The Best Weighted Cubature Formula Over the Space of S.L.Sobolev $\widetilde{W}_2^{(m)}(T_n)$.

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Abstract. When studying the best formulas for approximate integration, the question of the existence of such formulas first of all arises. This issue has been investigated quite fully, although it is rather complicated, as evidenced, for example, in the book [1], where significant progress was achieved in its solution. Until recently, studies of the best quadrature formulas were based on the study of spline functions of a special type (monosplines) associated with the formulas under consideration. These research methods have not yet been effectively applied in the study of cubature formulas in $\widetilde{W}_2^{(m)}(T_1)$, which are multidimensional analogs of the spaces $\widetilde{W}_2^{(m)}(T_1)$.

INTRODUCTION

The book [2] considers the question of the existence of the best cubature formulas of general form over the space of S.L.Sobolev $\widetilde{W}_2^{(m)}(T_1)$.

Consider the cubature formula

$$\int_{T_n} p(x)\,f(x)\,dx \approx \sum_{\lambda=1}^{N} c_\lambda f\left(x^{(\lambda)}\right), \tag{1}$$

with the error functional

$$\ell_N(x) = \varepsilon_{T_n}(x)\,p(x) - \sum_{\lambda=1}^{N} c_\lambda \delta\left(x - x^{(\lambda)}\right) \tag{2}$$

where, c_λ are coefficients (weights), and $x^{(\lambda)} = \left(x_1^{(\lambda)}, x_2^{(\lambda)}, \dots, x_n^{(\lambda)}\right)$ are nodes of cubature formula (1), $p(x) \in L_2(T_n)$, i.e. weighting function, $\varepsilon_{T_n}(x)$is the indicator of the area $T_n$and $\delta(x)$is the Dirac delta function.

The quality of the cubature formula is characterized by the norm of the error functional, which is determined by the formula

$$\left\|\ell(x)|\widetilde{W}_2^{(m)^*}\right\| = \sup_{\|f\|\neq 0} \frac{|<\ell_N, f>|}{\left\|f|\widetilde{W}_2^{(m)}\right\|}, \tag{3}$$

and is a function of unknown coefficients and nodes. Therefore, for computational practice, it is useful to be able to calculate the norm of the error functional (3) and estimate it. Finding the minimum of the norm of the error functional for c_λ and $x^{(\lambda)}$ is the problem of studying a function of many variables for an extremum. The values c_λ and $x^{(\lambda)}$, realizing this minimum, determine the optimal formula. In the one-dimensional case for weighted quadrature formulas over the space of S.L.Sobolev $\widetilde{W}_2^{(m)}(T_1)$ - the optimal coefficients and nodes are found [7]. And in the multidimensional case, for weighted cubature formulas (1), the norms of the error functional (2) over the space $\widetilde{W}_2^{(m)}(T_n)$ are calculated [6].

STATEMENT OF THE PROBLEM

In this paper, we will find the optimal coefficients and nodes of the weighted cubature formula (1) over the space of S.L. Sobolev $\widetilde{W}_2^{(m)}(T_n)$.

Definition. The space $\widetilde{W}_2^{(m)}(T_n)$ is defined as the space of functions defined on the torus T_n, having all generalized derivatives of order m and square summable with the norm

$$\left\| f(x) | \widetilde{W}_2^{(m)}\ (T_n) \right\| = \left(\int\limits_{T_n} f(x) dx \right)^2 + \sum_{k \neq 0} |2\pi k|^{2m} \cdot \left| \hat{f}_k \right|^2 \tag{4}$$

where $\hat{f}_k$ are Fourier coefficients of the function $f(x)$.

It is known that if $2m > n$, then $\widetilde{W}_2^{(m)}(T_n) \subset C(T_n)$.

The following theorem was proved in [3].

Theorem 1. The squared norm of the error functional (2) of the weighted cubature formula (1) in the Sobolev space $\widetilde{W}_2^{(m)}(T_n)$ is:

$$\left\| \ell(x) | \widetilde{W}_2^{(m)^*} \right\|^2 = \left| \hat{p}_0 - \sum_{\lambda=1}^{N} c_\lambda \right|^2 \frac{1}{(2\pi)^m} \sum_{k \neq 0} \frac{\left| \hat{p}_k - \sum\limits_{\lambda=1}^{N} c_\lambda e^{2\pi i \left(k, x^{(\lambda)}\right)} \right|^2}{|k|^{2m}}, \tag{5}$$

where $(k,x) = \sum\limits_{j=1}^{n} k_j x_j$, $|k| = \left(\sum_{j=1}^{n} k_j^2 \right)$ and $\hat{p}_k = \int\limits_{T_n} p(x) e^{2\pi i (k,x)} dx$.

