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Spectrum of a Three-Particle Model Hamiltonian on a One-Dimensional Lattice with Non-Local Potentials

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Abstract. We have analyzed the model Hamiltonian operator $H_{\mu,\lambda}$, $\mu,\lambda > 0$ related to the three particle system on a 1D lattice interacting via non-local potentials. The two channel operators $H_\mu^{(1)}$ and $H_\lambda^{(2)}$, which correspond to $H_{\mu,\lambda}$ are singled out, their spectra are determined. For the eigenfunctions of $H_{\mu,\lambda}$, we construct an analogue of the Faddeev equation. It is shown that $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is equal to the union of $\sigma(H_\mu^{(1)})$ and $\sigma(H_\lambda^{(2)})$. We establish that $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is consist of at most 3 segments.

INTRODUCTION

The most actively investigated objects in operator theory are the essential spectrum of the Hamiltonians connected with the 3 particle system on lattices. One of the most challenging aspects of these operators' spectral analysis is describing the essential spectrum's position. Many studies, for example, [1, 2] are devoted to the study of the essential spectrum of discrete Schrö dinger operators with local potentials. In particular, it was demonstrated in [1] that the essential spectrum of a three-particle discrete Schrödinger operator is the union of at most finitely many segments, even if the corresponding two-particle discrete Schrödinger operator has an unlimited number of eigenvalues. The Weyl criteria and the Hunziker-van Winter-Zhislin theorem [3] are two well-known approaches for determining the position of the essential spectra of such operators.

In the following article we have investigated the model operator (Hamiltonian) $H_{\mu,\lambda}$ related with 3 particle system on a 1D lattice and interacting via non-local potentials. Such operators are commonly used in the Hubbard model [4, 5]. Although the Hubbard model is now one of the most frequently studied many-electron metal models, very few exact results for the spectrum and wave functions of the crystal described by this model have been achieved. As a result, obtaining exact findings, at least in certain instances, such as non-local potentials, is very appealing.

For learning the location of $\sigma_{\text{ess}}(H_{\mu,\lambda})$, first of all we should introduce two channel operators $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ related to $H_{\mu,\lambda}$. When we use the theorem on the spectrum of decomposable operators, we depict the sets $\sigma(H_\mu^{(1)})$ and $\sigma(H_\lambda^{(2)})$ through the spectra of the families Friedrichs models. We then show that $\sigma_{\text{ess}}(H_{\mu,\lambda}) = \sigma(H_\mu^{(1)}) \cup \sigma(H_\lambda^{(2)})$, and that the set $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is consist of at most 3 segments. In addition, we determine the new two-particle and three-particle branches of $\sigma_{\text{ess}}(H_{\mu,\lambda})$.

The research paper is consist of the following: Section 1 is an introduction to the whole work. In Section 2, the model operator $H_{\mu,\lambda}$ is described as a bounded self-adjoint operator in the Hilbert space. In Section 3, we have considered the channel operators $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ related to $H_{\mu,\lambda}$ and the corresponding families of Friedrichs models, as well as defined their spectrum. Section 4 is dedicated to the derivation an analogue of the Faddeev equation for the eigenfunctions of $H_{\mu,\lambda}$. Section 5 is devoted to $\sigma_{\text{ess}}(H_{\mu,\lambda})$, as well as its new branches are studied.

A LATTICE THREE-PARTICLE HAMILTONIAN (MODEL OPERATOR)

Let $\mathbf{T}^1$ be 1D torus. The Hilbert space $L_2^{\text{sym}}(\mathbf{T}^2)$ is defined as a space of square-integrable symmetric (in general complex valued) functions with domain $\mathbf{T}^2$. We study the model Hamiltonian $H_{\mu,\lambda}$ defined by

$$H_{\mu,\lambda} := H_0 - \mu(V_1 + V_2) - \lambda V_3 \tag{1}$$

in $L_2^{\text{sym}}(\mathbf{T}^2)$, where H_0 is a non perturbed operator, i.e. the multiplication operator:

$$(H_0 f)(x,y) = u(x,y) f(x,y);$$

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the operators V_α, $\alpha = 1,2,3$ are partial integral operators of the form:

$$(V_1 f)(x,y) = v(y)\int_{\mathbf{T}^1} v(t)f(x,t)dt,$$
$$(V_2 f)(x,y) = v(x)\int_{\mathbf{T}^1} v(t)f(t,y)dt,$$
$$(V_3 f)(x,y) = \int_{\mathbf{T}^1} f(t,x+y-t)dt,$$

so called non-local interaction operators.

Here $f \in L_2^{\mathrm{sym}}(\mathbf{T}^2)$, is the kernel function $v(\cdot)$ is a continuous function on $\mathbf{T}^1$ with real values, and the multiplied function $u(\cdot,\cdot)$ is continuous symmetric on $\mathbf{T}^2$ with real values.

The boundedness and self-adjointness of the model Hamiltonian $H_{\mu,\lambda}$ in $L_2^{\mathrm{sym}}(\mathbf{T}^2)$ defined by formula (1) can be shown easily.

