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ANALYSIS OF THE DISCRETE SPECTRUM OF THE FAMILTY OF 3×3 OPERATOR MATRICES

MUKHIDDIN I. MUMINOV*
Faculty of Sciences
Universiti Teknologi Malaysia (UTM)
81310 Skudai, Johor Bahru, Malaysia

TULKIN H. RASULOV†
Faculty of Physics and Mathematics
Bukhara State University
M. Ikbol str. 11, 200100 Bukhara, Uzbekistan

NARGIZA A. TOSHEVA‡
Faculty of Physics and Mathematics
Bukhara State University
M. Ikbol str. 11, 200100 Bukhara, Uzbekistan

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Abstract

We consider the family of 3×3 operator matrices $\mathbf{H}(K)$, $K\in\mathbb{T}^3 := (-\pi;\pi]^3$ associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. We find a finite set $\Lambda\subset\mathbb{T}^3$ to prove the existence of infinitely many eigenvalues of $\mathbf{H}(K)$ for all $K\in\Lambda$ when the associated Friedrichs model has a zero energy resonance. It is found that for every $K\in\Lambda$, the number $N(K,z)$ of eigenvalues of $\mathbf{H}(K)$ lying on the left of z, $z<0$, satisfies the asymptotic relation $\lim\limits_{z\to -0} N(K,z)|\log|z||^{-1} = \mathcal{U}_0$ with $0<\mathcal{U}_0<\infty$, independently on the cardinality of Λ. Moreover, we prove that for any $K\in\Lambda$ the operator $\mathbf{H}(K)$ has a finite number of negative eigenvalues if the associated Friedrichs model has a zero eigenvalue or a zero is the regular type point for positive definite Friedrichs model.

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*E-mail address: mmuminov@mail.ru
†E-mail address: rth@mail.ru
‡E-mail address: nargiza_n@mail.ru

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1 Introduction

The main objective of the present paper is to establish the finiteness or infiniteness of the number of eigenvalues for a family of 3×3 operator matrices $\mathbf{H}(K)$, $K \in \mathbb{T}^3 := (-\pi;\pi]^3$ and especially the asymptotics for the number of infinitely many eigenvalues (Efimov's effect case). These operator matrices are associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles.

The Efimov effect is one of the most remarkable results in the spectral analysis for continuous three-particle Schrödinger operators: if none of the three two-particle Schrödinger operators (corresponding to the two-particle subsystems) has negative eigenvalues but at least two of them have zero energy resonance, then the three-particle Schrödinger operator has infinitely many negative eigenvalues accumulating at zero.

For the first time the Efimov effect has been discussed in [9]. Then this problem has been studied on a physical level of rigor in [2, 6]. A rigorous mathematical proof of the existence of Efimov's effect was originally carried out in [30] and then many works devoted to this subject, see for example [8, 21, 25, 26, 27]. The main result obtained by Sobolev [25] (see also [27]) is an asymptotics of the form $\mathcal{U}_0|\log|z||$ for the number $N(z)$ of eigenvalues on the left of z, $z < 0$, where the coefficient $\mathcal{U}_0$ does not depend on the two-particle potentials v_α and is a positive function of the ratios m_1/m_2 and m_2/m_3 of the masses of the three particles.

In a system of three-particles on three-dimensional lattices, due to the fact that the discrete analogue of the Laplacian or its generalizations are not rationally invariant, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-mass motion and the other one to the internal degrees of freedom. In particular, in this case the Efimov effect exists only for the zero value of the three-particle quasi-momentum $K \in \mathbb{T}^3$ (see [1, 3, 13]). An asymptotics analogous to [25, 27] was obtained in [1, 3] for the number of eigenvalues.

In all above mentioned papers devoted to the Efimov effect, the systems where the number of quasi-particles is fixed have been considered. In the theory of solid-state physics [18], quantum field theory [10], statistical physics [16, 17], fluid mechanics [7], magnetohydrodynamics [15] and quantum mechanics [28] some important problems arise where the number of quasi-particles is finite, but not fixed. In [24] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number.

In the present paper we consider the family of 3×3 operator matrices $\mathbf{H}(K)$, $K \in \mathbb{T}^3$ associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. This operator acts in the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space and it is arising in the spectral analysis of the energy operator of the spin-boson model of radioactive decay with two bosons on the torus [19, 22]. We discuss the case where the dispersion function has form $\varepsilon(p) = \sum_{i=1}^{3}(1-\cos(np^{(i)}))$ with $n > 1$. We denote by Λ the set of points $\mathbb{T}^3$ where the function $\varepsilon(\cdot)$ takes its (global) minimum. Under some smoothness assumptions on the parameters of a family of Friedrichs models $\mathbf{h}(k)$, $k \in \mathbb{T}^3$, we obtain the following results:

(i) We describe the location of the essential spectrum $\sigma_{\text{ess}}(\mathbf{H}(K))$ of $\mathbf{H}(K)$, $K \in \mathbb{T}^3$ via the spectrum of $\mathbf{h}(k)$, $k \in \mathbb{T}^3$;

(ii) We prove that for all $K \in \Lambda$ the $\mathbf{H}(K)$ has infinitely many negative eigenvalues accumulating at zero, if the operator $\mathbf{h}(\mathbf{0})$, $\mathbf{0} = (0,0,0)$ has a zero energy resonance (Efimov's effect). Moreover, for any $K \in \Lambda$ we establish the asymptotics $N(K;z) \sim \mathcal{U}_0|\log|z||$ with $0 < \mathcal{U}_0 < \infty$ for the number $N(K;z)$ of eigenvalues of $\mathbf{H}(K)$ lying on the left of z, $z < \min\sigma_{\text{ess}}(\mathbf{H}(K)) = 0$;

(iii) We prove the finiteness of negative eigenvalues of $\mathbf{H}(K)$ for $K \in \Lambda$, if the operator $\mathbf{h}(\mathbf{0})$ has a zero eigenvalue or a zero is the regular type point for $\mathbf{h}(\mathbf{0})$ with $\mathbf{h}(\mathbf{0}) \geq 0$.

We remark that for the Friedrichs model $\mathbf{h}(\mathbf{0})$ the presence of a zero energy resonance (consequently the existence of the Efimov effect for $\mathbf{H}(K)$, $K \in \Lambda$) is due to the annihilation and creation operators.

We point out that the operator $\mathbf{H}(K)$ has been considered before in [4, 5, 14] for $K = 0$ and $n = 1$, where proven the existence of Efimov's effect. Similar asymptotics for the number of eigenvalues was obtained in [4]. We recall that the main results (without proofs) of this paper has been announced in [20]. This paper is devoted to the detailed proof of these results.

It surprising that in the assertion (ii) the asymptotics for $N(K;z)$ is the same for all $K \in \Lambda$ and is stable with respect to the number n. Recall that in all papers devoted to Efimov's effect for lattice systems the existence of this effect have been proved only for zero value of the quasi-momentum ($K = 0$) and for the case $n = 1$.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the operator matrices $\mathbf{H}(K)$, $K \in \mathbb{T}^3$ are described as the family of bounded self-adjoint operators in the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space and the main results are formulated. In Section 3, we discuss some results concerning threshold analysis of the Friedrichs model $\mathbf{h}(k)$, $k \in \mathbb{T}^3$. In Section 4 we give a modification of the Birman-Schwinger principle for $\mathbf{H}(K)$, $K \in \mathbb{T}^3$. Section 5 we establish the finiteness of the number of eigenvalues of the operator $\mathbf{H}(K)$, $K \in \Lambda$. In section 6 we obtain the asymptotic formula for the number of negative eigenvalues of $\mathbf{H}(K)$, $K \in \Lambda$.

We adopt the following conventions throughout the present paper. Let $\mathbb{T}^3$ be the three-dimensional torus, the cube $(-\pi,\pi]^3$ with appropriately identified sides equipped with its Haar measure. Denote by $\sigma(\cdot)$, $\sigma_{\text{ess}}(\cdot)$ and $\sigma_{\text{disc}}(\cdot)$, respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator. In what follows we deal with the operators in various spaces of vector-valued functions. They will be denoted by bold letters and will be written in the matrix form.

2 Family of 3×3 operator matrices and main results

Let $\mathbb{C}$ be the field of complex numbers, $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^{\text{s}}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_1 = L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 = L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$, that is, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. The spaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are called zero-, one- and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\mathrm{s}}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively. It is well-known that if $\mathcal{A}$ is a bounded linear in a Hilbert space $\mathcal{H}$ and a decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ into three Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ is given, then $\mathcal{A}$ always admits [12, 29] a block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix}$$

with linear operators $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0,1,2$.

