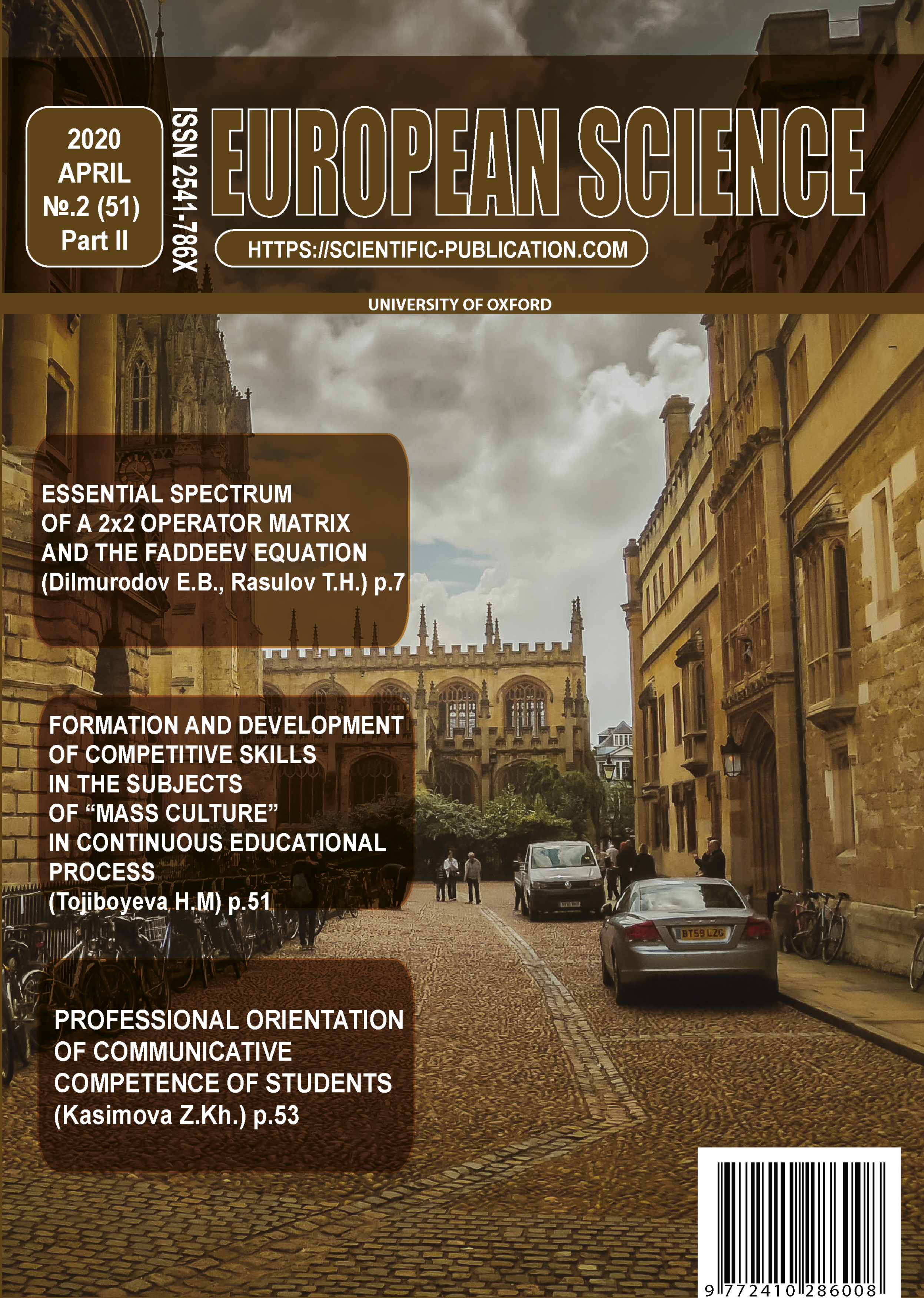

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Содержание

STRUCTURE OF THE NUMERICAL RANGE OF FRIEDRICHS MODEL WITH RANK TWO PERTURBATION

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Abstract: *In this paper we consider a Friedrichs model H with rank two perturbations. It is associated with a system of two quantum particles on $d-$ dimensional lattice. We investigate the structure of the closure of numerical range $W(H)$ of this operator in detail by terms of its parameters for all dimensions d of the torus T^d. For $d \geq 3$ and $\alpha = 1,2$ we find the critical value μ_α^0 of the parameter $\mu_\alpha > 0$ such that for all $\mu_\alpha \in (0; \mu_\alpha^0]$ there is no eigenvalues outside of the essential spectrum of H and for all $\mu_\alpha > \mu_\alpha^0$ there are two eigenvalues of H. Moreover, we find the conditions, which guarantees that the set $W(H)$ is closed.*

Keywords: *Friedrichs model, perturbation, quantum particles, numerical range, spectrum, threshold eigenvalues, virtual level.*

СТРУКТУРА ЧИСЛОВОЙ ОБЛАСТИ ЗНАЧЕНИЙ МОДЕЛИ ФРИДРИХСА С ДВУМЕРНЫМ ВОЗМУЩЕНИЕМ

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Аннотация: *в работе рассматривается модель Фридрихса H с двумерным возмущением. Оно ассоциировано с системой двух квантовых частиц на d –мерной решетке. Структура замыкания числовой области значений $W(H)$ этого оператора подробно исследована в терминах его параметров при всех размерностях d тора T^d. Для $d \geq 3$ и $\alpha = 1,2$ находим критическое значение μ_α^0 параметра $\mu_\alpha > 0$ такое, что при всех $\mu_\alpha \in (0; \mu_\alpha^0]$ не существуют собственные значения, лежащие вне существенного спектра оператора H и для любых $\mu_\alpha > \mu_\alpha^0$ оператор H имеет два собственных значения. Найдем условие, гарантирующее замкнутость множества $W(H)$.*