Note that the model Hamiltonian $H_{\mu,\lambda}$ is related with the system of 3 quantum particles on 1D lattice $\mathbf{Z}^1$. Indeed. Let us consider the operator energy $\widehat{H}$ of a 3 arbitrary particle system on $\mathbf{Z}^1$. This Hamiltonian acts in $l_2(\mathbf{Z}^3)$ and acting as

$$\widehat{H}\psi(n_1,n_2,n_3) = \sum_{s\in\mathbf{Z}}[\widehat{\varepsilon}_1(s)\psi(n_1+s,n_2,n_3)+\widehat{\varepsilon}_2(s)\psi(n_1,n_2+s,n_3)+$$
$$\widehat{\varepsilon}_3(s)\psi(n_1,n_2,n_3+s)] - [\mu_1\delta_{n_2n_3}+\mu_2\delta_{n_1n_3}+\mu_3\delta_{n_1n_2}]\psi(n_1,n_2,n_3).$$

Here for $\alpha = 1,2,3$ the function $\widehat{\varepsilon}_\alpha(\cdot)$, $\alpha = 1,2,3$ is defined on $\mathbf{Z}^1$ with real values, the number μ_α is the real (interaction energy of the particles β and γ), and δ_{nm} is the Kronecker delta.

We assume that $\widehat{\varepsilon}_\alpha(s)$ depends only on $|s|$, $s \in \mathbf{Z}^1$, is positive only for $s = 0$, and moreover, satisfies the inequality $|\widehat{\varepsilon}_\alpha(s)| \le C\exp(-a|s|)$ for some $a > 0$ and $C > 0$.

The boundedness and self-adjointness of the operator $\widehat{H}$ in $l_2(\mathbf{Z}^3)$ is clear.

Along with the 3 particle Hamiltonian $\widehat{H}$ in $l_2(\mathbf{Z}^3)$, we study 2 particle Hamiltonians $\widehat{h}_\alpha$, $\alpha = 1,2,3$ in $l_2(\mathbf{Z}^2)$ as

$$\widehat{h}_\alpha\psi(n_\beta,n_\gamma) = \sum_{s\in\mathbf{Z}}[\widehat{\varepsilon}_\beta(s)\psi(n_\beta+s,n_\gamma)+\widehat{\varepsilon}_\gamma(s)\psi(n_\beta,n_\gamma+s)] - \mu_\alpha\delta_{n_\beta n_\gamma}\psi(n_\beta,n_\gamma),$$
$$\alpha,\beta,\gamma = 1,2,3,\ \alpha\neq\beta, \beta\neq\gamma, \gamma\neq\alpha.$$

Applying the direct integral expansion and Fourier transform, one can reduce the problem of studying of the spectrum of $\widehat{H}$ and $\widehat{h}_\alpha, \alpha = 1,2,3$, to before analyzing families bounded self-adjoint operators $H(K)$, $K \in \mathbf{T}^1$ (3 particle Schrödinger operators on a lattice) and $h_\alpha(k)$, $k \in \mathbf{T}$ (2 particle Schrödinger operators on a lattice) in $L_2(\mathbf{T}^2)$ and $L_2(\mathbf{T}^1)$, respectively (see [8, 9]), having the form

$$(H(K)f)(x,y) = \varepsilon_K(x,y)f(x,y) - \mu_1\int_{\mathbf{T}^1} f(x,t)dt - \mu_2\int_{\mathbf{T}^1} f(t,y)dt -$$
$$\mu_3\int_{\mathbf{T}^1} f(t,x+y-t)dt,\ \ f \in L_2(\mathbf{T}^2),$$

where

$$\varepsilon_K(x,y) := \varepsilon_1(x)+\varepsilon_2(y)+\varepsilon_3(K-x-y),$$

and

$$(h_\alpha(k))f(x) = \varepsilon_k^{(\alpha)}(x)f(x) - \mu_\alpha\int_{\mathbf{T}^1} f(t)dt,\ f \in L_2(\mathbf{T}^1)$$

with

$$\varepsilon_k^{(\alpha)}(x) := \varepsilon_\beta(x)+\varepsilon_\gamma(k-x),\ \{\alpha,\beta,\gamma\} = \{1,2,3\},\ \beta<\gamma.$$

By virtue of the assumptions imposed on the function $\widehat{\varepsilon}_\alpha(\cdot)$, its Fourier transform ε_α is real analytic as well even function and has a unique global min at the fixed point $x = 0 \in \mathbf{T}^1$.

One can prove that if $\varepsilon_1(x) = \varepsilon_2(x)$ and $\mu_1 = \mu_2$, then the subspace $L_2^{\mathrm{sym}}(\mathbf{T}^2)$ is an invariant for $H(K)$ Therefore, the operator $H_{\mu,\lambda}$ is a more general model than this restricted Hamiltonian.

