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EXISTENCE OF EIGENVALUES OF A OPERATOR MATRIX WITH RANK 3

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Abstract

In the present paper, we consider a block operator matrix $V_{\mu,\lambda},\ \mu,\lambda>0$ associated to a system describing three particles in interaction, without conservation of the number of particles, in the quasi-momentum representation. The number, location and sign of eigenvalues of $V_{\mu,\lambda},\ \mu,\lambda>0$ are defined.

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Block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1]. One special class of block operator matrices are Hamiltonians associated with systems of nonconserved number of quasi-particles on a lattice. Their number can be unbounded as in the case of spin-boson models [2] or bounded as in the case of "truncated" spin-boson models [3, 4, 5]. They arise, for example, in the theory of solid-state physics, quantum field theory, and statistical physics. The authors of [6] have developed geometric and commutator techniques to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number.

Let us introduce some notations used in this work. Denote by $\mathbb{T}^3$ the three dimensional torus, the cube $(-\pi;\pi]^3$ with appropriately identified sides, $\mathbb{C}$ −be the field of complex numbers. $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable (complex) functions defined on $\mathbb{T}^3$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_0:=\mathbb{C}$ and $\mathcal{H}_1:=L_2(\mathbb{T}^3)$, that is, $\mathcal{H}:=\mathcal{H}_0\oplus\mathcal{H}_1$.

The space $\mathcal{H}$ is called the two particle cut subspace of Fock space $\mathcal{F}_s\big(L_2(\mathbb{T}^3)\big)$ over $L_2(\mathbb{T}^3)$, where

$$\mathcal{F}_s\big(L_2(\mathbb{T}^3)\big)=\mathbb{C}\oplus L_2(\mathbb{T}^3)\oplus L_2^s((\mathbb{T}^3)^2)\oplus\ldots$$

The elements f of $\mathcal{H}$ can be written as $f=(f_0,\ f_1),\ f_0\in\mathcal{H}_0, f_1\in\mathcal{H}_1$. The norm of $f=(f_0,\ f_1)\in\mathcal{H}$ can be defined as

$$\|f\|=\left(|f_0|^2+\int\limits_{\mathbb{T}^3}|f_1(t)|^2dt\right)^{\frac{1}{2}}.$$

The scalar product of $f=(f_0,\ f_1)\in\mathcal{H}$ and $g=(g_0,\ g_1)\in\mathcal{H}$ is defined as

$$(f,g)=f_0\overline{g_0}+\int\limits_{\mathbb{T}^3} f_1(t)\,\overline{g_1(t)}dt$$

In the present paper we consider the 2×2 block operator matrix $V_{\mu,\lambda}$ acting in the Hilbert space $\mathcal{H}$ given by

$$V_{\mu,\lambda}=\begin{pmatrix} h_{00} & \frac{\lambda}{\sqrt{2}}h_{01} \\ \frac{\lambda}{\sqrt{2}}h_{01}^{*} & \mu v \end{pmatrix}, \qquad (1)$$

where its components are defined by the rules

$$h_{00}f_0=af_0, \qquad h_{01}f_1=\int\limits_{\mathbb{T}^3} v(t)f_1(t)dt,$$

$$(h_{01}^{*}f_0)(y)=v(y)f_0, \quad (vf_1)(y)=\int\limits_{\mathbb{T}^3} f_1(t)dt. \qquad (2)$$

Here $\mu,\lambda>0$; $f_i\in\mathcal{H}_i, i=0,1$; $v(\cdot)$ is a real-valued continuous function on $\mathbb{T}^3$. Recall that the block operator matrix $V_{\mu,\lambda}$ is associated to a system describing two particles in interaction, without conservation of the number of particles, acts in the Hilbert space $\mathcal{H}$.

It is easy to show that the perturbation $V_{\mu,\lambda}$ is a bounded self-adjoint operator of rank of no more than 3.

Indeed,

$$ImV_{\mu,\lambda}=\{(a,bv(y)+c\colon a,b,c\in\mathbb{C})\}.$$

Let

$$f\in ImV_{\mu,\lambda}.$$

Then $f^{(1)}=(1,0)$, $f^{(2)}=(0,v(y))$ and $f^{(3)}=(0,1)$ are linearly independent if the functions $\phi(x)=1$ and $v(x)$ are linearly independent, and every element $f\in ImV_{\mu,\lambda}$ can be written as

$$f=af^{(1)}+bf^{(2)}+cf^{(3)}.$$

It means that

$$\dim ImV_{\mu,\lambda}\leq 3,$$

so rank

$$Rank\,V_{\mu,\lambda}\leq 3.$$

Therefore, the block operator matrix $V_{\mu,\lambda}$ is a finite dimensional operator.