Let us consider the following family of 3×3 operator matrices $\mathbf{H}(K)$, $K \in \mathbb{T}^3$ acting in the Hilbert space $\mathcal{H}$ as

$$\mathbf{H}(K) := \begin{pmatrix} H_{00}(K) & H_{01} & 0 \\ H_{01}^* & H_{11}(K) & H_{12} \\ 0 & H_{12}^* & H_{22}(K) \end{pmatrix}$$

with the entries

$$H_{00}(K)f_0 = w_0(K)f_0, \quad H_{01}f_1 = \int_{\mathbb{T}^3} v_0(s)f_1(s)ds, \quad (H_{11}(K)f_1)(p) = w_1(K;p)f_1(p),$$

$$(H_{12}f_2)(p) = \int_{\mathbb{T}^3} v_1(s)f_2(p,s)ds, \quad (H_{22}(K)f_2)(p,q) = w_2(K;p,q)f_2(p,q),$$

where H_{ij}^* $(i < j)$ denotes the adjoint operator to H_{ij} and $f_i \in \mathcal{H}_i$, $i = 0,1,2$.

Here $w_0(\cdot)$ and $v_i(\cdot)$, $i = 0,1$ are real-valued bounded functions on $\mathbb{T}^3$, the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ are defined by the equalities

$$w_1(K;p) := l_1\varepsilon(p) + l_2\varepsilon(K-p) + 1, \quad w_2(K;p,q) := l_1\varepsilon(p) + l_1\varepsilon(q) + l_2\varepsilon(K-p-q),$$

respectively, with $l_1, l_2 > 0$ and

$$\varepsilon(q) := \sum_{i=1}^{3}(1-\cos(nq^{(i)})), \quad q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3, \quad n \in \mathbb{N}.$$

Under these assumptions the operator $\mathbf{H}(K)$ is bounded and self-adjoint.

We remark that the operators H_{01} and H_{12} resp. H_{01}^* and H_{12}^* are called annihilation resp. creation operators, respectively. In this paper we consider the case, where the number of annihilations and creations of the particles of the considering system is equal to 1. It means that $H_{ij} \equiv 0$ for all $|i-j| > 1$.

To study the spectral properties of the operator $\mathbf{H}(K)$ we introduce a family of bounded self-adjoint operators (Friedrichs models) $\mathbf{h}(k)$, $k \in \mathbb{T}^3$, which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathbf{h}(k) := \begin{pmatrix} h_{00}(k) & h_{01} \\ h_{01}^* & h_{11}(k) \end{pmatrix},$$

where

$$h_{00}(k)f_0 = (l_2\varepsilon(k)+1)f_0, \quad h_{01}f_1 = \frac{1}{\sqrt{2}}\int\limits_{\mathbb{T}^3} v_1(s)f_1(s)ds,$$

$$(h_{11}(k)f_1)(q) = E_k(q)f_1(q), \quad E_k(q) := l_1\varepsilon(q)+l_2\varepsilon(k-q).$$

The following theorem [4, 5] describes the location of the essential spectrum of the operator $\mathbf{H}(K)$ by the spectrum of the family $\mathbf{h}(k)$ of Friedrichs models.

Theorem 2.1. *For the essential spectrum of* $\mathbf{H}(K)$ *the equality*

$$\sigma_{\mathrm{ess}}(\mathbf{H}(K)) = \bigcup_{p\in\mathbb{T}^3}\{\sigma_{\mathrm{disc}}(\mathbf{h}(K-p))+l_1\varepsilon(p)\}\cup[m_K;M_K] \tag{2.1}$$

holds, where the numbers m_K *and* M_K *are defined by*

$$m_K := \min_{p,q\in\mathbb{T}^3} w_2(K;p,q) \quad \textit{and} \quad M_K := \max_{p,q\in\mathbb{T}^3} w_2(K;p,q).$$

Let Λ a subset of $\mathbb{T}^3$ given by

$$\Lambda := \left\{(p^{(1)},p^{(2)},p^{(3)}) : p^{(i)} \in \left\{0,\pm\frac{2}{n}\pi;\pm\frac{4}{n}\pi;\ldots;\pm\frac{n'}{n}\pi\right\}\cup\Pi_n,\ i=1,2,3\right\},$$

where

$$n' := \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd}\end{cases} \quad \text{and} \quad \Pi_n := \begin{cases} \{\pi\}, & \text{if } n \text{ is even} \\ \emptyset, & \text{if } n \text{ is odd}\end{cases}$$

Direct calculation shows that the cardinality of Λ is equal to n^3. It is easy to check that for any $K\in\Lambda$ the function $w_2(K;\cdot,\cdot)$ has non-degenerate zero minimum at the points of $\Lambda\times\Lambda$, that is, $m_K = 0$ for $K\in\Lambda$.

The following assumption we be needed throughout the paper: the function $v_1(\cdot)$ is either even or odd function on each variable and there exist all second order continuous partial derivatives of $v_1(\cdot)$ on $\mathbb{T}^3$.

Since $\mathbf{0} = (0,0,0)\in\Lambda$ the definition of the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ implies the identity $\mathbf{h}(\mathbf{0}) \equiv \mathbf{h}(k)$ for all $k\in\Lambda$.

Let us denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 2.2. The operator $\mathbf{h}(\mathbf{0})$ is said to have a zero energy resonance, if the number 1 is an eigenvalue of the integral operator given by

$$(G\psi)(q) = \frac{v_1(q)}{2(l_1+l_2)}\int\limits_{\mathbb{T}^3}\frac{v_1(s)\psi(s)}{\varepsilon(s)}ds, \quad \psi\in C(\mathbb{T}^3)$$

and at least one (up to a normalization constant) of the associated eigenfunctions ψ satisfies the condition $\psi(p')\neq 0$ for some $p'\in\Lambda$. If the number 1 is not an eigenvalue of the operator G, then we say that $z=0$ is a regular type point for the operator $\mathbf{h}(\mathbf{0})$.

We notice that in Definition 2.2 the requirement of the existence of the eigenvalue 1 of G corresponds to the existence of a solution of $\mathbf{h}(\mathbf{0})f = 0$ and the condition $\psi(p') \neq 0$ for some $p' \in \Lambda$ implies that the solution f of this equation does not belong to $\mathcal{H}_0 \oplus \mathcal{H}_1$. More precisely, if the operator $\mathbf{h}(\mathbf{0})$ has a zero energy resonance, then the solution $\psi(\cdot)$ of $G\psi = \psi$ is equal to $v_1(\cdot)$ (up to constant factor) and the vector $f = (f_0, f_1)$, where

$$f_0 = \text{const} \neq 0, \quad f_1(q) = -\frac{v_1(q) f_0}{\sqrt{2}(l_1 + l_2)\varepsilon(q)}, \tag{2.2}$$

obeys the equation $\mathbf{h}(\mathbf{0})f = 0$ such that $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ (see Lemma 3.3). If the operator $\mathbf{h}(\mathbf{0})$ has a zero eigenvalue, then the vector $f = (f_0, f_1)$, where f_0 and f_1 are defined by (2.2), again obeys the equation $\mathbf{h}(\mathbf{0})f = 0$ and $f_1 \in L_2(\mathbb{T}^3)$ (see proof of the assertion (i) of Lemma 3.2).

As in the introduction, let us denote by $\tau_{\text{ess}}(K)$ the bottom of the essential spectrum of $\mathbf{H}(K)$ and by $N(K, z)$ the number of eigenvalues of $\mathbf{H}(K)$ on the left of z, $z \leq \tau_{\text{ess}}(K)$.

Note that if the operator $\mathbf{h}(\mathbf{0})$ has either a zero energy resonance or a zero eigenvalue, then for any $K \in \Lambda$ and $p \in \mathbb{T}^3$ the operator $\mathbf{h}(K - p) + l_1\varepsilon(p)\mathbf{I}$ is non-negative (see Lemma 3.4), where $\mathbf{I}$ is the identity operator in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Hence Theorem 2.1 and equality $m_K = 0$, $K \in \Lambda$ imply that $\tau_{\text{ess}}(K) = 0$ for all $K \in \Lambda$.

The main results of the present paper as follows.

