Кыргыз Республикасынын Улуттук илимдер академиясы
Национальная академия наук Кыргызской Республики
National Academy of Sciences of Kyrgyz Republic

ISSN 1694-8173

КР УИА Математика институтунун
Кабарлары

Вестник
Института математики НАН КР

Herald
of Institute of Mathematics of NAS of KR

№2

БИШКЕК • 2020

MSC 54B20,54E15,54E35 DOI: 10.52448/16948173_2022_2_27

ON A METRIC ON A HYPERSPACE

Zaitov A.A, Beshimova D.R, Jumaev D.I.
Tashkent Institute of Architecture and Civil Engineering, Uzbekistan;
Bukhara State University, Uzbekistan;
Tashkent Institute of Architecture and Civil Engineering, Uzbekistan.

In the paper we had constructed a function being a metric on the hyperspace of a metrizable space. Then it had established that this metric generates the Vietoris topology on the hyperspace. Further on the hyperspace of a uniform space we have allocated a (ε,δ)-uniform family of pseudo metrics, which generates a uniformity on the hyperspace.

Keywords: metric, hypespace, uniformity.

Бул макалада биз метрикалуу мейкиндиктин гипер мейкиндигинде метрика болгон функцияны курабыз. Бул метрика гипер мейкиндикте Vietoris топологиясын түзөрү аныкталган. Андан ары бир тектүү мейкиндиктин гипер мейкиндигинде ага бирдейликти жараткан псевдометриянын (ε,δ)-униформдуу үй-бүлөсү айырмаланат.

Урунтуу сөздөр: метрика, гипермейкиндик, бирдейлик.

В работе построена функция, являющаяся метрикой на гиперпространстве метризуемого пространства. Установлено, что эта метрика порождает топологию Вьеториса на гиперпространстве. Далее, на гиперпространстве равномерного пространства выделено (ε,δ)-равномерное семейство псевдометрик, порождающее равномерность на нем.

Ключевые слова: метрика, гиперпространство, равномерность.

As it is known, the Hausdorff metric evaluates how two sets differ in their emplacement in the metric space. Let (X,ρ) be a metric space. The Hausdorff distance (metric) ρ_H between two compact subsets F_1, F_2 of the space X is the number $\rho_H(F_1,F_2)=\max\{\inf\{\varepsilon:\varepsilon>0, F_2\subset O_\varepsilon F_1\}, \inf\{\varepsilon:\varepsilon>0, F_1\subset O_\varepsilon F_2\}\}$.

In this paper, we introduce another metric on the hyperspace, denoted by ρ_Z. For this purpose, we present the following constructions.

Let X be a compact Hausdorff space. Denote (see, [3])

$X_1=X_2=X_3=X$, $X_{12}=X_1\times X_2$, $X_{13}=X_1\times X_3$, $X_{23}=X_2\times X_3$,

$\Delta = \{(x,y) \in X^2 : x = y\}$, $X_{123} = X_1 \times X_2 \times X_3$.

Consider projections

$\pi_{12}^{123} : X_{123} \to X_{12}$, $\quad \pi_{13}^{123} : X_{123} \to X_{13}$, $\quad \pi_{23}^{123} : X_{123} \to X_{23}$,

$\pi_1^{12} : X_{12} \to X_1$, $\quad \pi_1^{13} : X_{13} \to X_1$, $\quad \pi_2^{23} : X_{23} \to X_2$,

$\pi_2^{12} : X_{12} \to X_2$, $\quad \pi_3^{13} : X_{13} \to X_3$, $\quad \pi_3^{23} : X_{23} \to X_3$.

Lemma 1. Let $\Phi_{12} \subset X_{12}$ and $\Phi_{23} \subset X_{23}$ be an arbitrary pair of subsets such that

$\pi_2^{12}(\Phi_{12}) = \pi_2^{23}(\Phi_{23})$.