Lemma 1. $z=0$ *is an eigenvalue of the operator* $V_{\mu,\lambda}$ *with infinite multiples.*

Proof. According to the definition, if $\dim Ker\left(V_{\mu,\lambda}-zI\right)=\infty$, then $z=0$ is an eigenvalue with infinite multiples. In order to find the kernel of the operator $V_{\mu,\lambda}$, we consider the following equation:

$$Ker\left(V_{\mu,\lambda}\right)=\{f\colon V_{\mu,\lambda}f=\theta\}. \qquad (3)$$

Let's check if there is an element $f = (0, f_1)$ satisfying the equation $V_{\mu,\lambda} f = \theta$. Then we get the following system of equations:

$$\begin{cases} \int\limits_{\mathbb{T}^3} v(t) f_1(t) dt = 0 \\ \int\limits_{\mathbb{T}^3} f_1(t) dt = 0 \end{cases} \qquad (4)$$

For any $n \in N$, there exists a system of n linearly independent elements $\{f_1^{(1)}, f_1^{(2)}, \dots, f_1^{(n)}\} \subset L_2(\mathbb{T}^3)$ satisfying the system of equations (4) and the condition $\dim Ker\left(V_{\mu,\lambda}\right) = \infty$ holds. The lemma is proved.

Theorem 1. *The block operator matrix $V_{\mu,\lambda}$ has no more than two positive and one negative simple eigenvalues.*

Proof. Let us consider the equation $V_{\mu,\lambda} f = zf, z \neq 0, f \in \mathcal{H}$ or the system of equations

$$\begin{cases} (a - z) f_0 + \frac{\lambda}{\sqrt{2}} (v, f_1)_1 = 0 \\ \frac{\lambda}{\sqrt{2}} v(y) f_0 + \mu \int\limits_{\mathbb{T}^3} f_1(t) dt = z f_1, \end{cases} \qquad (5)$$

where $(\cdot,\cdot)_1$ is the scalar product in $\mathcal{H}_1$.

Since $z \neq 0$ from the second equation of (5) we find

$$f_1(y) = \frac{\lambda}{\sqrt{2}} \frac{v(y)}{z} f_0 + \frac{\mu C_{f_1}}{z}, \qquad (6)$$

where

$$C_{f_1} = \int\limits_{\mathbb{T}^3} f_1(t) dt. \qquad (7)$$

Substituting the expression (6) for f_1 into the first equation of the system of equations (5) and the equality (7) we have that the system of equations (5) has a solution if and only if

$$P_{(1,v)_1}^{(\mu,\lambda)}(z) = 0,$$

where

$$P_{(1,v)_1}^{(\mu,\lambda)}(x, z) = \left(az - z^2 + \frac{\lambda^2}{2} \|v\|^2\right)(z - 8\mu\pi^3) + \frac{\mu\lambda^2}{2} (1, v)^2, \qquad z \neq 0.$$

Here by $\|\cdot\|_1$ we denoted the norm in $\mathcal{H}_1$.

We note that, if 1 and v are linearly dependent, then

$$|(1, v)_1| = \|1\|_1 \|v\|_1.$$

Therefore,

$$P_{(1,v)_1}^{(\mu,\lambda)}(z) = P_0^{(\mu,\lambda)}(z) + \frac{\mu\lambda^2}{2}(1,v)^2.$$

By the inequality

$$|(1,v)_1| \leq \|1\|_1 \|v\|_1$$

we obtain that

$$P_0^{(\mu,\lambda)}(z) \leq P_{(1,v)_1}^{(\mu,\lambda)}(z) \leq P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z).$$

Here,

$$P_0^{(\mu,\lambda)}(z) = \left(az - z^2 + \frac{\lambda^2}{2}\|v\|^2\right)(z - 8\mu\pi^3),$$

$$P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z) = \left(-z^3 + 8\mu\pi^3 z^2 + az^2 + 8\mu\pi^3 az + \frac{\lambda^2}{2}\|v\|^2 z\right).$$

There are cases possible 1) 1 and v are orthogonal; 2) 1 and v are parallel; 3) 1 and v are neither orthogonal and nor parallel.

1) Let 1 and v are orthogonal.