Theorem 2.3. *Let $K \in \Lambda$ and one of the following assumptions hold:*
(i) *the operator $\mathbf{h}(\mathbf{0})$ has a zero eigenvalue;*
(ii) $\mathbf{h}(\mathbf{0}) \geq 0$ *and a zero is the regular type point for* $\mathbf{h}(\mathbf{0})$.

Then the operator $\mathbf{H}(K)$ has finitely many negative eigenvalues.

Theorem 2.4. *Let $K \in \Lambda$. If the operator $\mathbf{h}(\mathbf{0})$ has a zero energy resonance, then the operator $\mathbf{H}(K)$ has infinitely many negative eigenvalues accumulating at zero and the function $N(K, \cdot)$ obeys the relation*

$$\lim_{z \to -0} \frac{N(K, z)}{|\log |z||} = \mathcal{U}_0, \quad 0 < \mathcal{U}_0 < \infty. \tag{2.3}$$

Remark 2.5. The constant $\mathcal{U}_0$ does not depend on the function $v_1(\cdot)$. It is positive and depends only on the ratio l_2/l_1.

Remark 2.6. Clearly, by equality (2.3) the infinite cardinality of the negative discrete spectrum of $\mathbf{H}(K)$ follows automatically from the positivity of $\mathcal{U}_0$.

Remark 2.7. It is surprising that the asymptotics (2.3) doesn't depends on the cardinality of Λ, that is, this asymptotics is the same for all $n \in \mathbb{N}$. Since $\Lambda|_{n=1} = \{\mathbf{0}\}$ in fact, a result similar to Theorem 2.4 was proved in [4] for $n = 1$ and $K = 0$.

3 Some spectral properties of the family of Friedrichs models $\mathbf{h}(k)$

In this section we study some spectral properties of the family of Friedrichs models $\mathbf{h}(k)$, which plays an important role in the study of spectral properties of $\mathbf{H}(K)$.

Let the operator $\mathbf{h}_0(k)$, $k \in \mathbb{T}^3$ acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathbf{h}_0(k) := \begin{pmatrix} 0 & 0 \\ 0 & h_{11}(k) \end{pmatrix}.$$

The perturbation $\mathbf{h}(k) - \mathbf{h}_0(k)$ of the operator $\mathbf{h}_0(k)$ is a self-adjoint operator of rank 2, and thus, according to the Weyl theorem, the essential spectrum of the operator $\mathbf{h}(k)$ coincides with the essential spectrum of $\mathbf{h}_0(k)$. It is evident that $\sigma_{\mathrm{ess}}(\mathbf{h}_0(k)) = [E_{\min}(k); E_{\max}(k)]$, where the numbers $E_{\min}(k)$ and $E_{\max}(k)$ are defined by

$$E_{\min}(k) := \min_{q \in \mathbb{T}^3} E_k(q) \quad \text{and} \quad E_{\max}(k) := \max_{q \in \mathbb{T}^3} E_k(q).$$

This yields $\sigma_{\mathrm{ess}}(\mathbf{h}(k)) = [E_{\min}(k); E_{\max}(k)]$.

For any $k \in \mathbb{T}^3$ we define an analytic function $\Delta(k;\cdot)$ (the Fredholm determinant associated with the operator $\mathbf{h}(k)$) in $\mathbb{C} \setminus [E_{\min}(k); E_{\max}(k)]$ by

$$\Delta(k;z) := l_2\varepsilon(k) + 1 - z - \frac{1}{2}\int\limits_{\mathbb{T}^3} \frac{v_1^2(s)ds}{E_k(s) - z}.$$

The following lemma [4] is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

Lemma 3.1. *For any* $k \in \mathbb{T}^3$ *the operator* $\mathbf{h}(k)$ *has an eigenvalue* $z \in \mathbb{C} \setminus [E_{\min}(k); E_{\max}(k)]$ *if and only if* $\Delta(k;z) = 0$.

Since for any $k \in \Lambda$ the function $E_k(\cdot)$ has non-degenerate zero minimum at the points of Λ and the function $v_1(\cdot)$ is a continuous on $\mathbb{T}^3$, for any $k \in \mathbb{T}^3$ the integral

$$\int\limits_{\mathbb{T}^3} \frac{v_1^2(s)ds}{E_k(s)}$$

is positive and finite. The Lebesgue dominated convergence theorem and the equality $\Delta(\mathbf{0};0) = \Delta(k;0)$ for $k \in \Lambda$ yield

$$\Delta(\mathbf{0};0) = \lim_{k \to k'} \Delta(k;0), \quad k' \in \Lambda.$$

For some $\delta > 0$ and $p_0 \in \mathbb{T}^3$ we set

$$U_\delta(p_0) := \{p \in \mathbb{T}^3 : |p - p_0| < \delta\}, \quad \mathbb{T}_\delta := \mathbb{T}^3 \setminus \bigcup_{q' \in \Lambda} U_\delta(q').$$

The following lemma establishes in which cases the bottom of the essential spectrum is a threshold energy resonance or eigenvalue.

Lemma 3.2. (i) *The operator* $\mathbf{h}(\mathbf{0})$ *has a zero eigenvalue if and only if* $\Delta(\mathbf{0};0) = 0$ *and* $v_1(q') = 0$ *for all* $q' \in \Lambda$;

(ii) *The operator* $\mathbf{h}(\mathbf{0})$ *has a zero energy resonance if and only if* $\Delta(\mathbf{0};0) = 0$ *and* $v_1(q') \neq 0$ *for some* $q' \in \Lambda$.

Proof. (i) "Only If Part". Suppose $f=(f_0,f_1)\in\mathcal{H}_0\oplus\mathcal{H}_1$ is an eigenvector of the operator $\mathbf{h}(\mathbf{0})$ associated with the zero eigenvalue. Then f_0 and f_1 satisfy the system of equations

$$\begin{cases} f_0+\frac{1}{\sqrt{2}}\int\limits_{\mathbb{T}^3} v_1(s)f_1(s)ds=0 \\ \frac{1}{\sqrt{2}}v_1(q)f_0+(l_1+l_2)\varepsilon(q)f_1(q)=0. \end{cases} \tag{3.1}$$

From (3.1) we find that f_0 and f_1 are given by (2.2) and from the first equation of (3.1) we derive the equality $\Delta(\mathbf{0};0)=0$.

Now we show that $f_1\in L_2(\mathbb{T}^3)$ if and only if $v_1(q')=0$ for all $q'\in\Lambda$. Indeed. If for some $q'\in\Lambda$ we have $v_1(q')=0$ (resp. $v_1(q')\neq 0$), then there exist the numbers $C_1,C_2,C_3>0$, $\alpha\geq 1$ and $\delta>0$ such that

$$C_1|q-q'|^\alpha\leq|v_1(q)|\leq C_2|q-q'|^\alpha,\quad q\in U_\delta(q'), \tag{3.2}$$

respectively

$$|v_1(q)|\geq C_3,\quad q\in U_\delta(q'). \tag{3.3}$$

The definition of the function $\varepsilon(\cdot)$ implies that there exist the numbers $C_1,C_2,C_3>0$ and $\delta>0$ such that

$$C_1|q-q'|^2\leq\varepsilon(q)\leq C_2|q-q'|^2,\quad q\in U_\delta(q'),\quad q'\in\Lambda \tag{3.4}$$

$$\varepsilon(q)\geq C_3,\quad q\in\mathbb{T}_\delta. \tag{3.5}$$

We have

$$\int\limits_{\mathbb{T}^3}|f_1(s)|^2ds=\frac{|f_0|^2}{2(l_1+l_2)^2}\sum_{q'\in\Lambda}\int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{\varepsilon^2(s)}+\frac{|f_0|^2}{2(l_1+l_2)^2}\int\limits_{\mathbb{T}_\delta}\frac{v_1^2(s)ds}{\varepsilon^2(s)}. \tag{3.6}$$

If $v_1(q')=0$ for all $q'\in\Lambda$, then using estimates (3.2)-(3.5) we obtain that

$$\int\limits_{\mathbb{T}^3}|f_1(s)|^2ds\leq C_1\sum_{q'\in\Lambda}\int\limits_{U_\delta(q')}\frac{|s-q'|^{2\alpha}}{|s-q'|^4}ds+C_2<\infty.$$

In the case $v_1(q')\neq 0$ for some $q'\in\Lambda$, an application of estimates (3.3), (3.4) imply

$$\int\limits_{\mathbb{T}^3}|f_1(s)|^2ds\geq C_1\int\limits_{U_\delta(q')}\frac{ds}{|s-q'|^4}=\infty.$$

Therefore $f_1\in L_2(\mathbb{T}^3)$ if and only if $v_1(q')=0$ for all $q'\in\Lambda$.

