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Содержание

MAIN PROPERTY OF REGULARIZED FREDHOLM DETERMINANT CORRESPONDING TO A FAMILY OF 3×3 OPERATOR MATRICES

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Abstract: *a bounded self-adjoint family of 3×3 operator matrices $A(K)$, $K \in T^d := (-\pi;\pi]^d$ acting in the cut subspace of standard Fock space is considered. Such matrices arising in the spectral analysis of the spin-boson model with two bosons on the torus T^d. An information about the essential spectrum of $A(K)$ is given. We construct the "regularized Fredholm determinant" in terms of Fredholm's minor and determinant corresponding to Faddeev type integral operator. It is shown that zeros of this determinant coincide with the discrete eigenvalues of $A(K)$ lying outside the essential spectrum.*

Keywords: *operator matrix, cut subspace, standard Fock space, annihilation and creation operators, essential spectrum, the Fredholm theorem, the Fredolm determinant and minor.*

ОСНОВНЫЕ СВОЙСТВА РЕГУЛЯРИЗОВАННОГО ОПРЕДЕЛИТЕЛЯ ФРЕДГОЛЬМА, СООТВЕТСТВУЮЩИЕ СЕМЕЙСТВУ 3×3 ОПЕРАТОРНЫХ МАТРИЦ

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Аннотация: *рассматривается ограниченное самосопряженное семейства 3×3 операторных матриц $A(K)$, $K \in T^d := (-\pi;\pi]^d$, действующий в обрезанном подпространстве стандартного фоковского пространства. Такие матрицы возникают в спектральном анализе модели спин-бозона с не более чем двумя фотонами на d-мерной торе T^d. Дано информация об существенном спектре оператора $A(K)$. Построен регуляризованный определитель Фредгольма в терминах минора и определителя Фредгольма соответствующий интегрального оператора Фредгольма. Установлено, что нули этого детерминанта совпадают дискретными собственными значениями оператора $A(K)$, лежащих вне существенного спектра.*

Ключевые слова: *операторная матрица, обрезанное подпространство, стандартное пространство Фока, операторы уничтожения и рождения, существенный спектр, теорема Фредгольма, определитель и минор Фредгольма.*

Block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1]. One special class of block operator matrices are Hamiltonians associated with systems of non-conserved number of quasi-particles on a lattice. Their number can be unbounded as in the case of spin-boson models [2] or bounded as in the case of "truncated" spin-boson models [3-6].

In the present paper we consider a family of 3×3 tridiagonal operator matrices of the form

$$A(K):=\begin{pmatrix} A_{00}(K) & A_{01} & 0 \\ A_{01}^{*} & A_{11}(K) & A_{12} \\ 0 & A_{12}^{*} & A_{22}(K) \end{pmatrix}, \quad K\in T^{d} \qquad (1)$$

in the so-called truncated Fock space

$$H:=H_0\oplus H_1\oplus H_2$$

with $H_0:=C$, $H_1:=L^2(T^d)$ and $H_2:=L^2_{sym}((T^d)^2)$. Here T^d is a d-dimensional torus, the cube $(-\pi;\pi]^d$ with appropriately identified sides equipped with its Haar measure, C be the field of complex numbers (channel 1), $L^2(T^d)$ be the Hilbert space of square integrable (complex) functions defined on T^d (channel 2) and $L^2_{sym}((T^d)^2)$ stands for the subspace of $L_2((T^d)^2)$ consisting of symmetric functions (with respect to the two variables) (channel 3).

The matrix entries $A_{ii}(K):H_i\to H_i$, $i=0,1,2$ and $A_{ij}:H_j\to H_i$, $i<j$, $i,j=0,1,2$ are given by

$$A_{00}(K)f_0=\omega_0(K)f_0,\; A_{01}f_1=\int_{T^d} v_0(t)f_1(t)dt,$$

$$(A_{11}(K)f_1)(p)=\omega_1(K;p)f_1(p),\; (A_{12}f_2)(p)=\int_{T^d} v_1(t)f_2(p,t)dt,$$

$$(A_{22}(K)f_2)(p,q)=\omega_2(K;p,q)f_2(p,q),\; f_i\in H_i,\, i=0,1,2.$$

Throughout the paper we assume that the parameter functions $\omega_0(\cdot)$, $v_i(\cdot)$, $i=0,1$; $\omega_1(\cdot;\cdot)$ and $\omega_2(\cdot;\cdot,\cdot)$ are real-valued continuous functions on T^d; $(T^d)^2$ and $(T^d)^3$, respectively. In addition, for any fixed $K\in T^d$ the function $\omega_2(K;\cdot,\cdot)$ is a symmetric, that is, the equality $\omega_2(K;p,q)=\omega_2(K;q,p)$ holds for any $p,q\in T^d$.