The lattice model operators (more general model than $H_{\mu,0}$) of the form

$$A = A_0 - K_1 - K_2 : L_2((\mathbf{T}^{\mathrm{d}})^2) \to L_2((\mathbf{T}^{\mathrm{d}})^2) \tag{2}$$

with

$$(A_0 f)(x,y) = w(x,y)f(x,y), \quad f \in L_2((\mathbf{T}^{\mathrm{d}})^2);$$

$$(K_1 f)(x,y) = \int_{\mathbf{T}^{\mathrm{d}}} k_1(x,t)f(t,y)ds, \quad (K_2 f)(x,y) = \int_{\mathbf{T}^{\mathrm{d}}} k_2(t,y)f(x,t)dt, \quad f \in L_2((\mathbf{T}^{\mathrm{d}})^2)$$

are discussed by many authors, see for instance the papers [6, 7, 8, 9, 10, 11, 12]. Here $w(\cdot,\cdot)$ and $k_\alpha(\cdot,\cdot)$, $\alpha = 1,2$ are function with real-values and continuous on $(\mathbf{T}^{\mathrm{d}})^2$. In [13, 14, 15, 16] the spectrum of the matrices, where one of the diagonal elements has form (2) and if this diagonal operator is a multiplication operator was discussed in [17, 18, 19].

The main objectives of this article are as follows:
(i) to study the subsets of spectrum of the family of Friedrichs models;
(ii) to determine the so called channel operators $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ corresponding to $H_{\mu,\lambda}$ and establish their spectra;
(iii) to construct the Faddeev type integral equation for the eigenfunctions $H_{\mu,\lambda}$;
(iv) to prove that $\sigma_{\mathrm{ess}}(H_{\mu,\lambda})$ is equal to the union of $\sigma(H_\mu^{(1)})$ and $\sigma(H_\lambda^{(2)})$;
(v) to show that $\sigma_{\mathrm{ess}}(H_{\mu,\lambda})$ as a set consists of at most 3 segments with finite length;
(vi) to determine the subsets (branches) of $\sigma_{\mathrm{ess}}(H_{\mu,\lambda})$.

In the following sections we discuss above mentioned objectives.

CHANNEL OPERATORS AND FAMILIES OF FRIEDRICHS MODELS.

To obtain an exact information about $\sigma_{\mathrm{ess}}(H_{\mu,\lambda})$ in this section we determine two operators $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ (so-called channel operators). They act in $L_2(\mathbf{T}^2)$ by

$$H_\mu^{(1)} = H_0 - \mu V_1, \quad H_\lambda^{(2)} = H_0 - \lambda V_3.$$

The boundedness and self-adjointness of $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ in $L_2(\mathbf{T}^2)$ can be proven easily.

For the bounded function $u_1(\cdot)$ on $\mathbf{T}^1$ we determine the multiplication operator U_1:

$$(U_1 g)(x,y) = u_1(x)g(x,y), \quad g \in L_2(\mathbf{T}^2).$$

Then the operator $H_\mu^{(1)}$ commutes with U_1.

Analogously the operator $H_\lambda^{(2)}$ commutes with any multiplication operator U_2 defined as

$$(U_2 g)(x,y) = u_2(x+y)g(x,y), \quad g \in L_2(\mathbf{T}^2),$$

where $u_2(\cdot)$ is the bounded function on $\mathbf{T}^1$.

By this reason from

$$L_2(\mathbf{T}^2) = \int_{k \in \mathbf{T}^1} \oplus L_2(\mathbf{T}^1)dk \tag{3}$$

we get the decompositions

$$H_\mu^{(1)} = \int_{k \in \mathbf{T}^1} \oplus h_\mu^{(1)}(k)dk \text{ and } H_\lambda^{(2)} = \int_{k \in \mathbf{T}^1} \oplus h_\lambda^{(2)}(k)dk. \tag{4}$$

In the decomposition (4) the fiber operators (families of bounded self-adjoint operators (Friedrichs models)) $h_\mu^{(1)}(k)$, $h_\lambda^{(2)}(k)$, $k\in\mathbf{T}^1$, act on $L_2(\mathbf{T}^1)$ by

$$h_\mu^{(1)}(k):=h_0^{(1)}(k)-\mu v_1,\quad h_\lambda^{(2)}(k):=h_0^{(2)}(k)-\lambda v_2,$$

where $h_0^{(\alpha)}(k)$, $\alpha=1,2$ are the multiplication operators on $L_2(\mathbf{T}^1)$:

$$(h_0^{(1)}(k)\psi)(x)=u(k,x)\psi(x),\quad \psi\in L_2(\mathbf{T}^1);$$
$$(h_0^{(2)}(k)\psi)(x)=u(x,k-x)\psi(x),\quad \psi\in L_2(\mathbf{T}^1),$$

the operators v_α, $\alpha=1,2$ are integral operators on $L_2(\mathbf{T})$:

$$(v_1\psi)(x)=v(x)\int_{\mathbf{T}^1}v(t)\psi(t)dt,\quad (v_2\psi)(x)=\int_{\mathbf{T}^1}\psi(t)dt,\quad \psi\in L_2(\mathbf{T}^1).$$

They are usually called the non-local interaction operators.