As we stated above, equality (5) indeed shows that the norm of the error functional is a function of unknown coefficients and nodes. In what follows, we will find the norms of the error functional $\left\| \ell(x) | \widetilde{W}_2^{(m)}(T_n) \right\|$ that realize the minimum using the coefficients c_λ and nodes $x^{(\lambda)}$.

Now we will deal with this problem, i.e. finding the minimum of the norm of the error functional by c_λ and $x^{(\lambda)}$.

Values c'_λ and $x^{(\lambda)}$realizing this minimum determine the best weighted cubature formula.

OPTIMAL WEIGHTED CUBATURE FORMULA IN THE SPACE $\widetilde{W}_2^{(m)}(T_n)$.

Let us introduce the notation $D^N(x) = \sum\limits_{\lambda=1}^{N} c_\lambda \delta\left(x - x^{(\lambda)}\right)$, then the following theorem holds for the error functional of the cubature formula (1).

Theorem 2. The functional of the error of the weighted cubature formula (1) at $\ell(x) = P(x)\varepsilon_{T_n}(x) - \prod\limits_{i=1}^{n} D^{N_i}(x_i)$, and $N_1 \cdot N_2 \cdot \ldots \cdot N_n = N$, $N_1 = N_2 = \ldots = N_n$ has the smallest norm in the Sobolev space $\widetilde{W}_2^{(m)}(T_n)$ when the nodes are the image of a lattice on the torus T_n and equal coefficients $c_1 = c_2 = .. = c_N = \overset{o}{c}$, which are expressed by the formula

$$\overset{o}{c} = \frac{\hat{p}_0 + \frac{1}{(2\pi)^{2m}} \cdot \frac{1}{N^{\frac{2m}{n}}} \sum\limits_{k \neq 0} \frac{\hat{p}_k}{|k|^{2m}}}{N\left(1 + \frac{1}{(2\pi)^{2m}} \frac{1}{N^{\frac{2m}{n}}} \sum\limits_{k \neq 0} \frac{1}{|k|^{2m}}\right)}$$

moreover,

$$\left\| \overset{0}{\ell}(x) | \widetilde{W}_2^{(m)^*}\ (T_n) \right\|^2 = \frac{A}{N^{\frac{2m}{n}}} + \frac{B}{N^{\frac{4m}{n}}}$$

where

$$A = \frac{1}{D(2\pi)^{2m}} \sum_{k \neq 0} \frac{(\hat{p}_k - \hat{p}_0)^2}{|k|^{2m}}, B = \frac{1}{D(2\pi)^{4m}} \left[\sum_{k' \neq 0} \frac{1}{|k'|^{2m}} \sum_{k \neq 0} \frac{\hat{p}_k^2}{|k|^{2m}} - \left(\sum_{k \neq 0} \frac{\hat{p}_k^2}{|k|^{2m}} \right)^2 \right], D = 1 + \frac{1}{(2\pi)^{2m}} \cdot \frac{1}{N^{\frac{2m}{n}}} \sum_{k \neq 0} \frac{1}{|k|^{2m}}.$$

Proof. Let $\sum\limits_{\lambda=1}^{N} c_\beta \neq 0$, then multiplying the numerator and denominator of the second term in equality (5) by $\left(\sum\limits_{\lambda=1}^{N} c_\lambda \right)$ and as a result we get

$$\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{p}_k-\sum\limits_{\lambda=1}^{N}c_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|^2}{|k|^{2m}}=\frac{1}{(2\pi)^{2m}}\cdot\sum_{k\neq 0}\frac{\left|\hat{p}_k-\left(\sum\limits_{\beta=1}^{N}c_\beta\right)\sum\limits_{\lambda=1}^{N}\frac{c_\lambda e^{2\pi\left(k,x^{(\lambda)}\right)}}{\sum\limits_{\beta=1}^{N}c_\beta}\right|^2}{|k|^{2m}}=$$

$$=\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{p}_k-\left(\sum\limits_{\beta=1}^{N}c_\beta\right)\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|^2}{|k|^{2m}},$$

where

$$c'_\lambda=\frac{c_\lambda}{\sum\limits_{\beta=1}^{N}c_\beta}. \tag{6}$$

It is obvious that

$$\sum_{\lambda=1}^{N}c'_\lambda=1. \tag{7}$$

Taking into account (6) and (7), we rewrite equality (5) in the form

$$\left\|\ell(x)|\widetilde{W}_2^{(m)^*}(T_n)\right\|=\left|\hat{p}_0-\sum_{\beta=1}^{N}c_\beta\right|^2+\frac{1}{(2\pi)^{2m}}\cdot\sum_{k\neq 0}\frac{\left|\hat{p}_k-\left(\sum\limits_{\beta=1}^{N}c_\beta\right)\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|^2}{|k|^{2m}} \tag{8}$$

Introducing the notation $\sum\limits_{\beta=1}^{N}c_\beta=t$, after some simplifications of equality (8), we rewrite it as a polynomial of the second degree with respect to t.