Ключевые слова: *модель Фридрихса, возмущения, квантовые частицы, поля значений, спектр, пороговые собственные значение, виртуальный уровень.*

In a well-known monograph [1], Friedrichs considered the operator of multiplication by an independent variable as an unperturbed operator and chose a perturbation to be given by an integral operator. This model was subsequently called the model of the perturbation theory of the continuous spectrum because the continuous (essential) spectrum of a self-adjoint operator is unchanged under compact perturbation. The present paper is devoted to the study the numerical range of a Friedrichs model, where as a perturbation chosen an integral operator with rank two.

First, we recall the definition of numerical range. Let L be a complex Hilbert space with inner product $(\cdot;\cdot)$ and H be a linear operator in L with domain $D(H)\subset L.$ Then the numerical range of an L is the subset of the complex numbers C, given by $W(H):=\{(Hx,x): x\in D(H),\ \|x\|=1\}$. It was first studied by O.Toeplitz in [2]. In [3] F.Hausdorff showed that indeed the set $W(H)$ is convex. We recall that the numerical range of a bounded linear operator satisfies the so-called *spectral inclusion property*

$$W(H)\subset\sigma_p(H),\quad \overline{W(H)}\subset\sigma(H)$$

for the point spectrum $\sigma_p(H)$ (or set of eigenvalues) and the spectrum $\sigma(H)$ of H; note that $W(H)$ is closed if $\dim L<\infty$.

The notion of numerical range is generalized by the different ways, see for example [4-7]. One important use of $W(H)$ is to bound the spectrum $\sigma(H)$.

For positive integer number d let T^d be the d-dimensional torus, the cube $(-\pi,\pi]^d$ with appropriately identified sides equipped with its Haar measure and $L_2(T^d)$ be the Hilbert space of square integrable (complex) functions defined on T^d.