Then there exists a set $\Phi_{123} \subset X_{123}$ such that

$\pi_3^{23} : X_{23} \to X_3$, $\quad \pi_{23}^{123}(\Phi_{123}) = \Phi_{23}$ v

Proof. Putting $\Phi_{123} = (\pi_{12}^{123})^{-1}(\Phi_{12}) \cap (\pi_{23}^{123})^{-1}(\Phi_{23})$ we get the searching set.

Lemma 2. Let F_1, F_2 and F_3 be an arbitrary triple of subsets of X, and $\Phi_{12} \subset X_{12}$ and $\Phi_{23} \subset X_{23}$ sets such that

$\pi_1^{12}(\Phi_{12}) = F_1$, $\quad \pi_2^{23}(\Phi_{23}) = F_2$,

$\pi_2^{12}(\Phi_{12}) = F_2$, $\quad \pi_3^{23}(\Phi_{23}) = F_3$.

Then there exists a set $\Phi_{13} \subset X_{13}$ such that

$\pi_1^{13}(\Phi_{13}) = F_1$, $\quad \pi_3^{13}(\Phi_{13}) = F_3$.

Proof. According to Lemma 1, we construct a set Φ_{123}, and then put $\Phi_{13} = \pi_{13}^{123}(\Phi_{123})$. This set has the required conditions.

For a compact Hausdorff space X let $\exp X$ be the hyperspace of X, i. e. a set of all closed nonempty subsets of X equipped with the Vietoris topology.

For a compact metric space X we define a function $\rho_Z : \exp X \times \exp X \to \mathbb{R}$ by the equality

$$\rho_Z(F_1, F_2) = \inf\left\{\sup\{\rho(x,y) : (x,y) \in \Phi\} : \Phi \in \exp X^2, \pi_1^{12}(\Phi) = F_1,\ \pi_2^{12}(\Phi) = F_2\right\}.$$

Lemma 3. For every couple of sets F_1, $F_2 \in \exp X$ there exists a set

$\Phi_{12} \in \exp X^2$ such that $\pi_1^{12}(\Phi_{12}) = F_1$, $\pi_2^{12}(\Phi_{12}) = F_2$ and

$\rho_Z(F_1, F_2) = \sup\{\rho(x,y):(x,y) \in \Phi_{12}\}$.

Proof. Clearly, for the product $F_1 \times F_2$ equalities $\pi_1^{12}(F_1 \times F_2) = F_1$, $\pi_2^{12}(F_1 \times F_2) = F_2$ hold. Let $b = \sup\{\rho(x,y):(x,y) \in F_1 \times F_2\}$. Since F_1 and F_2 are compact and the metric ρ is a continuous function on $X \times X$, we have $b = \max\{\rho(x,y):(x,y) \in F_1 \times F_2\} < \infty$. For every $\Phi \subset F_1 \times F_2$ such that $\pi_1^{12}(\Phi) = F_1$, $\pi_2^{12}(\Phi) = F_2$ one has $0 \le \sup\{\rho(x,y):(x,y) \in \Phi\} \le b$. So, for every net $\{\Phi^\alpha\}$ such that $\Phi^\alpha \subset F_1 \times F_2$, $\pi_1^{12}(\Phi^\alpha) = F_1$, $\pi_2^{12}(\Phi^\alpha) = F_2$, $\alpha \in \mathfrak{A}$, the number net $\{\sup\{\rho(x,y):(x,y) \in \Phi^\alpha\}\}$ has a limit. Consider such a net $\{\Phi^\alpha\}$ which besides listed above properties satisfies the following condition: $\sup\{\rho(x,y):(x,y) \in \Phi^\alpha \ge \sup\{\rho(x,y):(x,y)\} \in \Phi^\beta\}$ for all α, β, $\alpha \prec \beta$. The number net $\{\sup\{\rho(x,y):\ (x,y) \in \Phi^\alpha\}\}$ has a limit. Now using the Zorn's lemma, we complete the proof.

