RESEARCH ARTICLE | MARCH 11 2024

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AIP Conf. Proc. 3004, 060026 (2024)
https://doi.org/10.1063/5.0199850

Construction of the Optimal Cubature Formula in the Space $\tilde{H}_p^{\mu}(T_n)$

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Abstract. The modern statement of the problem of optimization of approximate integration formulas consists in minimizing the norm of the error functional formula on chosen normed spaces. This paper is devoted to cubature formulas over space $\tilde{H}_p^{\mu}(T_{n)})$. It was proved that the cubature formula of rectangles (4) is optimal among cubature formulas of the form (6) and exact at constants over space $\tilde{H}_p^{\mu}(T_n)$.

23 March 2024 01:18:13

INTRODUCTION

When studying the best formulas for approximate integration, the first question that arises is whether such formulas exist. This issue was studied quite extensively. Although it is rather complicated, as evidenced by [1], significant progress has been made in its solution. Until recently, considerations of the best quadrature formulas were based on the study of spline-functions of a special form (monosplines) related to the considered formulas [2, 3, 4, 5, 6, 7, 8] . These research methods have not yet been effectively applied in the study of cubature formulas in $\tilde{W}_2^{(m)}(T_n)$, which are multidimensional analogs of spaces $W_2^{(m)}(T_1)$.

In [9, 10, 11, 12, 13, 14, 15, 16], the problem of optimality of cubature and quadrature formulas with respect to some specific space was studied. Most of them are considered in the Sobolev space [1]. Multidimensional cubature formulas differ from one-dimensional ones in two ways:

1) the forms of multidimensional domains of integration are infinitely diverse;

2) the number of integration nodes grows rapidly with increasing space dimension.

Problem 2) requires special attention to the construction of the most economical formulas.

The main problem of multidimensional integration is to find an integral of a given accuracy.

$$I_{\Omega}(x)=\int\limits_{\Omega} f(x)dx=\int\limits_{E_n}\varepsilon_{\Omega} f(x)dx, \tag{1}$$

where x - is the point of n-dimensional space E_n, $\varepsilon_{\Omega}(x)$- is the characteristic function of the bounded integration domain Ω, and function $f(x)$ is continuous in the closure of domain Ω.

The multidimensional integral (1) is approximately expressed by the sum

$$\int\limits_{\Omega} f(x)dx \approx \sum_{\lambda=1}^{N} C_{\lambda} f(x^{(\lambda)}), \tag{2}$$

$x^{(\lambda)}$ are the nodes, C_{λ} are the coefficients of formula (2) and N is the number of nodes.
The error of the cubature formula of the form (2) is determined by the following equality

$$R(f)=\int\limits_{\Omega} f(x)dx-\sum_{\lambda=1}^{N} C_{\lambda} f(x^{(\lambda)}).$$

In what follows, n-dimensional torus T_n and space $\tilde{H}_p^{\mu}$ are taken as domain Ω and n-dimensional space E_n, respectively.

Definition 1. The set $T_n=\{x=(x_1,x_2,\ldots,x_n)\}$; $\quad x_k=\{t_k\}, t_k\in R,$ where $\{t_k\}=t_k-[t_k]$, i.e. fractional part t_k, is called n - dimensional torus T_n [1].

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 060026-1–060026-6; https://doi.org/10.1063/5.0199850
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

STATEMENT OF THE PROBLEM

The present study is devoted to cubature formulas over space $\tilde{H}_p^{\mu}(T_n)$.

Let continuous positive function $\mu(\xi)$ be given, whose growth does not exceed the power law. Function $\mu(\xi)$ is called the weight function.

Let H be a matrix of order n and $detH > 0$.