"If Part". Let $\Delta(\mathbf{0};0)=0$ and $v_1(q')=0$ for all $q'\in\Lambda$. Then the vector $f=(f_0,f_1)$, where f_0 and f_1 are defined by (2.2), obeys the equation $\mathbf{h}(\mathbf{0})f=0$ and as we show in "Only If Part" that $f_1\in L_2(\mathbb{T}^3)$.

(ii) "Only If Part". Let the operator $\mathbf{h}(\mathbf{0})$ have a zero energy resonance. Then by Definition 2.2 the equation

$$\psi(q)=\frac{v_1(q)}{2(l_1+l_2)}\int\limits_{\mathbb{T}^3}\frac{v_1(s)\psi(s)ds}{\varepsilon(s)},\quad \psi\in C(\mathbb{T}^3) \tag{3.7}$$

has a simple solution $\psi \in C(\mathbb{T}^3)$ and $\psi(q') \neq 0$ for some $q' \in \Lambda$. It is easy to see that this solution is equal to $v_1(\cdot)$ (up to a constant factor) and hence $\Delta(\mathbf{0}\,;0) = 0$.

"If Part". Let the equality $\Delta(\mathbf{0}\,;0) = 0$ hold and $v_1(q') \neq 0$ for some $q' \in \Lambda$. Then the function $v_1 \in C(\mathbb{T}^3)$ is a solution of the equation (3.7), that is, the operator $h(\mathbf{0})$ has a zero energy resonance. □

Set

$$\Lambda_0 := \{q' \in \Lambda : v_1(q') \neq 0\}.$$

Lemma 3.3. *If the operator $h(\mathbf{0})$ has a zero energy resonance, then the vector $f = (f_0, f_1)$, where f_0 and f_1 are given by* (2.2)*, obeys the equation $h(\mathbf{0})f = 0$ and $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$.*

Proof. Since the fact that the vector f defined as in Lemma 3.3 satisfies $h(\mathbf{0})f = 0$ is obvious, we show that $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$.

Let the operator $h(\mathbf{0})$ have a zero energy resonance. Then by the assertion (ii) of Lemma 3.2 we have $v_1(q') \neq 0$ for some $q' \in \Lambda$. Using the estimates (3.2)–(3.5) we have

$$\int_{\mathbb{T}^3} |f_1(s)|^2 ds \geq \frac{|f_0|^2}{2(l_1+l_2)^2} \int_{U_\delta(q')} \frac{v_1^2(s)ds}{\varepsilon^2(s)} \geq C_2 \int_{U_\delta(q')} \frac{ds}{|s-q'|^4} = \infty;$$

$$\int_{\mathbb{T}^3} |f_1(s)| ds = \frac{|f_0|}{\sqrt{2}(l_1+l_2)} \Big(\sum_{q' \in \Lambda_0} \int_{U_\delta(q')} \frac{|v_1(s)|ds}{\varepsilon(s)} + \sum_{q' \in \Lambda \setminus \Lambda_0} \int_{U_\delta(q')} \frac{|v_1(s)|ds}{\varepsilon(s)} + \int_{\mathbb{T}_\delta} \frac{|v_1(s)|ds}{\varepsilon(s)} \Big)$$
$$\leq C_1 \sum_{q' \in \Lambda_0} \int_{U_\delta(q')} \frac{ds}{|s-q'|^2} + C_2 \sum_{q' \in \Lambda \setminus \Lambda_0} \int_{U_\delta(q')} \frac{ds}{|s-q'|^{2-\alpha}} + C_3 < \infty.$$

Therefore, $f_1 \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$. □

Lemma 3.4. *If the operator $h(\mathbf{0})$ has either a zero energy resonance or a zero eigenvalue, then for any $K \in \Lambda$ and $p \in \mathbb{T}^3$ the operator $\mathbf{h}(K-p) + l_1\varepsilon(p)\mathbf{I}$ is non-negative.*

Similar lemma were proved in [5] and we refer to this paper for the proof.

Now we formulate a lemma (zero energy expansion for the Fredholm determinant, leading to behaviors of the zero energy resonance), which is important in the proof of Theorem 2.4, that is, the asymptotics (2.3).

Lemma 3.5. *Let the operator $h(\mathbf{0})$ have a zero energy resonance and $K, p' \in \Lambda$. Then the following decomposition*

$$\Delta(K-p\,; z - l_1\varepsilon(p)) = \frac{2\pi^2}{n^2(l_1+l_2)^{3/2}} \Big(\sum_{q' \in \Lambda_0} v_1^2(q') \Big) \sqrt{\frac{l_1^2 + 2l_1 l_2}{l_1+l_2} |p-p'|^2 - \frac{2z}{n^2}}$$
$$+ O(|p-p'|^2) + O(|z|)$$

holds for $|p-p'| \to 0$ and $z \to -0$.

Proof. Let us sketch the main idea of the proof. Assume the operator $h(\mathbf{0})$ have a zero energy resonance and $K, p' \in \Lambda$. Using the additivity property of the integral we represent the function $\Delta(K-p\,; z-l_1\varepsilon(p))$ as

$$\Delta(K-p\,;z-l_1\varepsilon(p)) = w_1(K;p) - z - \frac{1}{2}\sum_{q'\in\Lambda_0}\int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p,s)-z}$$

$$-\frac{1}{2}\sum_{q'\in\Lambda\setminus\Lambda_0}\int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p,s)-z} - \frac{1}{2}\int\limits_{\mathbb{T}_\delta}\frac{v_1^2(s)ds}{w_2(K;p,s)-z}, \tag{3.8}$$

where $\delta > 0$ is a sufficiently small number.

Since the function $w_2(K;\cdot,\cdot)$ has non-degenerate minimum at the points (p',q'), $p',q' \in \Lambda$, analysis similar to [4] show that

$$\int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p,s)-z} = \int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p',s)} - \frac{4\pi^2 v_1^2(q')}{n^2(l_1+l_2)^{3/2}}\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p'|^2 - \frac{2z}{n^2}}$$
$$+O(|p-p'|^2) + O(|z|), \quad q' \in \Lambda_0;$$

$$\int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p,s)-z} = \int\limits_{U_\delta(q')}\frac{v_1^2(s)ds}{w_2(K;p',s)} + O(|p-p'|^2) + O(|z|), \quad q' \in \Lambda\setminus\Lambda_0;$$

$$\int\limits_{\mathbb{T}_\delta}\frac{v_1^2(s)ds}{w_2(K;p,s)-z} = \int\limits_{\mathbb{T}_\delta}\frac{v_1^2(s)ds}{w_2(K;p',s)} + O(|p-p'|^2) + O(|z|)$$

as $|p-p'| \to 0$ and $z \to -0$. Here we remind that $v_1(q') = 0$ for all $q' \in \Lambda\setminus\Lambda_0$ and hence by the estimate (3.2) we have $v_1(q) = O(|q-q'|^\alpha)$ as $|q-q'| \to 0$ for some $\alpha \geq 1$. Now substituting the last three expressions and the the expansion

$$w_1(K;p) = 1 + \frac{(l_1+l_2)n^2}{2}|p-p'|^2 + O(|p-p'|^4)$$

as $|p-p'| \to 0$, to the equality (3.8) we obtain

$$\Delta(K-p\,;z-l_1\varepsilon(p)) = \Delta(\mathbf{0}\,;0) + \frac{2\pi^2}{n^2(l_1+l_2)^{3/2}}\Big(\sum_{q'\in\Lambda_0} v_1^2(q')\Big)\sqrt{\frac{l_1^2+2l_1l_2}{l_1+l_2}|p-p'|^2 - \frac{2z}{n^2}}$$
$$+O(|p-p'|^2) + O(|z|)$$

as $|p-p'| \to 0$ and $z \to -0$. Since the operator $h(\mathbf{0})$ has a zero energy resonance by the assertion (ii) of Lemma 3.2 we have the equality $\Delta(\mathbf{0}\,;0) = 0$, which completes the proof of the Lemma 3.5. □

Corollary 3.6. *Let the operator $h(\mathbf{0},\mathbf{0})$ have a zero energy resonance and $K \in \Lambda$. Then there exist the numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that*
(i) $C_1|p-p'| \leq \Delta(K-p\,;-l_1\varepsilon(p)) \leq C_2|p-p'|$, $p \in U_\delta(p')$, $p' \in \Lambda$;
(ii) $\Delta(K-p\,;-l_1\varepsilon(p)) \geq C_3$, $p \in \mathbb{T}_\delta$.

