** It should be noted that the following conditions must be fulfilled:

$$\lambda_k^{-1} \neq 0 \text{ and } d_k(\beta) \neq 0$$

for all $k \in N$.

**Remark 3.2.** The problem considered in this article is of an applied nature and describes the movement of gas in a pipe and outside the pipe, where discontinuity coefficients: $D$ is the diffusion coefficient, and , a $c$ is the gas velocity.

First we check the condition $\lambda_k^{-1} \neq 0$. Indeed,

$$\lambda_k^{-1} = e^{-D\omega_k^2 \beta} - \left(\cos\left(\frac{\omega_k}{c}\alpha\right) + Dc\omega_k \sin\left(\frac{\omega_k}{c}\alpha\right)\right) = e^{-D\omega_k^2 \beta} -$$

$$-\sqrt{1 + (Dc\omega_k)^2} \sin\left(\frac{\omega_k}{c}\alpha + \arcsin\left(\frac{1}{\sqrt{1 + (Dc\omega_k)^2}}\right)\right)$$

Let $\alpha = lcp$, $p \in N$, $\beta \in R$.

$$|\lambda_k^{-1}| = \left| e^{-D\omega_k^2 \beta} - \sqrt{1 + (Dc\omega_k)^2} \sin\left(\frac{\omega_k}{c}\alpha + \arcsin\left(\frac{1}{\sqrt{1 + (Dc\omega_k)^2}}\right)\right) \right| =$$

$$= \left| e^{-D\omega_k^2 \beta} \pm 1 \right| \geq C_0 > 0,$$

where $C_0 = const.$

Now let $\alpha = lc\frac{n}{m}$, $n, m \in N$, and $(n,m) = 1$ $\beta \in R$.

$$|\lambda_k^{-1}| = \left| e^{-D\omega_k^2\beta} - \sqrt{1+(Dc\omega_k)^2}\sin\left(\pi k\frac{n}{m} + \arcsin\left(\frac{1}{\sqrt{1+(Dc\omega_k)^2}}\right)\right)\right|,$$

$kn$ is divided by $m$ with remainder, $kn = sm + r$, $s, r \in N\cup\{0\}$,$0 \le r < m$.

$$|\lambda_k^{-1}| = \left| e^{-D\omega_k^2\beta} - \sqrt{1+(Dc\omega_k)^2}\sin\left(\pi\frac{r}{m} + \arcsin\left(\frac{1}{\sqrt{1+(Dc\omega_k)^2}}\right)\right)\right|,$$

Then, due to $\arcsin\left(\frac{1}{\sqrt{1+(Dc\omega_k)^2}}\right) \to 0$ and $0 \le \frac{\pi r}{m} < \pi$, the condition follows, $|\lambda_k^{-1}| \ge C_0 > 0$ i.e. $|\lambda_k| \le C_0$.

Now let's evaluate $d_k(\beta) \neq 0$.

$$|d_k(\beta)| = \left|\frac{1}{D} - \mu_k\lambda_k\frac{1-D}{D}e^{-D\omega_k^2\beta}\right| \ge \left|\frac{1}{D} - 2C_0\left|\frac{1-D}{D}\right|\right| \ge C_1 > 0,$$

where $|\mu_k| = \left|1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right| \le 2$. To mean $|d_k(\beta)| \ge C_1 > 0$.

Let the following conditions be satisfied: $|\mu_k| = \left|1-\cos\left(\frac{\omega_k}{c}\alpha\right)\right| \le 2$, $|\lambda_k| \le C_0$, $|d_k(\beta)| \ge C_1 > 0$.

We find the condition for the convergence of the resulting series (3.19)

$$|g(x)| = \left|\sum_{k=1}^{\infty} g_k\sin(\omega_k x)\right| = \left|\sum_{k=1}^{\infty}\omega_k^2\left(\frac{\psi_k}{d_k(\beta)} - \frac{\lambda_k\varphi_k e^{-D\omega_k^2\beta}}{d_k(\beta)}\right)\sin(\omega_k x)\right| \le$$

$$\le \left|\sum_{k=1}^{\infty}\omega_k^2\left(\frac{\psi_k}{d_k(\beta)} - \frac{\lambda_k\varphi_k e^{-D\omega_k^2\beta}}{d_k(\beta)}\right)\right|$$

$$\le \sum_{k=1}^{\infty}\omega_k^2\frac{|\psi_k|}{|d_k(\beta)|} + \sum_{k=1}^{\infty}\omega_k^2\frac{|\lambda_k||\varphi_k|e^{-D\omega_k^2\beta}}{|d_k(\beta)|}$$

$$\le \sum_{k=1}^{\infty}\omega_k^2\frac{|\psi_k|}{C_1} + \sum_{k=1}^{\infty}\omega_k^2\frac{C_0|\varphi_k|e^{-D\omega_k^2\beta}}{C_1}$$