Using $(v_\alpha)^*=v_\alpha$, $\mathrm{rankv}_\alpha=1$ and Weyl's theorem, we conclude that

$$\sigma_{\mathrm{ess}}(h_\mu^{(1)}(k))=[m_1(k);M_1(k)]$$

and

$$\sigma_{\mathrm{ess}}(h_\lambda^{(2)}(k))=[m_2(k);M_2(k)].$$

where

$$m_1(k):=\min_{x\in\mathbf{T}}u(k,x),\quad M_1(k):=\max_{x\in\mathbf{T}}u(k,x),$$
$$m_2(k):=\min_{x\in\mathbf{T}}u(x,k-x),\quad M_2(k):=\max_{x\in\mathbf{T}}u(x,k-x).$$

In order to study $\sigma_{\mathrm{disc}}(h_\mu^{(1)}(k))$ and $\sigma_{\mathrm{disc}}(h_\lambda^{(2)}(k))$ we determine the analytic functions on $\mathbf{C}\setminus[m_\alpha(k);M_\alpha(k)]$ by

$$\Delta_\mu^{(1)}(k;z):=1-\mu\int_{\mathbf{T}^1}\frac{v^2(t)dt}{u(k,t)-z};$$
$$\Delta_\lambda^{(2)}(k;z):=1-\lambda\int_{\mathbf{T}^1}\frac{dt}{u(t,k-t)-z}.$$

Simple calculations show that for any fixed $k\in\mathbf{T}$ the quantity $z_\alpha(k)\in\mathbf{C}\setminus[m_\alpha(k);M_\alpha(k)]$ is a discrete eigenvalue of $h_\mu^{(1)}(k)$ (respectively $h_\lambda^{(2)}(k)$) iff $\Delta_\mu^{(1)}(k;z_1(k))=0$ (respectively $\Delta_\lambda^{(2)}(k;z_2(k))=0$). As conclusion for $\sigma_{\mathrm{disc}}(h_\mu^{(1)}(k))$ and $\sigma_{\mathrm{disc}}(h_\lambda^{(2)}(k))$ we receive

$$\sigma_{\mathrm{disc}}(h_\mu^{(1)}(k))=\{\xi\in\mathbf{C}\setminus[m_1(k);M_1(k)]:\Delta_\mu^{(1)}(k;\xi)=0\},$$
$$\sigma_{\mathrm{disc}}(h_\lambda^{(2)}(k))=\{\xi\in\mathbf{C}\setminus[m_2(k);M_2(k)]:\Delta_\lambda^{(2)}(k;\xi)=0\}.$$

Using the essential and discrete spectra of $h_\mu^{(1)}(k)$ and $h_\lambda^{(2)}(k)$, we may precisely describe the sets $\sigma(H_\mu^{(1)})$ and $\sigma(H_\lambda^{(2)})$, respectively. It is established in the following assertion.

Lemma 1. We have

$$\sigma(H_\mu^{(1)})=\bigcup_{k\in\mathbf{T}}\sigma(h_\mu^{(1)}(k))=\bigcup_{k\in\mathbf{T}}\sigma_{\mathrm{disc}}(h_\mu^{(1)}(k))\cup[m;M];$$
$$\sigma(H_\lambda^{(2)})=\bigcup_{k\in\mathbf{T}}\sigma(h_\lambda^{(2)}(k))=\bigcup_{k\in\mathbf{T}}\sigma_{\mathrm{disc}}(h_\lambda^{(2)}(k))\cup[m;M],$$

where

$$m:=\min_{k,x\in\mathbf{T}}u(k,x),\quad M:=\max_{k,x\in\mathbf{T}}u(k,x).$$

Proof. Using the theorem about the spectra of the so called decomposable operators (see, for example, [3]) and taking into account the structure obtained above (4) for $H_\mu^{(1)}$ and $H_\lambda^{(2)}$ we get assertions of Lemma 1.

THE FADDEEV EQUATION FOR THE EIGENFUNCTIONS OF $H_{\mu,\lambda}$.

We construct the Faddeev type operator equations for eigenfunctions corresponding to discrete eigenvalues of the model Hamiltonian $H_{\mu,\lambda}$ within this section.

We determine the sets:

$$\Omega_{\mu,\lambda} := \bigcup_{k\in\mathbf{T}} \{\sigma_{\rm disc}(h_\mu^{(1)}(k)) \cup \sigma_{\rm disc}(h_\lambda^{(2)}(k))\}, \quad \Sigma_{\mu,\lambda} := \Omega_{\mu,\lambda} \cup [m;M]$$

and the space

$$L_2^{(2)}(\mathbf{T}^1) := \{\varphi = (\varphi_1, \varphi_2) : f_\alpha \in L_2(\mathbf{T}^1),\ \alpha = 1,2\}.$$

Let $\mu, \lambda > 0$ and $z \in \mathbf{C} \setminus \Sigma_{\mu,\lambda}$ be fixed. We determine the matrix $T_{\mu,\lambda}(z)$ in $L_2^{(2)}(\mathbf{T}^1)$ as

$$T_{\mu,\lambda}(z) := \begin{pmatrix} T_{11}(\mu; z) & T_{12}(\mu,\lambda; z) \\ T_{21}(\mu,\lambda; z) & 0 \end{pmatrix}.$$