Theorem 1. For any metric ρ on a compact metrizable space X the function ρ_Z is a metric on $\exp X$.

Proof. It is easy to see that $\rho_Z \ge 0$. For any $F \in \exp X$ we have

$$\rho_Z(F,F) = \inf\{\sup\{\rho(x,y):(x,y) \in \Phi\} : \Phi \in \exp X^2, \pi_1^{12}(\Phi) = F, \pi_2^{12}(\Phi) = F\} \le$$

$$\le \sup\{\rho(x,y):(x,y) \in \Delta\} = 0,$$

i. e. $\rho_Z(F,F) = 0$. Let now $\rho_Z(F_1,F_2) = 0$ for some F_1, $F_2 \in \exp X$. It should exist a nonempty closed subset $\Phi \subset X^2$ such that $\pi_1^{12}(\Phi) = F_1$, $\pi_2^{12}(\Phi) = F_2$ and $\Phi \subset \Delta$. The last inclusion implies $\pi_1^{12}(\Phi) = \pi_2^{12}(\Phi)$. Consequently, $F_1 = F_2$.

Thus $\rho_Z(F_1,F_2) = 0$ if and only if $F_1 = F_2$.

Clearly, the function ρ_Z is symmetric.

It reminds to establish the triangle rule. Take any triple F_1, F_2, $F_3 \in \exp X$. Then

$$\rho_Z(F_1,F_2)+\rho_Z(F_2,F_3)=\inf\left\{\sup\{\rho(x,y):(x,y)\in\Phi\}:\Phi\in\exp X^2,\pi_i^{12}(\Phi)=F_i,i=1,2\right\}$$

$$+\inf\left\{\sup\{\rho(y,z):(y,z)\in\Phi\}:\Phi\in\exp X^2,\pi_i^{23}(\Phi)=F_i,i=2,3\right\}$$

$$=(by\ \ force\ of\ \ Lemma\ 3)=\sup\{\rho(x,y):(x,y)\in\Phi_{12}\}+\sup\{\rho(y,z):(y,z)\in\Phi_{23}\}$$
$$=(by\ \ force\ of\ \ Lemma\ 1)=\sup\{\rho(x,y):(x,y,z)\in\Phi_{123}\}+\sup\{\rho(y,z):(x,y,z)\in\Phi_{123}\}$$
$$\geq(by\ the\ property\ of\ \ \sup)\geq\sup\{\rho(x,y)+\rho(y,z):(x,y,z)\in\Phi_{123}\}$$

$$\geq(by\ the\ triangle\ axiom)\geq\sup\{\rho(x,z):(x,y,z)\in\Phi_{123}\}$$

$$=(by\ \ force\ of\ \ Lemma\ 2)=\sup\{\rho(x,z):(x,z)\in\Phi_{13}\}$$
$$\geq(by\ the\ property\ of\ \ \inf)\geq\inf\left\{\sup\{\rho(x,z):(x,z)\in\Phi\}:\Phi\in\exp X^2,\pi_i^{13}(\Phi)=F_i,i=1,3\right\}$$
$$=(by\ the\ definition)=\rho_Z(F_1,F_3).$$

Theorem 1 is proved.

Theorem 2. For a compact metric space (X,ρ) the metric ρ_Z generates the Vietoris topology on $\exp X$.

Proof. Consider a sequence $\{F_n\}$ of closed sets converging in the Vietoris topology to a closed set F_0. Suppose $\lim\limits_{n\to\infty}\rho_Z(F_n,F_0)=a>0$. Without losing generality we can assume that $\rho_Z(F_n,F_0)\geq\frac{a}{2}$ for all $n\in\mathbb{N}$. We claim that $F_n\not\subset O_{\frac{a}{4}}F_0$ and $F_0\not\subset O_{\frac{a}{4}}F_n$. Otherwise we get an inequality: $\rho_Z(F_n,F_0)\leq\frac{a}{4}$. Hence, $F_n\not\subset O_{\frac{a}{4}}F_0$ and $F_0\not\subset O_{\frac{a}{4}}F_n$ for all n. Consequently, the sequence $\{F_n\}$ does not converges to F_0 in the Vietoris topology. We have obtained a contradiction.