Let us introduce the notation

$\overline{C_1} = \frac{\pi}{lC_1}$, $\overline{C_2} = \frac{\pi}{lC_1}C_0$

Then

$$|g(x)| \leq \overline{C_1} \sum_{k=1}^{\infty} k^2 |\psi_k| + \overline{C_2} \sum_{k=1}^{\infty} k^2 |\varphi_k| e^{-D\omega_k^2 \beta}$$

According to the Cauchy-Bunyakovsky inequality, we obtain the following

$$|g(x)| \leq \overline{C_1} \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2}} \sqrt{\sum_{k=1}^{\infty} |\psi_k^{(3)}|^2} + \overline{C_2} \sqrt{\sum_{k=1}^{\infty} k^2 e^{-D\omega_k^2 \beta}} \sqrt{\sum_{k=1}^{\infty} |\varphi_k|^2} \leq$$

$$\leq \overline{C_1} ||\psi_k^{(3)}||_{L_2(0.l)} + \overline{C_2} ||\varphi_k^{(1)}||_{L_2(0.l)},$$

where $\varphi_k^{(1)} = \frac{2}{l\omega_k} \int_0^l \varphi'(x) \cos(\omega_k x)\, dx$, $\psi_k^{(3)} = \frac{2}{l\omega_k} \int_0^l \psi'''(x) \cos(\omega_k x)\, dx$.

Thus, we proved the following theorem for inverse problem 1, i.e., $f(t) = 1$.

**Theorem 3.3.** *Suppose the conditions of remark 3.1 and*

*(B1)* $\varphi \in C[0,l]$, $\varphi'(x) \in L_2(0,1)$; $\varphi(0) = \varphi(l) = 0$;

*(B2)* $\psi \in C^2[0,l]$, $\psi'''(x) \in L_2(0,1)$; $\psi(0) = \psi(l) = 0$, $\psi''(0) = \psi''(l) = 0$ *are fulfilled.*

*Then the inverse problem (2.1)-(2.7) has a unique solution, which is represented by series (3.18), (3.19), and the following stability estimate is also valid*

$$||g(x)||_{C[0,l]} \leq \overline{C_1} ||\psi_k^{(3)}||_{L_2(0.l)} + \overline{C_2} ||\varphi_k^{(1)}||_{L_2(0.l)}.$$

# 4 Solutions of the inverse problem for $f(t) \neq const.$

Now consider the case, $f(t) \neq const$. We solve equations (3.5) and(3.6) for this case.

Solving equations (3.5) и (3.6), we obtain the following solutions:

$$u_k(t) = C_k e^{-D\omega_k^2 t} + g_k \int_0^t e^{-D\omega_k^2 (t-\tau)} f(\tau) d\tau, \quad t > 0 \tag{4.1}$$

$$u_k(t) = A_k \cos\left(\frac{\omega_k}{c} t\right) + B_k \sin\left(\frac{\omega_k}{c} t\right) + \frac{g_k}{c\omega_k} \int_0^t f(\tau) \sin\left(\frac{\omega_k}{c}(t-\tau)\right) d\tau, \ t < 0 \tag{4.2}$$

Using conditions (3.7) and(3.12) we find the unknown coefficients:

$$\begin{cases} C_k = A_k, \\ -C_k D\omega_k^2 + g_k f(0) = \frac{\omega_k}{c} B_k \\ C_k e^{-D\omega_k^2 \beta} + g_k \int\limits_0^\beta e^{-D\omega_k^2(\beta-\tau)} f(\tau)d\tau - A_k \cos\left(\frac{\omega_k}{c}\alpha\right) - \\ +B_k \sin\left(\frac{\omega_k}{c}\alpha\right) + \frac{g_k}{c\omega_k} \int\limits_0^{-\alpha} f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau = \varphi_k \end{cases}$$

Received a system of linear equations for determining unknown coefficients.

We solve a system of linear equations and using the notation (3.13), we get:

$$C_k = \lambda_k \left[\varphi_k - \frac{cg_k}{\omega_k} f(0)\sin\left(\frac{\omega_k}{c}\alpha\right) - g_k \int\limits_0^\beta e^{-D\omega_k^2(\beta-\tau)} f(\tau)d\tau - \right.$$

$$\left. - \frac{g_k}{c\omega_k} \int\limits_0^{-\alpha} f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau\right],$$

$$A_k = \lambda_k \left[\varphi_k - \frac{cg_k}{\omega_k} f(0)\sin\left(\frac{\omega_k}{c}\alpha\right) - g_k \int\limits_0^\beta e^{-D\omega_k^2(\beta-\tau)} f(\tau)d\tau - \right.$$

$$\left. - \frac{g_k}{c\omega_k} \int\limits_0^{-\alpha} f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau\right],$$

$$B_k = -cD\omega_k \lambda_k \left[\varphi_k - \frac{cg_k}{\omega_k} f(0)\sin\left(\frac{\omega_k}{c}\alpha\right) - g_k \int\limits_0^\beta e^{-D\omega_k^2(\beta-\tau)} f(\tau)d\tau - \right.$$

$$\left. - \frac{g_k}{c\omega_k} \int\limits_0^{-\alpha} f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau\right] + \frac{cg_k}{\omega_k} f(0).$$