Proof. Lemma 3.5 yields the assertion (i) for some positive numbers C_1, C_2. The positivity and continuity of the function $\Delta(K-p\,;-l_1\varepsilon(p))$ on the compact set $\mathbb{T}_\delta$ imply the assertion (ii). □

Lemma 3.7. *Let the operator $h(\mathbf{0})$ have a zero eigenvalue and $K \in \Lambda$. Then there exist the numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that*
(i) $C_1|p-p'|^2 \leq \Delta(K-p\,;-l_1\varepsilon(p)) \leq C_2|p-p'|^2$, $p \in U_\delta(p')$, $p' \in \Lambda$;
(ii) $\Delta(K-p\,;-l_1\varepsilon(p)) \geq C_3$, $p \in \mathbb{T}_\delta$.

Proof. Let the operator $h(\mathbf{0})$ have a zero eigenvalue. Then by the assertion (i) of Lemma 3.2 we have $v_1(p') = 0$ for all $p' \in \Lambda$.

Let $K \in \Lambda$. Then $\Delta(K-p\,;-l_1\varepsilon(p)) = \Delta(p\,;-l_1\varepsilon(p))$ holds for any $p \in \mathbb{T}^3$. Proceeding analogously to the proof of Lemma 3.4 of [5] one can show that the function $\Delta(\cdot\,;-l_1\varepsilon(\cdot))$ has minimum at the points $p = p' \in \Lambda$. Here we prove that this function has non-degenerate minimum at the points $p = p' \in \Lambda$. Since the function $w_2(\mathbf{0};\cdot,\cdot)$ is positive on $(\mathbb{T}^3 \setminus \Lambda) \times \mathbb{T}^3$ the integrals

$$\lambda_{ij}^{(1)}(p) := \int\limits_{\mathbb{T}^3} \left(\frac{\partial^2 w_2(\mathbf{0};p,s)}{\partial p^{(i)} \partial p^{(j)}} \right) \frac{v_1^2(s)ds}{(w_2(\mathbf{0};p,s))^2}, \quad i,j = 1,2,3$$

and

$$\lambda_{ij}^{(2)}(p) := \int\limits_{\mathbb{T}^3} \left(\frac{\partial w_2(\mathbf{0};p,s)}{\partial p^{(i)}} \frac{\partial w_2(\mathbf{0};p,s)}{\partial p^{(j)}} \right) \frac{v_1^2(s)ds}{(w_2(\mathbf{0};p,s))^3}, \quad i,j = 1,2,3$$

are finite for any $p \in \mathbb{T}^3 \setminus \Lambda$. The condition $v_1(p') = 0$ for all $p' \in \Lambda$ implies finiteness of these integrals at the points of Λ. Thus the functions $\lambda_{ij}^{(l)}(\cdot)$, $l = 1,2$ are continuous on $\mathbb{T}^3$.

We define the function $I(\cdot)$ on $\mathbb{T}^3$ by

$$I(p) := \int\limits_{\mathbb{T}^3} \frac{v_1^2(s)ds}{w_2(\mathbf{0};p,s)}.$$

The function $I(\cdot)$ is a twice continuously differentiable function $\mathbb{T}^3$ and

$$\frac{\partial^2 I(p)}{\partial p^{(i)} \partial p^{(j)}} = -\lambda_{ij}^{(1)}(p) + 2\lambda_{ij}^{(2)}(p), \quad i,j = 1,2,3.$$

Simple calculations shows that

$$\frac{\partial w_2(\mathbf{0};p,q)}{\partial p^{(i)}} = n\Big[l_1 \sin(nq^{(i)}) + l_2 \sin(n(p^{(i)} + q^{(i)}))\Big], \quad i = 1,2,3;$$
$$\frac{\partial^2 w_2(\mathbf{0};p,q)}{\partial p^{(i)} \partial p^{(i)}} = n^2\Big[l_1 \cos(nq^{(i)}) + l_2 \cos(n(p^{(i)} + q^{(i)}))\Big], \quad i = 1,2,3;$$
$$\frac{\partial^2 w_2(\mathbf{0};p,q)}{\partial p^{(i)} \partial p^{(j)}} = 0, \quad i \neq j, \quad i,j = 1,2,3$$

and hence for $p' \in \Lambda$ we obtain

$$\frac{\partial^2 I(p')}{\partial p^{(i)}\partial p^{(i)}} = -\frac{n^2}{4}\int\limits_{\mathbb{T}^3}\left(\sum_{l=1,\, l\neq i}^{3}(1-\cos(ns^{(l)}))\right)\frac{(1+\cos(ns^{(i)}))v_1^2(s)}{\varepsilon^3(s)}ds,\ i=1,2,3;$$

$$\frac{\partial^2 I(p')}{\partial p^{(i)}\partial p^{(j)}} = \frac{n^2}{4}\int\limits_{\mathbb{T}^3}\frac{\sin(ns^{(i)})\sin(ns^{(j)})v_1^2(s)}{\varepsilon^3(s)}ds, \quad i\neq j, \quad i,j=1,2,3.$$

The last equalities and the evenness of $v_2^2(\cdot)$ on each variables imply

$$\frac{\partial^2 I(p')}{\partial p^{(i)}\partial p^{(i)}} < 0, \quad \frac{\partial^2 I(p')}{\partial p^{(i)}\partial p^{(j)}} = 0, \quad i\neq j, \quad i,j=1,2,3$$

for $p' \in \Lambda$. Since

$$\frac{\partial^2 w_1(\mathbf{0};p')}{\partial p^{(i)}\partial p^{(i)}} = (l_1+l_2)n^2, \quad \frac{\partial^2 w_1(\mathbf{0};p')}{\partial p^{(i)}\partial p^{(j)}} = 0, \quad i\neq j, \quad i,j=1,2,3$$

for all $p' \in \Lambda$ by definition of $\Delta(\cdot\,;\cdot)$ we have

$$\frac{\partial^2 \Delta(p';0)}{\partial p^{(i)}\partial p^{(i)}} > (l_1+l_2)n^2, \quad \frac{\partial^2 \Delta(p';0)}{\partial p^{(i)}\partial p^{(j)}} = 0, \quad i\neq j, \quad i,j=1,2,3$$

for all $p' \in \Lambda$. Therefore the function $\Delta(\cdot\,;-l_1\varepsilon(\cdot))$ has non-degenerate minimum at the points of $p = p' \in \Lambda$. This fact completes the proof of lemma. □

Lemma 3.8. *Let $K \in \Lambda$ and zero be the regular type point for $\mathbf{h}(\mathbf{0})$ with $\mathbf{h}(\mathbf{0}) \geq 0$. Then there exists a positive number C_1 such that the inequality*

$$\Delta(K-p\,;z-l_1\varepsilon(p)) \geq C_1$$

holds for any $p \in \mathbb{T}^3$ and $z < 0$.

Proof. Let zero be the regular type point of $\mathbf{h}(\mathbf{0})$, that is, $\Delta(\mathbf{0}\,;0) \neq 0$. Assume $\Delta(\mathbf{0}\,;0) < 0$. Then $\lim\limits_{z\to-\infty}\Delta(\mathbf{0}\,;z) = -\infty$ and the continuity of of the function $\Delta(\mathbf{0}\,;\dot{)}$ on $(-\infty;0]$ imply that there exists $z_0 < 0$ such that $\Delta(\mathbf{0}\,;z_0) = 0$. In this case by Lemma 3.1 the number z_0 is an eigenvalue of the operator $h(\mathbf{0})$. On the other hand by the assumption of the lemma we have $\mathbf{h}(\mathbf{0}) \geq 0$. Therefore the operator $h(\mathbf{0})$ has no negative eigenvalues. This contrary gives $\Delta(\mathbf{0}\,;0) > 0$.