Definition 2. Space $\tilde{H}_p^{\mu}(T_n)$ is defined as the space of periodic functions with period matrix H of the following form

$$f(x) = \sum_{\gamma} \hat{f}[\gamma] e^{-2\pi i \gamma^* H^{-1} x}$$

for which the sum is finite

$$\sum_{\gamma} \left|\hat{f}[\gamma]\right|^p \mu^p(\gamma H^{-1}).$$

The norm in space $\tilde{H}_p^{\mu}$ is determined by the following formulas

$$\left\|f(x)|\tilde{H}_p^{\mu}\right\| = \left\{\sum_{\gamma} \left|\hat{f}[\gamma]\right|^p \mu^p(\gamma H^{-1})\right\}^{\frac{1}{p}}, \ 1 \leq p < \infty, \tag{3}$$

and

$$\left\|f(x)|\tilde{H}_{\infty}^{\mu}\right\| = \sup_{\gamma} \left\{ \left|\hat{f}[\gamma]\right| \mu(\gamma H^{-1}) \right\}, \ p = \infty.$$

Space $\tilde{H}_p^{\mu}$ becomes isometrically isomorphic to space L_p and hence a complete Banach space.
In (3), the Fourier coefficients $\hat{f}[\gamma]$ are calculated by the following formula

$$\hat{f}[\gamma] = < f(x), e^{2\pi i \gamma H^{-1} x} > = \int_{\Omega_0} f(x) e^{2\pi i \gamma H^{-1} x} dx,$$

where Ω_0 is the basic parallelepiped for matrix H:

$$\Omega_0 = \left\{x : x = Hy; \ 0 \leq y_j < 1, \quad j = \overline{1,n}\right\}.$$

In [9, 10], the asymptotic optimality of the cubature formula of rectangles over space $\tilde{H}_p^{\mu}(T_n)$ was proved.

OPTIMAL CUBATURE FORMULA IN SPACE $\tilde{H}_p^{\mu}(T_n)$

In this paper, we prove that the cubature formula for rectangles

$$\int_{T_n} f(x) dx \approx \frac{1}{N} \sum_{1 \leq \lambda_i \leq N} f(h\lambda_i), \tag{4}$$

is optimal among cubature formulas of the form

$$\int_{T_n} f(x) dx \approx \sum_{1 \leq \lambda_i \leq N} C_{\lambda_i} f(h\lambda_i), \quad i = \overline{1,n}, \ h = \frac{1}{N}, \tag{5}$$

exact on constants over space $\tilde{H}_p^{\mu}(T_n)$, where

$T_n = \{x = (x_1, x_2, x_3, \ldots, x_n) : x_k = \{t_k\}, \ t_k \in R\}$ and $\{t_k\} = t_k - [t_k]$
i.e. fractional part t_k is called n -dimensional torus [1].

Definition 3. Cubature formula

$$\int_{T_n} f(x)dx \approx \sum_{\lambda=1}^{N} C_\lambda f(h\lambda),\ \lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{\text{n}}) \tag{6}$$

where λ_i are integers, is called exact on some set M if (6) becomes an exact equality for $f(x) \in M$.

In particular, when $M = R$, then $\sum\limits_{\lambda=1}^{N} C_\lambda = 1$.

Theorem 1. *The cubature formula of rectangles (4) is optimal among cubature formulas of the form (6) exact at constants over space* $\tilde{H}_p^\mu(T_n)$.

To prove this theorem, we consider the following lemmas, proved by M.D. Ramazanov and G.N. Salikhov (see [9]).

Lemma 1. *Properties over Banach space* $\tilde{B}$ $\sup\limits_{f(x)\in\tilde{B}} \frac{\|f(x)|C\|}{\|f(x)|\tilde{B}\|} < \infty$ *and* $\|f(x+a)|\tilde{B}\| = \|f(x)|\tilde{B}\|$

i.e. the norm of the shift-invariant space $\tilde{B}$ *for all* $f(x) \in \tilde{B}$ *and among all* $a \in R^n$*, one of the optimal error functionals has the following form*

$$\ell_h^{0,Q}(x) = \ell_h^{C_0} = \varepsilon_Q(x) - h^n C_0(h) \sum_{kh\in Q} \delta(x - kh) \tag{7}$$

Note. We assume that nodes $x^{(k)}$ are chosen according to the rule $x^{(k)} = kh$, where $h > 0$ is the parameter, tending to zero, so that numbers $\frac{1}{h}$ remain integers, $\varepsilon_Q(x)$ is the characteristic function of domain Q, $\delta(x)$ - is the Dirac delta function.

For completeness, we present the proof of the lemma.