Then

$$P_0^{(\mu,\lambda)}(z) = P_{(1,v)_1}^{(\mu,\lambda)}(z) < P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z).$$

In this case the numbers

$$\hat{z}_1(\mu) = \mu 8\pi^3 > 0, \qquad \hat{z}_2(\lambda) = \frac{a + \sqrt{a^2 + 2\lambda^2\|v\|^2}}{2} > 0,$$

and

$$\hat{z}_3(\lambda) = \frac{a - \sqrt{a^2 + 2\lambda^2\|v\|^2}}{2} < 0$$

are zeroes of

$$P_0^{(\mu,\lambda)}(z) = P_{(1,v)_1}^{(\mu,\lambda)}(z),$$

i.e., the eigenvalues of the operator $V_{\mu,\lambda}$.

We remark that the numbers $\hat{z}_1(\mu), \hat{z}_2(\lambda), \hat{z}_3(\lambda)$ are also zeroes of $P_0^{(\mu,\lambda)}(\cdot)$ in the case where 1 and v are not orthogonal.

2) Let 1 and v are be parallel. Then

$$P_0^{(\mu,\lambda)}(z) < P_{(1,v)_1}^{(\mu,\lambda)}(z) = P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z).$$

In this case the polynomial $P_{(1,v)_1}^{(\mu,\lambda)}(z)$ can be written in the form

$$P_{(1,v)_1}^{(\mu,\lambda)}(z) = -z\left(z^2 - 8\mu\pi^3 z - az + 8\mu\pi^3 a - \frac{\lambda^2}{2}\|v\|^2\right).$$

From here it follows that the numbers

$$\bar{z}_1 = 0, \qquad \bar{z}_2(\mu,\lambda) = \frac{a + 8\mu\pi^3 + \sqrt{(a - 8\mu\pi^3)^2 + 2\lambda^2\|v\|^2}}{2} > 0,$$

and

$$\bar{z}_3(\mu,\lambda) = \frac{a + 8\mu\pi^3 + \sqrt{(a - 8\mu\pi^3)^2 + 2\lambda^2\|v\|^2}}{2}$$

are zeroes of

$$P_{(1,v)_1}^{(\mu,\lambda)}(z) = P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z)$$

i.e., the eigenvalues of $V_{\mu,\lambda}(\gamma)$.

We remark that the numbers $\bar{z}_1, \bar{z}_2(\mu,\lambda), \bar{z}_3(\mu,\lambda)$ are also zeroes of $P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(\cdot)$ in the case where 1 and v are not parallel.

3) Let 1 and v be neither orthogonal and nor parallel. Then we have

$$P_0^{(\mu,\lambda)}(z) < P_{(1,v)_1}^{(\mu,\lambda)}(z) < P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(z).$$

Without loss of generality (otherwise we would be prove the following facts in the same way) we assume that the inequalities

$$0 = \bar{z}_1 < \hat{z}_1(\mu), \qquad \hat{z}_3(\lambda) < \bar{z}_1 = 0, \qquad 0 = \bar{z}_1 < \hat{z}_2(\lambda)$$

hold.

Since the numbers $\bar{z}_1$ and $\hat{z}_3(\mu)$ are zeroes of $P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(\cdot)$ and $P_0^{(\mu,\lambda)}(\cdot)$, respectively, we have

$$0 = P_0^{(\mu,\lambda)}(\hat{z}_3(\mu)) < P_{(1,v)_1}^{(\mu,\lambda)}(\hat{z}_3(\mu))$$

and

$$P_{(1,v)_1}^{(\mu,\lambda)}(\bar{z}_1) < P_{8\pi^3\|v\|_1}^{(\mu,\lambda)}(\bar{z}_1) = 0.$$

Then there exist $z_2(\mu,\lambda) \in (\hat{z}_3(\mu), 0)$ such that $P_{(1,v)_1}^{(\mu,\lambda)}(z_2(\mu,\lambda)) = 0$.

Analogously one can prove that there exist the numbers $z_1(\mu,\lambda) \in (0, \hat{z}_1(\mu))$ and $z_3 \in (0, \hat{z}_2(\lambda))$, which are zeroes of the polynomial $P_{(1,v)_1}^{(\mu,\lambda)}(\cdot)$.

Since $P_{(1,v)_1}^{(\mu,\lambda)}(\cdot)$ is a polynomial of order 3 $z_1(\mu,\lambda), z_2(\mu,\lambda), z_3(\mu,\lambda)$ are zeroes of $P_{(1,v)_1}^{(\mu,\lambda)}(\cdot)$, that is, they are eigenvalues of $V_{\mu,\lambda}$.

Theorem 1 is completely proved.

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