Since for any $K \in \Lambda$ the function $\Delta(K-p\,;-l_1\varepsilon(p))$ has minimum at the points $p = p' \in \Lambda$, for all $p \in \mathbb{T}^3$ and $z < 0$ we obtain

$$\Delta(K-p\,;z-l_1\varepsilon(p)) > \Delta(K-p\,;-l_1\varepsilon(p)) \geq \Delta(\mathbf{0}\,;0) > 0.$$

Setting $C_1 := \Delta(\mathbf{0}\,;0)$ we complete the proof of lemma. □

4 The Birman-Schwinger principle.

For a bounded self-adjoint operator A acting in the Hilbert space $\mathcal{R}$, we define the number $n(\gamma, A)$ by the rule

$$n(\gamma, A) := \sup\{\dim F : (Au, u) > \gamma,\, u \in F \subset \mathcal{R},\, ||u|| = 1\}.$$

The number $n(\gamma, A)$ is equal to the infinity if $\gamma < \max \sigma_{\text{ess}}(A)$; if $n(\gamma, A)$ is finite, then it is equal to the number of the eigenvalues of A bigger than γ.

By the definition of $N(K, z)$, we have

$$N(K, z) = n(-z, -\mathbf{H}(K)),\; -z > -\tau_{\text{ess}}(K).$$

Since for any $K \in \mathbb{T}^3$ the function $\Delta(K - p\,; z - l_1\varepsilon(p))$ is a positive on $(p, z) \in \mathbb{T}^3 \times (-\infty; \tau_{\text{ess}}(K))$, the positive square root of $\Delta(K - p\,; z - l_1\varepsilon(p))$ exists for any $K, p \in \mathbb{T}^3$ and $z < \tau_{\text{ess}}(K)$.

In our analysis of the discrete spectrum of $\mathbf{H}(K)$, $K \in \mathbb{T}^3$ the crucial role is played by the self-adjoint compact 2×2 block operator matrix $\widehat{\mathbf{T}}(K, z)$, $z < \tau_{\text{ess}}(K)$ acting on $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\widehat{\mathbf{T}}(K, z) := \begin{pmatrix} \widehat{T}_{00}(K, z) & \widehat{T}_{01}(K, z) \\ \widehat{T}^*_{01}(K, z) & \widehat{T}_{11}(K, z) \end{pmatrix}$$

with the entries

$$\widehat{T}_{00}(K, z) g_0 = (1 + z - w_0(K)) g_0, \quad \widehat{T}_{01}(K, z) g_1 = -\int\limits_{\mathbb{T}^3} \frac{v_0(s) g_1(s) ds}{\sqrt{\Delta(K - s\,; z - l_1\varepsilon(s))}};$$

$$(\widehat{T}_{11}(K, z) g_1)(p) = \frac{v_1(p)}{2\sqrt{\Delta(K - p\,; z - l_1\varepsilon(p))}} \int\limits_{\mathbb{T}^3} \frac{v_1(s) g_1(s) ds}{\sqrt{\Delta(K - s\,; z - l_1\varepsilon(s))}(w_2(K; p, s) - z)}.$$

The following lemma is a modification of the well-known Birman-Schwinger principle for the operator $\mathbf{H}(K)$ (see [1, 3, 4, 25]).

Lemma 4.1. *Let $K \in \mathbb{T}^3$. The operator $\widehat{\mathbf{T}}(K, z)$ is compact and continuous in $z < \tau_{\text{ess}}(K)$ and*

$$N(K, z) = n(1, \widehat{\mathbf{T}}(K, z)).$$

For the proof of this lemma, see Lemma 5.1 of [4].

5 Finiteness of the number of eigenvalues of $\mathbf{H}(K)$, $K \in \Lambda$

We starts the proof of the finiteness of the number of negative eigenvalues (Theorem 2.3) with the following two lemmas.

Lemma 5.1. *Let $K, p', q' \in \Lambda$. Then there exist the numbers $C_1, C_2 > 0$ and $\delta > 0$ such that*
(i) $C_1(|p - p'|^2 + |q - q'|^2) \le w_2(K; p, q) \le C_2(|p - p'|^2 + |q - q'|^2)$, $(p, q) \in U_\delta(p') \times U_\delta(q')$;
(ii) $w_2(K; p, q) \ge C_1$, $(p, q) \notin \bigcup\limits_{p' \in \Lambda} U_\delta(p') \times \bigcup\limits_{q' \in \Lambda} U_\delta(q')$.

Proof. Since for any $K \in \Lambda$ the function $w_2(K;\cdot,\cdot)$ has non-degenerate zero minimum at the points $(p',q') \in \Lambda \times \Lambda$, we obtain the following expansion

$$w_2(K;p,q) = \frac{n^2}{2}\left[(l_1+l_2)|p-p'|^2 + 2l_2(p-p',q-q') + (l_1+l_2)|q-q'|^2\right] + O(|p-p'|^4) + O(|q-q'|^4)$$

as $|p-p'|, |q-q'| \to 0$ for $p',q' \in \Lambda$. Then there exist positive numbers C_1, C_2 and δ so that (i) and (ii) hold true. □

Lemma 5.2. *Let $K \in \Lambda$ and one of the following assumptions hold:*
(i) *the operator $\mathbf{h}(\mathbf{0})$ has a zero eigenvalue;*
(ii) *a zero is the regular type point for $\mathbf{h}(\mathbf{0})$ and $\mathbf{h}(\mathbf{0}) \geq 0$.*
Then for any $z \leq 0$ the operator $\widehat{\mathbf{T}}(K,z)$ is compact and continuous from the left up to $z = 0$.

Proof. Let $K \in \Lambda$. Denote by $Q(K;p,q;z)$ the kernel of the integral operator $\widehat{T}_{11}(K,z)$, $z < 0$, that is,

$$Q(K;p,q;z) := \frac{v_1(p)v_1(q)}{2\sqrt{\Delta(K-p;z-l_1\varepsilon(p))}(w_2(K;p,q)-z)\sqrt{\Delta(K-q;z-l_1\varepsilon(q))}}.$$

If the operator $\mathbf{h}(\mathbf{0})$ has a zero eigenvalue, then by the assertion (i) of Lemma 3.2 we have $v_1(q') = 0$ for all $q' \in \Lambda$. By virtue of inequality (3.2), Corollary 3.6 and Lemma 5.1 the kernel $Q(K;p,q;z)$ is estimated by

$$C_1 \sum_{p',q' \in \Lambda} \left(\frac{\chi_\delta(p-p')}{|p-p'|}+1\right)\left(\frac{|q-q'|\chi_\delta(p-p')\chi_\delta(q-q')}{|p-p'|^2+|q-q'|^2}+1\right)\left(\frac{\chi_\delta(q-q')}{|q-q'|^{\frac{1}{2}}}+1\right), \tag{5.1}$$

where $\chi_\delta(\cdot)$ is the characteristic function of $U_\delta(\mathbf{0})$.

If a zero is the regular type point for $\mathbf{h}(\mathbf{0})$ and $\mathbf{h}(\mathbf{0}) \geq 0$, then by virtue of Lemmas 3.8 and 5.1 the kernel $Q(K;p,q;z)$ is estimated by

$$C_1 \sum_{p',q' \in \Lambda} \left(\frac{\chi_\delta(p-p')\chi_\delta(q-q')}{|p-p'|^2+|q-q'|^2}+1\right). \tag{5.2}$$

The functions (5.1) and (5.2) are square-integrable on $(\mathbb{T}^3)^2$ and hence for any $z \leq 0$ the operator $\widehat{T}_{11}(K,z)$ is Hilbert-Schmidt.

The kernel function of $\widehat{T}_{11}(K,z)$, $z < 0$ is continuous in $p,q \in \mathbb{T}^3$. Therefore the continuity of the operator $\widehat{T}_{11}(K,z)$ from the left up to $z = 0$ follows from Lebesgue's dominated convergence theorem.

Since for all $z \leq 0$ the operators $\widehat{T}_{00}(K,z)$, $\widehat{T}_{01}(K,z)$ and $\widehat{T}^*_{01}(K,z)$ are of rank 1 and continuous from the left up to $z = 0$ one concludes that $\widehat{\mathbf{T}}(K,z)$ is compact and continuous from the left up to $z = 0$. □

We are now ready for the

Proof of Theorem 2.3. Let the conditions of Theorem 2.3 be fulfilled. Using the Weyl inequality

$$n(\lambda_1+\lambda_2, A_1+A_2) \le n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for the sum of compact operators A_1 and A_2 and for any positive numbers λ_1 and λ_2 we have

$$n(1, \widehat{\mathbf{T}}(K,z)) \le n(1/2, \widehat{\mathbf{T}}(K,0)) + n(1/2, \widehat{\mathbf{T}}(K,z) - \widehat{\mathbf{T}}(K,0)) \tag{5.3}$$

for all $z < 0$.