The elements of the 2×2 matrix $T_{\mu,\lambda}(z)$ acting

$$(T_{11}(\mu; z)\varphi_1)(x) = \frac{\mu\upsilon(x)}{\Delta_\mu^{(1)}(x;z)} \int_{\mathbf{T}^1} \frac{v(t)\varphi_1(t)dt}{u(x,t)-z},$$
$$(T_{12}(\mu,\lambda; z)\varphi_2)(x) = \frac{\lambda}{\Delta_\mu^{(1)}(x;z)} \int_{\mathbf{T}^1} \frac{v(t-x)\varphi_2(t)dt}{u(x,t-x)-z},$$
$$(T_{21}(\mu,\lambda; z)\varphi_1)(x) = \frac{2\mu}{\Delta_\lambda^{(2)}(x;z)} \int_{\mathbf{T}^1} \frac{v(x-t)\varphi_1(t)dt}{u(t,x-t)-z}.$$

Here $\varphi_\alpha \in L_2(\mathbf{T}^1)$, $\alpha = 1,2$.

Note that for any $\mu, \lambda > 0$ and $z \in \mathbf{C} \setminus \Sigma_{\mu,\lambda}$ integral operators $T_{11}(\mu; z)$, $T_{12}(\mu,\lambda; d)$ and $T_{21}(\mu,\lambda; z)$ belong to the so called Hilbert-Schmidt class, so $T_{\mu,\lambda}(z)$ is a completely continuous operator.

We formulate one of the main results of the paper.

Theorem 1. The quantity $z \in \mathbf{C} \setminus \Sigma_{\mu,\lambda}$ is a discrete eigenvalue of $H_{\mu,\lambda}$ iff the quantity $\gamma = 1$ is the discrete eigenvalue of the matrix $T_{\mu,\lambda}(z)$. In addition, the multiplicities of discrete eigenvalues z and 1 are equal.

Proof. Assume $z \in \mathbf{C} \setminus \Sigma_{\mu,\lambda}$ is an discrete eigenvalue of $H_{\mu,\lambda}$. By $f \in L_2^{\rm sym}(\mathbf{T}^2)$ we denote the corresponding eigenfunction. It satisfies the operator equation $H_{\mu,\lambda} f = zf$ or

$$(u(x,y)-z)f(x,y) - \mu v(y)\int_{\mathbf{T}} v(t)f(x,t)dt - \mu v(x)\int_{\mathbf{T}} v(t)f(t,y)dt - \lambda\int_{\mathbf{T}} f(t,x+y-t)dt = 0. \quad (5)$$

For all $x,y \in \mathbf{T}^1$ the condition $u(x,y)-z \neq 0$ is true, because $z \notin [m;M]$. In this case for f from (5) we obtain

$$f(x,y) = \frac{\mu v(y)\varphi_1(x) + \mu v(x)\varphi_1(y) + \lambda\varphi_2(x+y)}{u(x,y)-z}, \quad (6)$$

in the last equality the functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ has form

$$\varphi_1(x) := \int_{\mathbf{T}} v(t)f(x,t)dt \quad (7)$$

and

$$\varphi_2(x) := \int_{\mathbf{T}} f(t,x-t)dt. \quad (8)$$

Putting (6) for f to (7) and (8), we get that the system of equations

$$\Delta_\mu^{(1)}(x;z)\varphi_1(x) = \mu v(x)\int_{\mathbf{T}} \frac{v(t)\varphi_1(t)dt}{u(x,t)-z} + \lambda\int_{\mathbf{T}} \frac{v(t-x)\varphi_2(t)}{u(x,t-x)-z}, \quad (9)$$
$$\Delta_\lambda^{(2)}(x;z)\varphi_2(x) = 2\mu\int_{\mathbf{T}} \frac{v(x-t)\varphi_1(t)dt}{u(t,x-t)-z},$$

has a non zero solution iff (6) has a non zero solution.

By construction of the set $\Omega_{\mu,\lambda}$ we obtain $\Delta_\mu^{(1)}(x;z) \neq 0$ and $\Delta_\lambda^{(2)}(x;z) \neq 0$ under the condition $z \notin \Omega_{\mu,\lambda}$ and $x \in \mathbf{T}$. Moreover, (9) has a non zero solution iff

$$\varphi_1(x) = \frac{\mu \upsilon(x)}{\Delta_\mu^{(1)}(x;z)} \int_{\mathbf{T}} \frac{\upsilon(t)\varphi_1(t)dt}{u(x,t)-z} + \frac{\lambda}{\Delta_\mu^{(1)}(x;z)} \int_{\mathbf{T}} \frac{\upsilon(t-x)\varphi_2(t)}{u(x,t-x)-z},$$
$$\varphi_2(x) = \frac{2\mu}{\Delta_\lambda^{(2)}(x;z)} \int_{\mathbf{T}} \frac{\upsilon(x-t)\varphi_1(t)dt}{u(t,x-t)-z}$$

or

$$\varphi = T_{\mu,\lambda}(z)\varphi, \quad \varphi \in L_2^{(2)}(\mathbf{T}) \tag{10}$$

has a non zero solution.