Proof. Let the optimal functional be given by the following formula

$$\ell_h^0(x) = \varepsilon_Q(x) - h^n \sum_{kh\in Q} C_k \delta(x - kh).$$

For any integer vector S under condition $S_j = 0, 1, \ldots, h^{-1} - 1$ (j = 1,…, n), functional $\{\ell_h^s(x)\}_{h\in H}$, defined by the rule

$$<\ell_h^s(x), f(x)> = <\ell_h^0(x), f(x+sh)>,$$

has a norm equal to the norm of functional $\{\ell_h^0(x)\}_{h\in H}$.
$\{\ell_h^s(x)\}_{h\in H}$ is the error functional

$$\ell_h^s(x) = \varepsilon_Q(x) - h^n \sum_{kh\in Q} C_{k-s} \delta(x - kh),$$

where C_k are the coefficients of the optimal functional, initially given at points $kh \in Q$ and periodically extended to the entire space. So, $\{\ell_h^s(x)\}_{h\in H}$ is also the optimal functional.

The arithmetic mean over all S of functionals $\{\ell_h^s(x)\}_{h\in H}$ (denote it by $\left\{\ell_h^{0,Q}(x)\right\}_{h\in H}$) is also an error functional. It is evident that all coefficients $C_{k,0}$ of functional $\ell_h^{0,Q}(x)$ are equal to each other, $C_{k,0} = (const\ k) = C_0$, norm $\ell_n^{0,Q}(x)$ does not exceed $\|\ell_h^s(x)|\tilde{B}^*\| = \|\ell_h^0(x)|\tilde{B}^*\|$, i.e. $\left\{\ell_h^{0,Q}(x)\right\}$ is also the optimal functional.

To understand the situation more clearly, we calculate the explicit form of the optimal functional (7), i.e., constant $C_0(h)$ for some specific spaces.

The following result for $p = 2$ was established by I. Babushka [11].

Lemma 2. *Let* $\mu(k)$ *be a function of discrete argument k with a property for some* P :

$$\left(\sum_k \frac{1}{|\mu(k)|^p}\right)^{1/p} < \infty,\ at\ \ 1 \le p < \infty.$$

$$\left(\sum_k |a_k|^p\right)^{1/p} = \sup_k\ |a_k|\ ,\ at\ p = \infty.$$

Then over space $\tilde{H}_p^{\mu}$ the error functional (7) with

$$C_0(h) = \frac{1}{1+\left(\sum\limits_{k\neq 0}\left|\frac{\mu(0)}{\mu(k/h)}\right|^q\right)^{p/q}},\ \frac{1}{p}+\frac{1}{q}=1 \tag{8}$$

is optimal.

Proof. The space of all generalized functions of the following form is a conjugate space of $\tilde{H}_p^{\mu}$

$$\ell(x) = \sum_s \ell_s e^{2\pi isx} \tag{9}$$

with finite norm

$$\left\|\ell(x)|\tilde{H}_p^{\mu*}\right\| = \left(\sum_s \frac{|\ell_s|^q}{|\mu(s)|^q}\right)^{1/q}.$$

Each generalized function $\ell(x)$ defines a functional on $\tilde{H}_p^{\mu}$ according to the rule $\langle \ell(x), f(x)\rangle = \sum\limits_s \ell_s(x) f_s(x)$.

In particular, the error functionals (9) are also expanded in the Fourier series. By Lemma 1, one of the optimal functionals can be written as

$$\ell_h^{0,Q}(x) = \varepsilon_Q(x) - h^n C_0 \sum_{kh\in Q} \delta(x-kh).$$

The Fourier coefficients of a functional of this type are $\ell_0^0 = 1-C_0$, $\ell_s^0 = -C_0$ for $s = r\cdot\frac{1}{h}$ (r is any integer vector), and for the rest, S , $\ell_s^0 = 0$. For any choice of C_0, the norm of this functional is:

$$g(C_0) = \left(\frac{|1-C_0|^q}{|\mu(0)|^q} + \sum_{s\neq 0}\frac{|C_0|^q}{|\mu(s/h)|^q}\right)^{1/q}, \tag{10}$$

where C_0 corresponding to the optimal functional, gives the minimum of function $g(C_0)$. From the last formula (10) it is directly seen that the optimal functional C_0 cannot be complex, and on the real axis it cannot be outside the segment [0,1].

It remains to find the points of the minimum of $g(C_0)$ on the segment [0,1].