Then (8) takes the form

$$\left\|\ell(x)|\widetilde{W}_2^{(m)^*}(T_n)\right\|^2=\hat{p}_0^2-2t\hat{p}_0+t^2+\frac{1}{(2\pi)^{2m}}\cdot\sum_{k\neq 0}\frac{\left|\hat{p}_k-t\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|^2}{|k|^{2m}}=\hat{p}_0^2-2t\hat{p}_0+t^2+\frac{1}{(2\pi)^{2m}}\times$$

$$\times\sum_{k\neq 0}\frac{\left|\hat{p}_k^2-2t\hat{p}_k\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}+t^2\left(\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i(k,x)}\right)^2\right|}{|k|^{2m}}=$$

$$=\hat{p}_0^2-2t\hat{p}_0+t^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\cdot\frac{\left|\hat{p}_k^2-2t\hat{p}_k\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}+t^2\left(\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right)^2\right|}{|k|^{2m}}=$$

$$=\hat{p}_0^2-2t\hat{p}_0+t^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{p}_k^2}{|k|^{2m}}-\frac{2t}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{p}_k\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}}{|k|^{2m}}+\frac{t^2}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left(\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right)^2}{|k|^{2m}}=$$

$$=\left[1+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left(\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right)^2}{|k|^{2m}}\right]t^2-2t\left[\hat{p}_0+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{p}_k\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|}{|k|^{2m}}\right]+\left[\hat{p}_0^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{p}_k^2}{|k|^{2m}}\right].$$

Denoting

$$a=1+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i\left(k,x^{(\lambda)}\right)}\right|^2}{|k|^{2m}},\ b=\hat{p}_0+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{p}_k\sum\limits_{\lambda=1}^{N}c'_\lambda e^{2\pi i(k,x)}\right|}{|k|^{2m}},\ d=\hat{p}_0^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{p}_k^2}{|k|^{2m}},$$

we have

$$\left\| \ell(x)|\widetilde{W}_2^{(m)^*}(T_n)^{\;2} \right\| = at^2 - 2bt + d, \tag{9}$$

The right side of (9) is denoted by $y$and we examine the expression:

$$y = at^2 - 2bt + d = a\left(t - \frac{b}{a}\right)^2 + d - \frac{b^2}{a} \tag{10}$$

Now let us prove that for $\overset{o}{a} = \min(a)$ and $\overset{o}{t} = \frac{\overset{o}{b}}{\overset{o}{a}}$, $\overset{o}{d}$, expression (10) reaches its lowest value, i.e.

$$\left\| \overset{o}{\ell}(x)|\widetilde{W}_2^{(m)^*}(T_n) \right\|^2 = \overset{o}{y} = \overset{o}{a}t^2 - 2\overset{o}{b} + \overset{0}{d}$$

By virtue of (11) and from the results of [4] it follows that the sum

$$\sum_{k\neq 0} \frac{\left| \sum\limits_{\lambda=1}^{N} c'_\lambda e^{2\pi i(k,x)} \right|^2}{|k|^{2m}}$$

reaches its lowest value of

$$\frac{2}{N^{\frac{2m}{n}}} \sum_{k=1}^{\infty} \frac{1}{|k|^{2m}} = \frac{1}{N^{\frac{2m}{n}}} \sum_{k\neq 0} \frac{1}{|k|^{2m}}, \tag{11}$$

if the nodes $x^{(\lambda)} = \left(x_1^{(\lambda)}, x_2^{(\lambda)}, ..., x_n^{(\lambda)}\right)$ of the cubature formula (1) are equidistant and all the coefficients c'_λ are equal to each other, i.e.

$$c'_\lambda = \frac{1}{N} \text{ and } x^{(\lambda)} = \frac{\lambda}{N},\ \lambda = 1, 2, ..., N$$