Let us show the opposite. Let $\rho_Z(F_n,F_0)\to 0$. Take sets $\Phi_{n0}\in\exp X^2$ satisfying the conclusion of Lemma 3. Each set Φ_{n0} is assigned a number $\sup\{\rho(x,y):(x,y)\in\Phi_{n0}\}$. It is understandable that the number 0 corresponds to the only set $\Phi_{00}=\Delta(X)\cap(F_0\times F_0)$. According to the data, the sequence $\{\sup\{\rho(x,y):(x,y)\in\Phi_{n0}\}\}_{n=1}^{\infty}$ has to converge to 0. Then for any $\varepsilon>0$ there exists

n_0 such that $F_n \subset O_\varepsilon F_0$ for all $n \geq n_0$. So, the sequence $\{F_n\}$ converges to F_0 in the Vietoris topology.

Theorem 2 is proved.

Lemma 4. For any pseudometric ρ on X the function ρ_Z is a pseudometric on $\exp X$.

Proof. Since the proof of the Lemma consists of repeating the reasonings carried out in the proof of Theorem 1, it suffices to note that in this case the function ρ_Z is not a metric on $\exp X$. Indeed, let x, $y \in X$, $x \neq y$ and $\rho(x,y) = 0$. Then $\rho_Z(\{x\},\{y\}) = \rho(x,y) = 0$ although $\{x\} \neq \{y\}$.

Consider a Tychonoff space X and define a set (see, [2], [5])

$\exp X = \{K \subset X : \ K \textit{ is compact and } K \neq \varnothing\}$.

Consider any family $\mathfrak{P}$ of pseudometrics and define

$U_d(r) = \{(x,y) : d(x,y) < r\}$

for $d \in \mathfrak{P}$ and $r > 0$. The resulting family $\{U_d(r) : d \in \mathfrak{P}, r > 0\}$ of entourages is a subbase for a uniformity in that the family of finite intersections is a base for a uniformity, denoted $\mathcal{U}_{\mathfrak{P}}$.

Given a sequence $\{V_n : n \in \mathbb{N}\}$ of entourages on a set X such that $V_0 = X^2$ and $V_{n+1}^3 \subset V_n$ for all n one can find a pseudometric d on X such that $U_d(2^{-n}) \subset V_n \subset \{(x,\ y) : d(y,x) \leq 2^{-n}\}$ for all n. Thus every uniform structure can be defined by a family of pseudometrics. The family of all pseudometrics d that satisfy $(\forall r > 0)(U_d(r) \in \mathcal{U})$ is denoted $\mathfrak{P}_{\mathcal{U}}$; it is the largest family of pseudometrics that generate $\mathcal{U}$ [1], [4]. The family $\mathfrak{P}_{\mathcal{U}}$ satisfies the following two properties.

$(P1)$ If d_1, $d_2 \in \mathfrak{P}_{\mathcal{U}}$ then $\max\{\rho_1, \rho_2\} \in \mathfrak{P}_{\mathcal{U}}$;

$(P2)$ If d is a pseudometric and for every $\varepsilon > 0$ there are $\rho \in \mathfrak{P}_{\mathcal{U}}$ and $\delta > 0$ such that always $\rho(x,y) < \delta$ implies $d(x,y) < \varepsilon$ then $d \in \mathfrak{P}_{\mathcal{U}}$.

A family $\mathfrak{P}$ of pseudometrics with these properties is called a pseudometric uniformity; it satisfies the equality $\mathfrak{P}_{\mathcal{U}_{\mathfrak{P}}} = \mathfrak{P}$ [4].