Under these assumptions the operator $A(K)$ is bounded and self-adjoint.

We write elements f of the space H in the form, $f=(f_0,f_1,f_2)$, $f_i\in H_i$, $i=0,1,2$ and for any two elements $f=(f_0,f_1,f_2)$, $g=(g_0,g_1,g_2)\in H$ their scalar product is defined by

$$(f,g):=f_0\overline{g}_0+\int_{T^d} f_1(t)\overline{g_1(t)}dt+\int_{(T^d)^2} f_2(s,t)\overline{g_2(s,t)}ds\,dt$$

The family of operator matrices of this form play a key role for the study of the energy operator of the spin-boson model with two bosons on the torus. In fact, the latter is a 6×6 operator matrix which is unitary equivalent to a 2×2 block diagonal operator with two copies of a particular case of $A(K)$ on the diagonal, see [5,6]. Consequently, the essential spectrum and finiteness of discrete eigenvalues of the spin-boson Hamiltonian are determined by the corresponding spectral information on the operator matrix $A(K)$ in (1).

Independently of whether the underlying domain is a torus or the whole space or the whole space R^d the full spin-boson Hamiltonian is an infinite operator matrix in Fock space for which rigorous results are very hard to obtain. One line of attack is to consider the compression to the truncated Fock space with a finite N of bosons, and in fact most of the existing literature concentrates on the case $N \le 2$. For the case of R^d there some exceptions, see e.g. [3] for $N = 2$ and [4] for $N = 3$, where a rigorous scattering theory was developed for small coupling constants.

For the case when the underlying domain is a torus, the spectral properties $A(K)$ for a fixed K were investigated in [5-14], see also the references therein. The results obtained in this paper for all K will play important role when we study the problem of finding subset $\Lambda \subset T^d$ such that the operator matrices $A(K)$ has a finitely or infinitely many eigenvalues for all $K \in \Lambda$. Spectral properties of the $2x2$ operator matrices are studied in [15-22].

Throughout this paper, we use the following notation. If A is a linear bounded self-adjoint operator from Hilbert space to another, then $\sigma(A)$ denotes its spectrum, $\sigma_{ess}(A)$ its essential spectrum and $\sigma_{disc}(A)$ its discrete spectrum.

We define the numbers $E_{\min}(K,k)$ and $E_{\max}(K,k)$ as

$$E_{\min}(K,k) := \min_{q \in T^d} \omega_2(K;k,q),\ E_{\max}(K,k) := \max_{q \in T^d} \omega_2(K;k,q)$$

For any fixed $K,k \in T^d$ we define an analytic function $\Delta_K(k;\cdot)$ in $C \setminus [E_{\min}(K,k)\,;\,E_{\max}(K,k)]$ by

$$\Delta_K(k;z) := \omega_1(K;k) - z - \frac{1}{2} \int_{T^d} \frac{v_1^2(t)\,dt}{\omega_2(K;k,t) - z}.$$

Let Λ_K be the set of all complex numbers $z \in C \setminus [E_{\min}(K,k)\,;\,E_{\max}(K,k)]$ such that the equality $\Delta_K(k;z) = 0$ holds for some $K,k \in T^d$ and

$$m_K := \min_{p,q \in T^d} \omega_2(K;p,q),\ M_K := \max_{p,q \in T^d} \omega_2(K;p,q),\ \Sigma_K := [m_k;M_K] \cup \Lambda_K$$

For any $z \in C \setminus \Sigma_K$ we define an integral operator $T(K,z)$ acting in the Hilbert space H_1 as

$$(T(K;z)g_1)(p) = \frac{v_1(p)}{2\Delta_K(p;z)} \int_{T^d} \frac{v_1(t)g_1(t)\,dt}{\omega_2(K;p,t) - z}.$$

Suppose that I is the unit operator in $L_2(T^d)$, $\widehat{\Delta}(K;z)$ and $\widehat{D}(K;x,y,z)$ are respectively the Fredholm determinant and minor of the operator $I - T(K;z)$

For any $z \in C \setminus \Sigma_K$, we define the regular function

$$\Omega_K(z) := \left(\omega_0(K) - z - \int_{T^d} \frac{v_0^2(t)\, dt}{\Delta_K(t,z)} \right) \Delta(K;z) + \int_{(T^d)^2} \frac{v_0(p) v_0(t) \widehat{D}(K;p,t;z)}{\Delta_K(t,z)} dt dp$$

Theorem 1. A number $z \in C \setminus \Sigma_K$ is an eigenvalue of the operator $A(K)$ if and only if $\Omega_K(z) = 0$.

By Theorem 1, the function $\Omega_K(\cdot)$ possesses the characteristic property of the Fredholm determinant. For this reason, we call it the "regularized Fredholm determinant" corresponding to $A(K)$.

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