Substituting the found coefficients (3.10) and (3.11) into (**??**) and (**??**) respectively we obtain a formal solution, imagining that $g(x)$ the given function.

For $t>0$ we rewrite the solution as follows:

$$u_k(t)=\lambda_k\varphi_k e^{-D\omega_k^2 t}-g_k\Bigg[\Bigg(\frac{c}{\omega_k}f(0)\sin\left(\frac{\omega_k}{c}\alpha\right)+\int\limits_0^{\beta}e^{-D\omega_k^2(\beta-\tau)}f(\tau)d\tau+$$

$$+\frac{1}{c\omega_k}\int\limits_0^{-\alpha}f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau\Bigg)\lambda_k e^{-D\omega_k^2 t}-\int\limits_0^{t}e^{-D\omega_k^2(t-\tau)}f(\tau)d\tau\Bigg], \tag{4.3}$$

let's introduce the notation:

$$\widetilde{d}_k(t)=\Bigg(\frac{c}{\omega_k}f(0)\sin\left(\frac{\omega_k}{c}\alpha\right)+\int\limits_0^{\beta}e^{-D\omega_k^2(\beta-\tau)}f(\tau)d\tau+$$

$$+\frac{1}{c\omega_k}\int\limits_0^{-\alpha}f(\tau)\sin\left(\frac{\omega_k}{c}(\alpha+\tau)\right)d\tau\Bigg)\lambda_k e^{-D\omega_k^2 t}-\int\limits_0^{t}e^{-D\omega_k^2(t-\tau)}f(\tau)d\tau \tag{4.4}$$

Then (4.3) takes the following form:

$$u_k(t)=\lambda_k\varphi_k e^{-D\omega_k^2 t}+g_k\widetilde{d}_k(t).$$

Now we will solve the inverse problem, with the given condition (2.7), i.e. if $u_k(\beta)=\psi_k$ find $g(x)$. Uniqueness of the solution (2.1)-(2.7) has been proven in [29], for $f(t)=1$ and for $D=\frac{1}{c^2}$. Consider the discontinuous case of coefficients, i.e. $D\neq\frac{1}{c^2}$ and $f(t)\neq const$.

We use the condition (2.7) and found formal solution (4.3), find $g_k$:

$$g_k=\frac{\psi_k}{\widetilde{d}_k(\beta)}-\frac{\lambda_k\varphi_k e^{-D\omega_k^2 t}}{\widetilde{d}_k(\beta)}, \tag{4.5}$$

where $\widetilde{d}_k(\beta)$ defined by (4.4).

Let $\beta$ be such that

$$\widetilde{d}_k(\beta)\neq 0. \tag{4.6}$$

Note, this set is not empty. We have recommended e.g., [2].

From here we obtain the solution of the given inverse problem for the case $f(t)\neq const$.

**Theorem 4.1.** *Let $\varphi(x)$, $\psi(x)$ functions are satisfied the conditions Theorem 3.3. Besides, $f(t) \in C[0,T]$ and the condition (4.6) satisfies. Then, there exists a uniquely solution the inverse problem II (2.1)-(2.7), which is represented by series*