Theorem 3. For a Tychonoff space X and a family of pseudometrics $\mathfrak{P} = \{\rho\}$ generating a uniformity on X the family

$$\mathfrak{P}_{\exp} = \{d : d \textit{ is a pseudometric on } \exp X \textit{ and}$$
$$\forall \varepsilon > 0, \exists \rho \in \mathfrak{P}, \exists \delta > 0 \textit{ that } \rho_Z(F_1, F_2) < \delta \Rightarrow d(F_1, F_2) < \varepsilon\}$$

generates a uniformity on $\exp X$.

Proof. We will show that the family $\mathfrak{P}_{\exp}$ satisfies the above two conditions (P1) and (P2).

Let ρ_1, $\rho_2 \in \mathfrak{P}$ be arbitrary pseudometrics and F_1, $F_2 \in \exp X$ arbitrary sets. Take sets Φ^1_{12}, Φ^2_{12} and $\Phi^{\vee}_{12}$ satisfying Lemma 3 for ρ_1, ρ_2 and $\max\{\rho_1, \rho_2\}$, respectively. We have

$$\max\{\rho_{1Z}, \rho_{2Z}\}(F_1, F_2) = \max\{\rho_{1Z}(F_1, F_2), \rho_{2Z}(F_1, F_2)\}$$

$$= \max\left\{\inf_{\Phi^1} \sup_{(x,y)\in\Phi^1} \rho_1(x,y), \inf_{\Phi^2} \sup_{(x,y)\in\Phi^2_{12}} \rho_2(x,y)\right\}$$

$$= \max\left\{\sup_{(x,y)\in\Phi^1_{12}} \rho_1(x,y), \sup_{(x,y)\in\Phi^2_{12}} \rho_2(x,y)\right\}$$

$$\le \max\left\{\sup_{(x,y)\in\Phi^{\vee}_{12}} \rho_1(x,y), \sup_{(x,y)\in\Phi^{\vee}_{12}} \rho_2(x,y)\right\}$$

$$= \sup_{(x,y)\in\Phi^{\vee}_{12}} \{\max\{\rho_1, \rho_2\}(x,y)\}$$

$$= (\max\{\rho_1, \rho_2\})_Z(F_1, F_2),$$

i. e. $\max\{\rho_{1Z}, \rho_{2Z}\}(F_1, F_2) \le (\max\{\rho_1, \rho_2\})_Z(F_1, F_2)$.

Let us show the reverse inequality. One can check that the following are valid.

$$(\max\{\rho_1, \rho_2\})_Z(F_1, F_2) = \inf_{\Phi} \sup_{(x,y)\in\Phi} \max\{\rho_1, \rho_2\}(x,y)$$

$$= \inf_{\Phi} \sup_{(x,y)\in\Phi} \max\{\rho_1(x,y), \rho_2(x,y)\}$$

$$=\inf_{\Phi}\max\left\{\sup_{(x,y)\in\Phi}\rho_1(x,y),\ \sup_{(x,y)\in\Phi}\rho_2(x,y)\right\}$$

$$\le\inf_{\Phi}\max\left\{\sup_{(x,y)\in\Phi_{12}^1}\rho_1(x,y),\ \sup_{(x,y)\in\Phi}\rho_2(x,y)\right\}$$

$$\le\max\left\{\sup_{(x,y)\in\Phi_{12}^1}\rho_1(x,y),\ \sup_{(x,y)\in\Phi_{12}^2}\rho_2(x,y)\right\}$$

$$=\max\left\{\rho_{1Z}(F_1,F_2),\rho_{2Z}(F_1,F_2)\right\}=\max\left\{\rho_{1Z},\rho_{2Z}\right\}(F_1,F_2).$$

Due to the arbitrariness of the sets F_1 and F_2 we get $\max\{\rho_{1Z},\rho_{2Z}\}=(\max\{\rho_1,\rho_2\})_Z$. On the other hand $\max\{\rho_1,\rho_2\}\in\mathfrak{P}$. So, by construction we obtain that $\max\{\rho_{1Z},\rho_{2Z}\}\in\mathfrak{P}_{\exp}$.