Consider the following function for $1 \le q < \infty$

$$g^q(C_0) = (1-C_0)^q\frac{1}{|\mu(0)|^q} + C_0^q\sum_{s\neq 0}\frac{1}{|\mu(s/h)|^q}.$$

It is continuously differentiable on [0,1], the point of the minimum is easily found and is given by the following formula (8).

As $q=\infty$

$g(C_0) = \sup\left\{(1-C_0)\frac{1}{|\mu(0)|}, C_0\frac{1}{|\mu(\frac{s}{h})|}\ \ (\forall S, S_i = 0,\pm 1,\pm 2,...,|S|\neq 0)\right\}.$

Let $a = \sup\limits_{|s|\neq 0}\frac{|\mu(0)|}{|\mu(s/h)|}$.

Then

$g(C_0) = \frac{1}{|\mu(0)|}\max\{(1-C_0), C_0\cdot a\}$

The minimum of the norm will be reached at $1-C_0 = C_0\cdot a$ or $C_0 = \frac{1}{1+a}$, which coincides with formula (8).

Thus, we see that for spaces $\tilde{H}_p^{\mu}$ $0 < C_0(h) < 1$, but $C_0(h)$ tends to 1 as $h\to 0$.

It turns out that these properties $C_0(h)$ are of a general nature (see [9]).

Proof of Theorem 1. It is known that the optimal cubature formula with coefficients $\sum\limits_{\lambda} C_\lambda = 1$ over space $\tilde{H}_p^{\mu}(T_n)$ exists (see [9, 10])

$$\inf \left\| \int_{T_n} f(x)dx - \sum_{\lambda} C_\lambda f(h\lambda) | \tilde{H}_p^\mu \right\| = c > 0$$

and the least value is reached on some system of numbers.

By Lemma 1, all coefficients C_λ are equal to each other and equal to $\frac{1}{N}$. This implies that assertions of the theorem are true.

Thus, the following theorem is true for the norm of the error functional of the cubature formula (6).

Theorem 2. *The norm of the error functional of the cubature formula (6) over space $\tilde{H}_p^\mu(T_n)$ is*

$$\left\| \int_{T_n} f(x)dx - \sum_{\lambda=1}^{N} \frac{1}{N} f(h\lambda) | \tilde{H}_p^\mu \right\| = \inf_{\chi} \left\{ \int_{T_n} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i \gamma x}}{\mu(h\gamma)} + \chi \right|^q dx \right\}^{\frac{1}{q}}$$

at $1 \leq q < \infty$ *and*

$$\left\| \int_{T_n} f(x)dx - \sum_{\lambda=1}^{N} \frac{1}{N} f(h\lambda) | \tilde{H}_p^\mu \right\| = \inf_{\chi} vrai\sup_{\gamma} \left| \sum_{\gamma \neq 0} \frac{e^{2\pi i \gamma x}}{\mu(h\gamma)} + \chi \right|$$

as $q = \infty$, which is what was required to be proved.

CONCLUSION

In studying various problems that arise in the theory of approximate integration and partial differential equations and related branches of analysis, the so-called functional approach proved to be very productive.

The essence of this approach (if we confine ourselves to the example of a boundary value problem for a differential equation) is that the differential equation with boundary conditions, is implemented as an operator acting in a specially selected functional space; the required information is extracted from the properties of this operator.

S.L.Sobolev developed an algorithm for constructing cubature formulas, which he called formulas with a regular boundary layer. He proved the asymptotic optimality of these formulas and estimated the upper bound of the norm of the error functional in space $U_2^m(\Omega)$, with separation of the leading term. V.D.Charushnikov proved similar statements for spaces $H_2^\mu(\Omega)$ and T.Kh.Sharipov established the validity of similar results for spaces $H_p^\mu(\Omega)$. Ramazonov established the optimality of the formula of rectangles for cubature formulas on a given lattice, set in one of the equivalent normalizations over the space of periodic functions, compactly embedded in the space of continuous functions, and with norm, invariant with respect to shifts of the function arguments. This paper is devoted to cubature formulas over space $\tilde{H}_p^\mu(T_{n)})$. It was proved that the cubature formula of rectangles (4) is optimal among cubature formulas of the form (6) and exact at constants over space $\tilde{H}_p^\mu(T_n)$.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the Institute of Mathematics named V.I.Romanovskiy for making convenient research facilities. We much appreciate the reviewers for their thoughtful comments and efforts toward improving our manuscript.

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