Now let us show that, in addition, the multiplicities of the discrete eigenvalues z and 1 are equal. Assume that the multiplicity of the discrete eigenvalue $z \in \mathbf{C} \setminus \Sigma_{\mu,\lambda}$ of $H_{\mu,\lambda}$ is equal to n, and the multiplicity of the discrete eigenvalue $\gamma = 1$ of $T_{\mu,\lambda}(z)$ is equal to m. We show $n = m$.

Suppose $n < m$. Then by the assumption for the discrete eigenvalue $\gamma = 1$ there are linearly independent eigenvectors $\varphi^{(i)} = (\varphi_1^{(i)}, \varphi_2^{(i)})$, $i = 1, ..., m$ of $T_{\mu,\lambda}(z)$. Determine the functions f_i, $i = 1, ..., m$ according to (6). In this for $i = 1, ..., m$ the equality $H_{\mu,\lambda} f_i = z f_i$ is valid. From $n < m$ we receive the existence of a non zero element $(c_1, \ldots, c_m) \in \mathbf{C}^m$ with $\sum\limits_{i=1}^{m} c_i \varphi^{(i)} \neq 0$, but $\sum\limits_{i=1}^{m} c_i f_i = 0$. We obtain

$$0 = \sum_{i=1}^{m} c_i f_i(x,y) = \frac{\mu \nu(y)}{u(x,y)-z} \sum_{i=1}^{m} c_i \varphi_1^{(i)}(x) + \frac{\mu \nu(x)}{u(x,y)-z} \sum_{i=1}^{m} c_i \varphi_1^{(i)}(y)$$
$$+ \frac{\lambda}{u(x,y)-z} \sum_{i=1}^{m} c_i \varphi_2^{(i)}(x+y) \neq 0.$$

This fact isn't valid because of $n < m$.

If $n > m$, then there are linearly independent elements f_i, $i = 1, ..., n$ corresponding to the discrete eigenvalue z of $H_{\mu,\lambda}$. We know that corresponding eigenvector function to the discrete eigenvalue $\gamma = 1$ of$T_{mu,\lambda}(z)$ is equal to $\varphi^{(i)} = (\varphi_1^{(i)}, \varphi_2^{(i)})$, $i = 1, ..., n$. From $n > m$ we receive the existence of a non zero element $(d_1, \ldots, d_n) \in \mathbf{C}^n$ with $\sum\limits_{i=1}^{n} d_i \varphi^{(i)} = 0$. The linearly independence of f_i, $i = 1, ..., n$ imply $\sum\limits_{i=1}^{n} d_i f_i \neq 0$. In this case

$$0 \neq \sum_{i=1}^{n} d_i f_i(x,y) = \frac{\mu \nu(y)}{u(x,y)-z} \sum_{i=1}^{n} d_i \varphi_1^{(i)}(x) + \frac{\mu \nu(x)}{u(x,y)-z} \sum_{i=1}^{n} d_i \varphi_1^{(i)}(y)$$
$$+ \frac{\lambda}{u(x,y)-z} \sum_{i=1}^{n} d_i \varphi_2^{(i)}(x+y) = 0.$$

This isn't valid because of $n > m$. So $n = m$. Theorem 1 is completely proved.

ESSENTIAL SPECTRUM OF $H_{\mu,\lambda}$.

This section is devoted to the study of $\sigma_{\text{ess}}(H_{\mu,\lambda})$.

In the corresponding complex Hilbert spaces the norm and the inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively.

The set $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is described in the following theorem.

Theorem 2. The statement $\sigma_{\text{ess}}(H_{\mu,\lambda}) = \sigma(H_\mu^{(1)}) \cup \sigma(H_\lambda^{(2)})$ is true. In addition, $\sigma_{\text{ess}}(H_{\mu,\lambda})$ consists no more than 3 segments.

Proof. Firstly, we prove $\Sigma_{\mu,\lambda} \subset \sigma_{\rm ess}(H_{\mu,\lambda})$. Using $\Sigma_{\mu,\lambda} = \Omega_{\mu,\lambda} \cup [m;M]$ in the beginning we establish $[m;M] \subset \sigma_{\rm ess}(H_{\mu,\lambda})$. Taking an arbitrary $z_0 \in [m;M]$ we prove $z_0 \in \sigma_{\rm ess}(H_{\mu,\lambda})$. To establish the last statement it suffices to find a sequence of orthonormal vector functions $\{F_n\} \subset L_2^{\rm sym}(\mathbf{T}^2)$ (Weyl's criterion [3]) that satisfying

$$\lim_{n\to\infty} \|(H_{\mu,\lambda} - zE)F_n\| = 0. \tag{11}$$

The symmetric function $u(\cdot,\cdot)$ is continuous on $\mathbf{T}^2$ and hence we receive the existence of some point $(x_0,y_0) \in \mathbf{T}^2$ with $z_0 = u(x_0,y_0)$.