By virtue of Lemma 5.2 the operator $\widehat{\mathbf{T}}(K,z)$ is continuous from the left up to $z = 0$, which implies that the second summand on the r.h.s. of (5.3) tends to zero as $z \to -0$. By Lemma 4.1 we have $N(K,z) = n(1, \widehat{\mathbf{T}}(K,z))$ as $z < 0$ and hence

$$\lim_{z\to -0} N(K,z) = N(K,0) \le n(1/2, \widehat{\mathbf{T}}(K,0)).$$

Thus $N(K,0) \le n(1/2, \widehat{\mathbf{T}}(K,0))$. By Lemma 5.2 the number $n(1/2, \widehat{\mathbf{T}}(K,0))$ is finite and hence $N(K,0) < \infty$. This completes the proof of Theorem 2.3. □

6 Asymptotics for the number of negative eigenvalues of $\mathbf{H}(K)$, $K \in \Lambda$

In this section first we derive the asymptotic relation (2.3) for the number of negative eigenvalues of $\mathbf{H}(K)$, $K \in \Lambda$.

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\sigma = L_2(\mathbb{S}^2)$. As we shall see, the discrete spectrum asymptotics of the operator $\widehat{\mathbf{T}}(K,z)$ $K \in \Lambda$ as $z \to -0$ is determined by the integral operator $S_{\mathbf{r}}$, $\mathbf{r} = 1/2|\log|z||$ in $L_2((0,\mathbf{r}),\sigma)$ with the kernel

$$S(y,t) := \frac{1}{4\pi^2} \frac{(l_1+l_2)^2}{\sqrt{l_1^2+2l_1l_2}} \frac{1}{(l_1+l_2)\cosh y + l_2 t},$$

where $y = x - x'$, $x, x' \in (0,\mathbf{r})$ and $t = \langle \xi, \eta \rangle$ is the inner product of the arguments $\xi, \eta \in \mathbb{S}^2$.

The eigenvalues asymptotics for the operator $S_{\mathbf{r}}$ have been studied in detail by Sobolev [25], by employing an argument used in the calculation of the canonical distribution of Toeplitz operators.

Let us recall some results of [25] which are important in our work.

The coefficient in the asymptotics (2.3) of $N(K,z)$ will be expressed by means of the self-adjoint integral operator $\widehat{S}(\theta)$, $\theta \in \mathbb{R}$, in the space σ, whose kernel is of the form

$$\widehat{S}(\theta,t) := \frac{1}{4\pi^2} \frac{(l_1+l_2)^2}{l_1^2+2l_1l_2} \frac{\sinh[\theta \arccos \frac{l_2}{l_1+l_2} t]}{\sinh(\pi\theta)},$$

and depends on $t = \langle \xi, \eta \rangle$. For $\gamma > 0$, define

$$U(\gamma) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} n(\gamma, \widehat{S}(\theta)) d\theta.$$

This function was studied in detail in [25]; where it was used in showing existence proof of the Efimov effect. In particular, as it was shown in [25], the function $U(\cdot)$ is continuous in $\gamma > 0$, and the limit

$$\lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}) = U(\gamma) \tag{6.1}$$

exists and the number $U(1)$ is positive.

For completeness, we reproduce the following lemma, which has been proven in [25].

Lemma 6.1. *Let* $A(z) = A_0(z) + A_1(z)$*, where* $A_0(z)$ $(A_1(z))$ *is compact and continuous for* $z < 0$ (*for* $z \le 0$). *Assume that the limit*

$$\lim_{z\to -0} f(z) n(\gamma, A_0(z)) = l(\gamma)$$

exists and $l(\cdot)$ *is continuous in* $(0;+\infty)$ *for some function* $f(\cdot)$*, where* $f(z) \to 0$ *as* $z \to -0$. *Then the same limit exists for* $A(z)$ *and*

$$\lim_{z\to -0} f(z) n(\gamma, A(z)) = l(\gamma).$$

Remark 6.2. Since the function $U(\cdot)$ is continuous with respect to γ, it follows from Lemma 6.1 that any perturbation of $A_0(z)$ treated in Lemma 6.1 (which is compact and continuous up to $z = 0$) does not contribute to the asymptotic relation (2.3). In the rest part of this subsection we use this fact without further comments.

Now we are going to reduce the study of the asymptotics for the operator $\widehat{\mathbf{T}}(K,z)$ with $K \in \Lambda$ to that of the asymptotics $S_{\mathbf{r}}$.

Let $\mathbf{T}(\delta;|z|)$ be the operator in $\mathcal{H}_0 \oplus \mathcal{H}_1$ defined by

$$\mathbf{T}(\delta;|z|) := \begin{pmatrix} 0 & 0 \\ 0 & T_{11}(\delta;|z|) \end{pmatrix},$$

where $T_{11}(\delta;|z|)$ is the integral operator in $\mathcal{H}_1$ with the kernel

$$D \sum_{p',q' \in \Lambda_0} \frac{v_1(p')v_1(q')\chi_\delta(p-p')\chi_\delta(q-q')(m|p-p'|^2 + 2|z|/n^2)^{-\frac{1}{4}}(m|q-p'|^2 + 2|z|/n^2)^{-\frac{1}{4}}}{(l_1+l_2)|p-p'|^2 + 2l_2(p-p',q-q') + (l_1+l_2)|q-q'|^2 + 2|z|/n^2}.$$

Here

$$D := \frac{(l_1+l_2)^{3/2}}{\pi^2}\Big(\sum_{q'\in\Lambda_0} v_1^2(q')\Big)^{-1} \quad \text{and} \quad m := \frac{l_1^2 + 2l_1 l_2}{l_1 + l_2}.$$

The operator $\mathbf{T}(\delta;|z|)$ is called singular part of $\widehat{\mathbf{T}}(K,z)$.

The main technical point to apply Lemma 6.1 is the following lemma.

Lemma 6.3. *Let* $K \in \Lambda$. *Then for any* $z \le 0$ *and small* $\delta > 0$ *the difference* $\widehat{\mathbf{T}}(K,z) - \mathbf{T}(\delta;|z|)$ *is compact and is continuous with respect to* $z \le 0$.

Proof. By Lemma 5.1 and Corollary 3.6, one can estimate the kernel of the operator $\widehat{T}_{11}(K,z)-T_{11}(\delta;|z|)$, $z\le 0$, by the square-integrable function

$$C_1\sum_{p',q'\in\Lambda_0}\Big[\frac{1}{|p-p'|^{1/2}}+\frac{1}{|q-q'|^{1/2}}+\frac{|p-p'|+|q-q'|}{|p-p'|^{1/2}(|p-p'|^2+|q-q'|^2)|q-q'|^{1/2}}$$
$$+\frac{|z|^{1/2}}{(|p-p'|^2+|z|)^{1/4}(|p-p'|^2+|q-q'|^2)(|q-q'|^2+|z|)^{1/4}}+1\Big].$$

Hence, the operator $\widehat{T}_{11}(K,z)-T_{11}(\delta;|z|)$ belongs to the Hilbert-Schmidt class for all $z\le 0$. In combination with the continuity of the kernel of the operator with respect to $z<0$, this implies the continuity of $\widehat{T}_{11}(K,z)-T_{11}(\delta;|z|)$ with respect to $z\le 0$.

It is easy to see that $\widehat{T}_{00}(K,z)$, $\widehat{T}_{01}(K,z)$ and $\widehat{T}^*_{01}(K,z)$ are rank 1 operators and they are continuous from the left up to $z=0$. Consequently $\widehat{\mathbf{T}}(K,z)-\mathbf{T}(\delta;|z|)$ is compact and continuous in $z\le 0$. □

From definition of $\mathbf{T}(\delta;|z|)$ it follows that $\sigma(\mathbf{T}(\delta;|z|))=\{0\}\cup\sigma(T_{11}(\delta;|z|))$ and hence $n(\gamma,\mathbf{T}(\delta;|z|))=n(\gamma,T_{11}(\delta;|z|))$ for all $\gamma>0$.

The following theorem is fundamental for the proof of the asymptotic relation (2.3).