$$g(x) = \sum_{k=1}^{\infty} g_k \sin(\omega_k x),$$

*where $g_k$ is defined by (4.4).*

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## References

1. Bzhikhatlov Kh.G., Nakhushev A.M. On a boundary value problem for an equation of mixed type, Dokl. AN SSSR 183 (2), 261-264 (1968).
2. Durdiev D.K., Merazhova Sh.B. On the solution of inverse problems for an equation of mixed parabolic-hyperbolic type: a one-dimensional case. // Bukhoro davlat university ilmiy ahborotnomasi, 2015 yil, 2-dream, 2-6 bb.
3. Dzhuraev T.D., Sopuev A., Mamazhanov A. Boundary value problems for equations of parabolic-hyperbolic type (FAN, Tashkent, 1986).
4. Egorov I.E. On a boundary value problem for an equation of mixed type with a spectral parameter, Mat. NEFU Notes 21(1), 11-17 (2014).
5. Egorov I.E. On a mixed problem for a hyperbolic-parabolic equation, Mat. notes 23(3), 389-400 (1978).
6. Frankl F.I. Selected Works on Gas Dynamics (Nauka, Moscow, 1973).
7. Fikera G., On a unified theory of boundary value problems for elliptic-parabolic equations of the second order, Sb. translations. Mathematics 7(6), 99-121 (1963).
8. Gelfand I.M. Some questions of analysis and differential equations, Uspekhi Mat. Nauk 14(3), 3-19 (2059).
9. Islomov B.I., Ubaidullaev U.Sh. Inverse problem for an equation of mixed type with fractional order operators in a rectangular region, Izv. universities. Mathematics 3, 29-46 (2021).
10. Jamalov S.Z., Ashurov R.R. On a linear inverse problem for a multidimensional equation of mixed type of the first kind of the second order, Izv. universities. Mathematics 6, 11-22 (2019).

11. Karchevsky A.L. Numerical solution of a one-dimensional inverse problem for an elasticity system, Dokl. AN 375 (2), 235-238 (2000).
12. Kurpinar E., Karchevsky A.L. Numerical solution of the inverse problem for the elasticity system for horizontally stratified media, Inverse Problems 20 (3), 953-976 (2004).
13. Karchevsky A.L. Numerical reconstruction of medium parameters of member of thin anisotropic layers, Journal of Inverse and Ill-Posed Problems 12 (5), 519-634 (2004).
14. Karchevsky A.L. Simultaneous reconstruction of permittivity and conductivity, Journal of Inverse and IllPosed Problems 17 (4), 385-402 (2009).
15. Karchevsky A.L. Determining the possibility of a rock burst in a coal seam, Siberian Journal of Industrial Mathematics 20 (4), 35-43 (2017).
16. Karchevsky A.L. Correct scheme of actions for numerical solution of the inverse problem by the optimization method, Siberian Journal of Computational Mathematics 11 (2), 139-149 (2008).
17. Kapustin N.Yu. The Tricomi problem for a parabolic-hyperbolic equation with a degenerate hyperbolic part, Differ. Equations 23 (1), 72-78 (1987).
18. Kozhanov A.I. On a nonlinear loaded parabolic equation and an inverse problem related to it, Mat. notes 76(6), 840-853 (2004).
19. Kozhanov A.I. Nonlinear loaded equations and inverse problems, Zh. Vychisl. math. and math.phys. 44(4), 694-716 (2004).
20. Nakhushev A.M. On the theory of boundary value problems for an equation of mixed hyperbolic-parabolic type, Dokl. AN SSSR 235 (2), 273-276 (1977).
21. Nazarova L.A., Nazarov L.A., Karchevsky A.L., Vandamm M. Estimation of diffusion-capacitive parameters of a coal seam from the data of gas pressure measurement in a well based on the solution of an inverse problem, Siberian Journal of Industrial Mathematics 17 ( 1), 78-85 (2014).
22. Rakhmanova L.Kh. Solution of a nonlocal problem by the spectral method for an equation of mixed parabolic-hyperbolic type in a rectangular region, Izv. universities. Mathematics 11, 36-40 (2007).
23. Romanov V.G. Inverse problems of mathematical physics. M.: Nauka, 1984, 264p.
24. Sabitov K.B. On the theory of equations of mixed parabolic-hyperbolic type with a spectral parameter, Differ. Equations 25 (1), 117-126 (1989).
25. Sabitov K.B., Safin E.M. Inverse problem for an equation of mixed parabolic-hyperbolic type in a rectangular region, Izv. universities. Mathematics 4, 55-62 (2010).

26. Sabitov K.B., Safin E.M. Inverse problem for an equation of mixed parabolic-hyperbolic type, Mat. notes 87(6), 907-918 (2010).
27. Sabitov K.B. Non-local problem for a non-homogeneous equation of parabolic-hyperbolic type in a rectangular domain, Mat. notes 89(4), 596-602 (2011).
28. Sabitov K.B., Sidirov S.N. An inverse problem for a degenerate parabolic-hyperbolic equation with a nonlocal boundary condition, Izv. universities. Mathematics 1, 46-59 (2015).
29. Sabitov K.B. Initial-boundary and inverse problems for an inhomogeneous equation of a mixed parabolic-hyperbolic equation, Mat. notes 102(3), 415-435 (2017).
30. Tricomi F., On linear equations of mixed type (Gostekhizdat, Moscow, 1947).
31. Tricomi F. Lectures on partial differential equations (Izd. inostr. lit., M., 1957).
32. Vragov V.N. Boundary Value Problems for Nonclassical Equations of Mathematical Physics (NSU Publishing House, Novosibirsk, 1983).

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