Put

$$W_n := V_n(x_0) \times V_n(y_0), \quad n \in \mathbf{N}$$

with

$$V_n(x_0) := \Big\{ x \in \mathbf{T} : \frac{1}{n+n_0+1} < |x-x_0| < \frac{1}{n+n_0} \Big\},$$

here the quantity $n_0 \in \mathbf{N}$ is selected from the condition $V_n(x_0) \cap V_n(y_0) = \emptyset$ for any $n \in \mathbf{N}$ (assumed $x_0 \neq y_0$).

We denote the Lebesgue measure of W by $\mathrm{mes}(W)$ and the characteristic function of the set W by $\chi_W(\cdot)$. Determine

$$F_n(x,y) := \frac{\chi_{W_n}(x,y) + \chi_{W_n}(y,x)}{\sqrt{2\mathrm{mes}(W_n)}}.$$

An orthonormality of $\{F_n\}$ follows from its construction.

We estimate $\|(H_{\mu,\lambda} - z_0E)F_n\|$:

$$\|H_{\mu,\lambda} - z_0E)F_n\|^2 \leq 2 \sup_{(x,y)\in W_n} |u(x,y) - z_0|^2 + \Big[8\mu^2 \max_{x\in\mathbf{T}} |v(x)|^2 + 2\lambda^2 \Big] \mathrm{mes}(V_n(x_0)).$$

Taking into account the determination of $V_n(x_0)$ and the fact that the function $u(\cdot,\cdot)$ is a continuous at $(x_0,y_0) \in \mathbf{T}^2$ we receive $\|(H_{\mu,\lambda} - z_0E)F_n\| \to 0$ with $n \to \infty$, it means $z_0 \in \sigma_{\rm ess}(H_{\mu,\lambda})$. Consequently $[m;M] \subset \sigma_{\rm ess}(H_{\mu,\lambda})$.

To show the inclusion $\Omega_{\mu,\lambda} \subset \sigma_{\rm ess}(H_{\mu,\lambda})$ we take an arbitrary point $z_{\mu,\lambda} \in \Omega_{\mu,\lambda}$ and we show that $z_{\mu,\lambda} \in \sigma_{\rm ess}(H_{\mu,\lambda})$. We differ the cases: $z_{\mu,\lambda} \in [m;M]$ and $z_{\mu,\lambda} \notin [m;M]$. For the case $z_{\mu,\lambda} \in [m;M]$, the fact $z_{\mu,\lambda} \in \sigma_{\rm ess}(H_{\mu,\lambda})$ is proved in the beginning of the proof.

Let

$$z_{\mu,\lambda} \in \bigcup_{k\in\mathbf{T}} \{\sigma_{\rm disc}(h_\mu^{(1)}(k))\} \setminus [m;M].$$

It follows from the determination of $\bigcup_{k\in\mathbf{T}}\{\sigma_{\rm disc}(h_\mu^{(1)}(k))\}$ that the statement $\Delta_\mu^{(1)}(k_1; z_\mu) = 0$ is valid for some $k_1 \in \mathbf{T}$.

We determine

$$\Phi_n(x,y) := \frac{v(y)\varphi_n(x) + v(x)\varphi_n(y)}{2(u(x,y) - z_{\mu,\lambda})}$$

with

$$\varphi_n(x) := \frac{c_n(x)\chi_{V_n(x_0)}(x)}{\sqrt{\mathrm{mes}(V_n(x_0))}}.$$

Here $c_n(\cdot) \in L_2(\mathbf{T})$ is chosen from

$$(\Phi_n, \Phi_m) = \frac{1}{2\sqrt{\mathrm{mes}(V_n(x_0))}\sqrt{\mathrm{mes}(V_m(y_0))}} \int_{V_n(x_0)} \int_{V_m(y_0)} \frac{v(s)v(t)c_n(s)c_m(t)}{(u(s,t) - z_{\mu,\lambda})^2} ds dt = 0 \tag{12}$$

with $n \neq m$, $\|\Phi_n\| = 1$.

For completeness we give the assertion about $\{c_n(\cdot)\}$:

Proposition 1. There is an ortho-normal system $\{c_n(\cdot)\} \subset L_2(\mathbf{T})$ satisfying $\mathrm{supp}\, c_n(\cdot) \subset V_n(x_0)$ with (12).

Similar Proposition is proven in [20].

Next we have to prove

$$\lim_{n\to\infty} \|(H_{\mu,\lambda} - z_{\mu,\lambda}E)\Phi_n\| = 0.$$

For positive integer n we estimate $\|(H_{\mu,\lambda} - z_{\mu,\lambda}E)\Phi_n\|$

$$\|(H_{\mu,\lambda} - z_{\mu,\lambda}E)\Phi_n\|^2 \leq C^{(1)}_{\mu,\lambda}\text{mes}(V_n(x_0)) + C^{(2)}_{\mu,\lambda} \sup_{x\in V_n(x_0)} |\Delta^{(1)}_\mu(x; z_{\mu,\lambda})|^2 \tag{13}$$

for some $C^{(\alpha)}_{\mu,\lambda} > 0$, $\alpha = 1,2$.