Theorem 6.4. *We have the relation*

$$\lim_{|z|\to 0}\frac{n(\gamma,T_{11}(\delta;|z|))}{|\log|z||}=U(\gamma),\quad \gamma>0. \tag{6.2}$$

Proof. The subspace of functions ψ, supported by the set $\bigcup_{q'\in\Lambda_0}U_\delta(q')$ is invariant with respect to the operator $T_{11}(\delta;|z|)$. Let $T^0_{11}(\delta;|z|)$ be the restriction of the integral operator $T_{11}(\delta;|z|)$ to the subspace $L_2(\bigcup_{q'\in\Lambda_0}U_\delta(q'))$, that is, the integral operator in $L_2(\bigcup_{q'\in\Lambda_0}U_\delta(q'))$ with the kernel $T^0_{11}(\delta;|z|;\cdot,\cdot)$ defined on $\bigcup_{p'\in\Lambda_0}U_\delta(p')\times\bigcup_{q'\in\Lambda_0}U_\delta(q')$ as

$$T^0_{11}(\delta;|z|;p,q)=\frac{D(m|p-p'|^2+2|z|/(n^2))^{-\frac14}(m|q-q'|^2+2|z|/(n^2))^{-\frac14}}{(l_1+l_2)|p-p'|^2+2l_2(p-p',q-q')+(l_1+l_2)|q-q'|^2+2|z|/(n^2)},$$

$(p,q)\in U_\delta(p')\times U_\delta(q')$ for $p',q'\in\Lambda_0$.

In the rest part of the proof we denote by n_0 the number of points of Λ_0 and for convenience we numerate the points of Λ_0 as $p_1,\dots,p_{n_0}$ and set $\overline{1,n_0}=1,\dots,n_0$.

Since $L_2(\bigcup_{q'\in\Lambda_0}U_\delta(q'))\cong\bigoplus_{q'\in\Lambda_0}L_2(U_\delta(q'))$, we can express the integral operator $T^0_{11}(\delta;|z|)$ as the $n_0\times n_0$ block operator matrix $\mathbf{T}_0(\delta;|z|)$ acting on $\bigoplus_{i=1}^{n_0}L_2(U_\delta(p_i))$ as

$$\mathbf{T}_0(\delta;|z|):=\begin{pmatrix} T_0^{(1,1)}(\delta;|z|) & \dots & T_0^{(1,n_0)}(\delta;|z|)\\ \vdots & \ddots & \vdots\\ T_0^{(n_0,1)}(\delta;|z|) & \dots & T_0^{(n_0,n_0)}(\delta;|z|)\end{pmatrix},$$

where for $i,j=\overline{1,n_0}$ the operator $T_0^{(i,j)}(\delta;|z|):L_2(U_\delta(p_j))\to L_2(U_\delta(p_i))$ is the integral operator with the kernel $T_0(\delta;|z|;p,q)$, $(p,q)\in U_\delta(p_i)\times U_\delta(p_j)$.

Set

$$L_2^{(n_0)}(U_r(\mathbf{0})) := \{\phi = (\phi_1, \cdots, \phi_{n_0}) : \phi_i \in L_2(U_r(\mathbf{0})), i = \overline{1, n_0}\}.$$

It is easy to show that $\mathbf{T}_0(\delta; |z|)$ is unitarily equivalent to the $n_0 \times n_0$ block operator matrix $\mathbf{T}_1(r)$, $r = |z|^{-\frac{1}{2}}$, acting on $L_2^{(n_0)}(U_r(\mathbf{0}))$ as

$$\mathbf{T}_1(r) := \begin{pmatrix} v_1(p_1)v_1(p_1)T_1(r) & \dots & v_1(p_1)v_1(p_{n_0})T_1(r) \\ \vdots & \ddots & \vdots \\ v_1(p_{n_0})v_1(p_1)T_1(r) & \dots & v_1(p_{n_0})v_1(p_{n_0})T_1(r) \end{pmatrix},$$

where $T_1(r)$ is the integral operator on $L_2(U_r(\mathbf{0}))$ with the kernel

$$\frac{D(m|p|^2 + 2/(n^2))^{-\frac{1}{4}}(m|q|^2 + 2/(n^2))^{-\frac{1}{4}}}{(l_1 + l_2)|p|^2 + 2l_2(p, q) + (l_1 + l_2)|q|^2 + 2/(n^2)}.$$

The equivalence is realized by the unitary dilation ($n_0 \times n_0$ diagonal matrix)

$$\mathbf{B}_r := \operatorname{diag}\{B_r^{(1)}, \dots, B_r^{(n_0)}\} : \bigoplus_{i=1}^{n_0} L_2(U_\delta(p_i)) \to L_2^{(n_0)}(U_r(\mathbf{0})),$$

Here for $i = \overline{1, n_0}$ the operator $B_r^{(i)} : L_2(U_\delta(p_i)) \to L_2(U_r(\mathbf{0}))$ acts as

$$(B_r^{(i)} f)(p) = (r/\delta)^{-3/2} f(\delta p/r + p_i).$$

Let $\mathbf{A}_r$ and $\mathbf{E}$ be the $n_0 \times 1$ and $1 \times n_0$ matrices of the form

$$\mathbf{A}_r := \begin{pmatrix} v_1(p_1)T_1(r) \\ \vdots \\ v_1(p_{n_0})T_1(r) \end{pmatrix}, \quad \mathbf{E} := (v_1(p_1)I \dots v_1(p_{n_0})I),$$

respectively, where I is the identity operator on $L_2(U_r(\mathbf{0}))$.

It is well known that if B_1, B_2 are bounded operators and $\gamma \neq 0$ is an eigenvalue of B_1B_2, then γ is an eigenvalue for B_2B_1 as well of the same algebraic and geometric multiplicities (see *e.g.* [11]). Therefore, $n(\gamma, \mathbf{A}_r\mathbf{E}) = n(\gamma, \mathbf{E}\mathbf{A}_r)$, $\gamma > 0$. Direct calculation shows that $\mathbf{T}_1(r) = \mathbf{A}_r\mathbf{E}$ and

$$\mathbf{E}\mathbf{A}_r = T_1^0(r) := \Big(\sum_{i=1}^{n_0} v_1^2(p_i)\Big)T_1(r).$$

So, for $\gamma > 0$ we have $n(\gamma, \mathbf{T}_1(r)) = n(\gamma, T_1^0(r))$.

Furthermore, replacing

$$(m|p|^2 + 2/(n^2))^{\frac{1}{4}}, \quad (m|q|^2 + 2/(n^2))^{\frac{1}{4}} \quad \text{and} \quad (l_1 + l_2)|p|^2 + 2l_2(p, q) + (l_1 + l_2)|q|^2 + 2/(n^2)$$

by the expressions

$$(m|p|^2)^{\frac{1}{4}}(1 - \chi_1(p))^{-1}, \quad (m|q|^2)^{\frac{1}{4}}(1 - \chi_1(q))^{-1} \quad \text{and} \quad (l_1 + l_2)|p|^2 + 2l_2(p, q) + (l_1 + l_2)|q|^2,$$

respectively, we obtain the integral operator $T_2(r)$. The error $T_1^0(r) - T_2(r)$ is a Hilbert-Schmidt operator and continuous up to $z = 0$.

Using the dilation

$$M : L_2(U_r(\mathbf{0}) \setminus U_1(\mathbf{0})) \to L_2((0,\mathbf{r}),\sigma), \quad (Mf)(x,w) = e^{3x/2} f(e^x w),$$

where $\mathbf{r} = 1/2|\log|z||$, $x \in (0,\mathbf{r})$, $w \in \mathbb{S}^2$, one sees that the operator $T_2(r)$ is unitarily equivalent to the integral operator $S_{\mathbf{r}}$.

Since the difference of the operators $S_{\mathbf{r}}$ and $T_{11}(\delta;|z|)$ is compact (up to unitary equivalence) and hence, since $\mathbf{r} = 1/2|\log|z||$, we obtain the equality

$$\lim_{|z|\to 0} \frac{n(\gamma, T_{11}(\delta;|z|))}{|\log|z||} = \lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, S_{\mathbf{r}}), \quad \gamma > 0.$$

Now Lemma 6.1 and the equality (6.1) complete the proof of Theorem 6.4. □

We are now ready for the

Proof of Theorem 2.4. Let the operator $\mathbf{h}(\mathbf{0})$ have a zero energy resonance and $K \in \Lambda$. Using Lemmas 6.1, 6.3 and Theorem 6.4 we have that

$$\lim_{|z|\to 0} \frac{n(1, \mathbf{T}(K,z))}{|\log|z||} = U(1).$$

Taking into account the last equality and Lemma 4.1, and setting $\mathcal{U}_0 = U(1)$, we complete the proof of Theorem 2.4. □

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