For any two pseudometrics d_1, $d_2\in\mathfrak{P}_{\exp}$ suppose

$\forall\varepsilon>0,\ \exists\rho_i\in\mathfrak{P},\ \exists\delta_i>0$ that $\rho_{iz}(F_1,F_2)<\delta_i\Rightarrow d_i(F_1,F_2)<\varepsilon,\ i=1,2;\ F_1,F_2\in\exp X.$

Then

$\max\{\rho_{1Z},\rho_{2Z}\}(F_1,F_2)<\min\{\delta_1,\delta_2\}\Rightarrow\max\{d_1,d_2\}(F_1,F_2)<\varepsilon,\ i=1,2;\ F_1,F_2\in\exp X.$

Consequently, $\max\{d_1,d_2\}\in\mathfrak{P}_{\exp}$.

there exist pseudometrics ρ_1, $\rho_2\in\mathfrak{P}$ such that $d_1\le\rho_{1Z}$ and $d_2\le\rho_{2Z}$. Then $\max\{d_1,d_2\}\le\max\{\rho_{1Z},\rho_{2Z}\}$. Hence, $\max\{d_1,d_2\}\in\mathfrak{P}_{\exp}$ Condition (P1) is verified.

Let now d be a pseudometric on $\exp X$ such that for every $\varepsilon>0$ there are $\tilde d\in\mathfrak{P}$ and $\delta>0$ such that $\tilde d_Z(F_1,F_2)<\delta\Rightarrow d(F_1,F_2)<\varepsilon,\qquad F_1,F_2\in\exp X$.

Then by construction there exists $d\in\mathfrak{P}_{\exp}$. Condition (P2) is established, and the proof of Theorem 3 is completed.

The work was carried out within the framework of the project of the Tashkent Institute of Architecture and Civil Engineering and Bukhara State University, Uzbekistan.

REFERENCES

1. Borubaev A.A. *Uniform topology and its applications.* – Bishkek, Publishing house "Ilim". 2021. – 345 p.

2. Zaitov A.A., Kurbanov Kh.Kh. When is the space of semi-additive functionals an absolute (neighbourhood) retract? *Proceedings of the International Geometry Center*. Vol. 15, no. 2 (2022), pp. 86-99.
3. Zaitov A.A. On a metric on the space of idempotent probability measures. *Appl. Gen. Topol*. Vol. 21, no. 1 (2020), pp. 35-51.
4. Hart K.P., Nagata J. and Vaughan J.E. *Encyclopedia of General Topology. Elsevier Science Ltd.* 2004.
5. Zaitov A.A., Jumaev D.I. Hyperspace of the π-complete spaces and maps. *Eurasian Mathematical Journal*. Vol. 12, no. 2 (2021), pp. 1-16.

Received: 20th October, 2022

Revised: 22th November, 2022

Accepted: 15th December, 2022

Information about the authors

Zaitov Adilbek Atakhanovich, Professor, Doctor of Sciences in Physics and Mathematics, Head of the Department "Mathematics and Natural Disciplines", Tashkent Institute of Architecture and Civil Engineering, 9, Yangi Shahar Str., Tashkent 100206, Uzbekistan, ORCID: 0000-0002-2248-0442
e-mail: adilbek_zaitov@mail.ru

Beshimova Dilorom Ruzinazarovna, PhD-student, Bukhara State University, 11, M. Iqbol Str., Bukhara 200100, Uzbekistan,e-mail: Drbeshimova@gmail.com

Jumaev Davron Ilxomovich, Senior Lecturer, Department "Mathematics and Natural Disciplines", Tashkent Institute of Architecture and Civil Engineering, 9, Yangi Shahar Str., Tashkent 100206, Uzbekistan, ORCID: 0000-0001-5387-3639,e-mail: d-a-v-ron@mail.ru