Hence, it is easy to see that a- reaches its minimum value, taking into account (11) from (9) it follows that

$$\overset{o}{a} = 1 + \frac{1}{(2\pi)^{2m}} \frac{1}{N^{\frac{2m}{n}}} \sum_{k\neq 0} \frac{1}{|k|^{2m}},\ \overset{o}{b} = \hat{p}_0 + \frac{1}{(2\pi)^{2m}} \frac{1}{N^{\frac{2m}{n}}} \sum_{k\neq o} \frac{|\hat{p}_k|}{|k|^{2m}},\ \overset{o}{d} = \hat{p}_0^2 + \frac{1}{(2\pi)^{2m}} \frac{1}{N^{\frac{2m}{n}}} \sum_{k\neq o} \frac{\hat{p}_k^2}{|k|^{2m}} \tag{12}$$

Hence

$$\min y = \overset{o}{y} = \overset{o}{a}\left(t - \frac{\overset{o}{b}}{\overset{o}{a}}\right)^2 + \overset{o}{d} - \frac{\overset{o}{b}{}^2}{\overset{o}{a}}$$

Indeed, we will show that for $\min a = \overset{o}{a}$ the equality
$t = \frac{\overset{o}{b}}{\overset{o}{a}}$ and $\left\| \overset{o}{\ell}(x)|\widetilde{W}_2^{(m)^*}(T_n) \right\|$ is the smallest value of $\left\| \ell(x)|\widetilde{W}_2^{(m)^*}(T_n) \right\|$.
Let

$$c'_\lambda = \frac{1}{N},\ \lambda = 1, 2, ..., N \tag{13}$$

It follows from (6) and (13) that

$$c_1 = c_2 = ... = c_N = \overset{o}{c}, \tag{14}$$

$$\overset{o}{y} = \overset{o}{a}\left(\sum_{\lambda=1}^{N} c_\lambda - \frac{\overset{o}{b}}{\overset{o}{a}}\right)^2 + \overset{o}{d} - \frac{\overset{o}{b}{}^2}{\overset{o}{a}} = \overset{o}{a}\left(N\overset{o}{c} - \frac{\overset{o}{b}}{\overset{o}{a}}\right)^2 + \overset{o}{d} - \frac{\overset{o}{b}{}^2}{\overset{o}{a}} \tag{15}$$

Now we will consider the right-hand side of (5) as a function of $\overset{0}{c}$ and denote it by (7) $\overset{o}{y}\left(\overset{o}{c}\right)$ i.e.

$$y^{\circ}\left(c^{\circ}\right)=\overset{o}{a}\left(N\overset{o}{c}-\frac{\overset{o}{b}}{\overset{o}{a}}\right)^{2}+\overset{o}{d}-\frac{\overset{o}{b}^{2}}{\overset{o}{a}}.$$

Then, from the necessary condition for an extremum, we have

$$\frac{\partial\overset{0}{y}}{\partial\overset{0}{c}}\left(\overset{o}{c}\right)=2N\overset{o}{a}\left(N\overset{o}{c}-\frac{\overset{o}{b}}{\overset{o}{a}}\right)=0 \text{ or } N\overset{o}{c}-\frac{\overset{o}{b}}{\overset{o}{a}}=0.$$

From here

$$\overset{o}{c}=\frac{\overset{o}{b}}{N\overset{o}{a}} \tag{16}$$

Taking into account (11) from (16) we obtain

$$\overset{o}{c}=\frac{\hat{p}_{0}+\frac{1}{(2\pi)^{2m}}\frac{1}{N^{\frac{2m}{n}}}\sum_{k\neq o}\frac{\hat{p}_{k}}{|k|^{2m}}}{N\left(1+\frac{1}{(2\pi)^{2m}}\frac{1}{N^{\frac{2m}{n}}}\sum_{k\neq 0}\frac{1}{|k|^{2m}}\right)} \tag{17}$$

Thus, taking into account (15) and (17), we find

$$t=\sum_{\lambda=1}^{N}c_{\lambda}=N\overset{o}{c}=N\frac{\overset{o}{b}}{N\overset{o}{a}}=\frac{\overset{o}{b}}{\overset{o}{a}}$$

Taking into account (11), (12), and (16), it is easy to see that for the found values $\overset{o}{a},\overset{o}{b},\overset{o}{d}$ and $\overset{o}{c}$ the quantity $\left\|\overset{o}{\ell}(x)|\widetilde{W}_{2}^{(m)^{*}}(T_{n})\right\|^{2}$ has the smallest value.

CONCLUSION

In this paper, on the basis of the norms of the error functional of the form (9) obtained earlier in [3], we construct an optimal cubature formula in the Sobolev space $\widetilde{W}_{2}^{(m)}(T_{1})$. Thus, using the necessary extremum condition, the optimal coefficients are found and the weighted optimal cubature formula is constructed.

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