We know $\text{mes}(V_n(x_0)) \to 0$ and $\sup_{x\in V_n(x_0)} |\Delta^{(1)}_\mu(x; z_{\mu,\lambda})|^2 \to 0$ with $n \to \infty$. From (13) we receive $\|(H_{\mu,\lambda} - z_{\mu,\lambda}E)\Phi_n\| \to 0$ for $n \to \infty$ and $z_{\mu,\lambda} \in \sigma_{\text{ess}}(H_{\mu,\lambda})$. From arbitrariness of $z_{\mu,\lambda}$ we receive

$$\bigcup_{k\in\mathbf{T}} \{\sigma_{\text{disc}}(h^{(1)}_\mu(k))\} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}).$$

The statement

$$\bigcup_{k\in\mathbf{T}} \{\sigma_{\text{disc}}(h^{(2)}_\lambda(k))\} \subset \sigma_{\text{ess}}(H_{\mu,\lambda})$$

can be proven similarly. Therefore, $\Sigma_{\mu,\lambda} \subset \sigma_{\text{ess}}(H_{\mu,\lambda})$ is valid.

We will show the statement $\sigma_{\text{ess}}(H_{\mu,\lambda}) \subset \Omega_{\mu,\lambda}$. For each $\mu,\lambda > 0$ and $z \in \mathbf{C} \setminus \Omega_{\mu,\lambda}$ the operator $T_{\mu,\lambda}(z)$ is a completely continuous operator-valued function on $\mathbf{C}\setminus\Omega_{\mu,\lambda}$. The Hamiltonian $H_{\mu,\lambda}$ is a self-adjoint, and hence from Theorem 1 we receive the existence of $(I - T_{\mu,\lambda}(z))^{-1}$, if $z \in \mathbf{R}$ and $|z|$ is a large. From Fredholm's analytic theorem [3] we receive the existence of a discrete set $S_{\mu,\lambda} \subset \mathbf{C} \setminus \Omega_{\mu,\lambda}$ as well existence and analyticity of $(I - T_{\mu,\lambda}(z))^{-1}$ on $\mathbf{C} \setminus (S_{\mu,\lambda} \cup \Omega_{mu,\lambda})$. It is meromorphic on $\mathbf{C} \setminus \Omega_{\mu,\lambda}$ with finite rank residues. Then $\sigma(H_{\mu,\lambda}) \setminus \Omega_{\mu,\lambda}$ is consist of isolated elements, and the boundary of $\Omega_{\mu,\lambda}$ maybe the only possible accumulation points. The statement

$$\sigma(H_{\mu,\lambda}) \setminus \Omega_{\mu,\lambda} \subset \sigma_{\text{disc}}(H_{\mu,\lambda}) = \sigma(H_{\mu,\lambda}) \setminus \sigma_{\text{ess}}(H_{\mu,\lambda})$$

valid. Consequently, $\sigma_{\text{ess}}(H_{\mu,\lambda}) \subset \Omega_{\mu,\lambda}$ is true. As a result, we receive $\sigma_{\text{ess}}(H_{\mu,\lambda}) = \Omega_{\mu,\lambda}$.

Using the monotonicity property of $\Delta^{(1)}_\mu(k;\cdot)$ (resp. $\Delta^{(2)}_\lambda(k;\cdot)$) on $(-\infty, m_1(k))$ (resp. $(-\infty, m_2(k))$) as well $\Delta^{(1)}_\mu(k;z) > 1$ and $\Delta^{(2)}_\lambda(k;z) > 1$ for all $z > M$ we receive that the operator $h^{(1)}_\mu(k)$ (resp. $h^{(2)}_\lambda(k)$) has no more than 1 simple discrete eigenvalue on $(-\infty; m)$ and hasn't discrete eigenvalues on $(M; +\infty)$. Applying well known theorem on the spectrum of decomposable operators [3] and the determination of $\Omega_{\mu,\lambda}$ we receive that $\Omega_{\mu,\lambda}$ is consist of the union of at most 2 segments. From here we obtain that the max number of segments of $\Sigma_{\mu,\lambda}$ is 3. Theorem 2 is proven.

CONCLUSION.

In the present paper the spectral properties of the model Hamiltonian $H_{\mu,\lambda}$, $\mu,\lambda > 0$ related to the three particle system on a 1D lattice interacting via non-local potentials is investigated. The relation between this Hamiltonian and the three-particle Schrödinger operator on a 1D lattice is established. The two channel operators $H^{(1)}_\mu$ and $H^{(2)}_\lambda$, which correspond to $H_{\mu,\lambda}$ are singled out, their spectra are determined by the spectrum of the family of Friedrichs model. For the eigenfunctions of $H_{\mu,\lambda}$, an analogue of the Faddeev type equation is constructed. It is shown that $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is equal to the union of $\sigma(H^{(1)}_\mu)$ and $\sigma(H^{(2)}_\lambda)$. We establish that $\sigma_{\text{ess}}(H_{\mu,\lambda})$ is consist of at most 3 segments.

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