Let us consider a so-called Friedrichs model H acting on the Hilbert space $L_2(T^d)$

$$H:=H_0-\mu_1V_1+\mu_2V_2, \qquad (1)$$

where the operators H_0 and V_α, $\alpha=1,2$ are defined by

$$(H_0f)(p)=u(p)f(p),\quad (V_\alpha f)(p)=\mu_\alpha\upsilon_\alpha(p)\int_{T^d}\upsilon_\alpha(t)f(t)dt,\quad \alpha=1,2.$$

Here $\mu_\alpha>0,\ \alpha=1,2$ are positive reals, $u(\cdot)$ and $\upsilon_i(\cdot)$, $i=1,2$ are real-valued continuous functions on T^d. Under these assumptions, operator H defined by (1) is bounded and self-adjoint.

Notice that when we study the model operators associated to a system of three particles on a d-dimensional lattice [8-19] and interacting via non-local potentials, the role of a twoparticle discrete Schrödinger operator is played the Friedrichs model. Such type operators are also important in the investigations of the essential spectrum and the number of eigenvalues, located inside (in the gap, in the below of the bottom) of the essential spectrum of operator matrices [20-25].

It is clear that the essential spectrum of the operator H coincides with the spectrum of $[m;M]$, where the numbers m and M are defined by

$$m:=\min_{p\in T^d}u(p),\quad M:=\max_{p\in T^d}u(p).$$

Throughout this paper, we assume that the function $u(\cdot)$ has the continuous partial derivatives up to the third-order inclusive on T^d, in addition it has an unique non-degenerate minimum at the point $p_1\in T^3$ and an unique non-degenerate maximum at the point $p_2\in T^3$.

For the case $d\geq 3$ it is easy to check that the integrals

$$\int_{T^d}\frac{\upsilon_1^2(t)dt}{u(t)-m},\quad \int_{T^d}\frac{\upsilon_2^2(t)dt}{M-u(t)}, \qquad (2)$$

are positive and finite. In this case we introduce the following quantities:

$$\mu_1^0:=\left(\int_{T^d}\frac{\upsilon_1^2(t)dt}{u(t)-m}\right)^{-1},\quad \mu_2^0:=\left(\int_{T^d}\frac{\upsilon_2^2(t)dt}{M-u(t)}\right)^{-1}.$$

One can show that under the condition

$$mes(\mathrm{supp}\,\{\upsilon_1(\cdot)\}\cap\mathrm{supp}\,\{\upsilon_2(\cdot)\})=0. \tag{3}$$

for any $\mu_\alpha \le \mu_\alpha^0,\ \alpha=1,2$ the operator H has no eigenvalues lying in the outside of $[m;M]$ and for any $\mu_\alpha > \mu_\alpha^0,\ \alpha=1,2$ the operator H has two simple eigenvalues $E_1\in(-\infty;m)$ and $E_2\in(M;\infty)$.

The following theorem describes the numerical range $W(H)$ of H.

Theorem 1. Let $d\ge 3$ and the condition (3) be fulfilled.

1) If $\mu_\alpha \le \mu_\alpha^0$ for any $\alpha=1,2$, then $\overline{W(H)}=[m;M]=\sigma(H)$;

2) If $\mu_1 \le \mu_1^0$ and $\mu_2 > \mu_2^0$, then $\overline{W(H)}=[m;E_2]\subset\sigma(H)$;

3) If $\mu_1 > \mu_1^0$ and $\mu_2 \le \mu_2^0$, then $\overline{W(H)}=[E_1;M]\subset\sigma(H)$;

4) If $\mu_\alpha > \mu_\alpha^0$ for any $\alpha=1,2$, then $W(H)=[E_1;E_2]\subset\sigma(H)$.

A simple calculations show that if $d=1,2$ and $\upsilon_\alpha(p_\alpha)\ne 0,\ \alpha=1,2$, then the integrals (2) are infinite. Therefore, for any $\mu_\alpha>0,\ \alpha=1,2$ the operator H has two simple eigenvalues $E_1\in(-\infty;m)$ and $E_2\in(M;\infty)$. From these fact it follows that $W(H)=[E_1;E_2]\subset\sigma(H)$.

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