ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2024, Vol. 45, No. 10, pp. 5254–5263. © Pleiades Publishing, Ltd., 2024.

Optimal Quadrature Formulas with Derivative for Calculating Integrals of Strongly Oscillating Functions

Kh. M. Shadimetov[1,2*], F. I. Davlatova[2], and N. H. Mamatova[2,3***]**

(Submitted by T. K. Yuldashev)

[1]*Tashkent State Transport University, Tashkent, 100167 Uzbekistan*

[2]*Romanovskii Institute of Mathematics, Tashkent, 100174 Uzbekistan*

[3]*Bukhara State University, Bukhara, 200118 Uzbekistan*

Received April 24, 2024; revised August 9, 2024; accepted August 26, 2024

Abstract—In this work, in the space of differentiable functions, we consider the construction of weighted optimal quadrature formulas with derivative for the approximate calculation of integrals of fast oscillating functions. Here, using the extremal function, the squared norm of the error functional of the quadrature formula under consideration is calculated. Minimizing this norm by the coefficients of the quadrature formula, a system of the Wiener-Hopf type is obtained. By solving this system using a discrete analogue of an differential operator, the coefficients of optimal quadrature formulas are obtained.

DOI: 10.1134/S1995080224602698

Keywords and phrases: *optimal quadrature formula, extremal function, optimal coefficients, the error of the quadrature formula, the error functional, oscillating function*

1. INTRODUCTION

It is known that fractional integration and fractional differentiation developed slowly for a long time. However, in recent years, interest in fractional calculus has increased due to its applications in science and technology. Fractional derivatives have become an excellent tool for characterizing the memory and hereditary properties of various materials and processes. There are analytical methods such as the Fourier transform, Laplace transform, Mellin transform and Green's function methods that are used to solve fractional differential and integral equations. Solutions of integral and differential equations are mainly expressed in terms of definite integrals with different weight functions. To obtain the final result in most cases, these integrals must be calculated approximately (see, for example, [1, 2]).

The work [3] is devoted to study of periodic solutions for an impulsive system of fractional order integro-differential equations. The existence and uniqueness of the solution of the (ω, c)—periodic boundary value problem are reduced to the investigation of solvability of the system of nonlinear functional integral equations. The method of contracted mapping is used in the proof of one-valued solvability of nonlinear functional integral equations. Obtained some estimates for the (ω, c)—periodic solution of the studying problem. In [4] authors considered an inverse boundary value problem for a mixed type partial differential equation with Hilfer operator of fractional integro-differentiation in a positive rectangular domain and with spectral parameter in a negative rectangular domain. There, using the Fourier series method, the solutions of direct and inverse boundary value problems were constructed in the form of a Fourier series. Fourier series are expressed by Fourier coefficients that can be considered as strongly oscillating integrals.

*E-mail: kholmatshadimetov@mail.ru
**E-mail: fotimadavlatova733@gmail.com
***E-mail: nilufar.mamatova.76@mail.ru

Construction of optimal, in some sense, numerical methods for approximate calculating integrals of strongly oscillating functions is an interesting and important problem in computer sciences and mathematics, since many practically important problems, such as digital signal processing, pattern recognition, radar, medical electronics, computed tomography, etc., require the calculation of such integrals.

One of the first works devoted to numerical calculation of the integral

$$I([0,1],\omega)=\int_0^1 e^{2\pi i\omega x}\varphi(x)dx$$

is the work of Filon [5], which proposes a method reminiscent of Simpson's method. However, while in Simpson's method the entire integrand is replaced by a quadratic function, in Filon's method only the function $\varphi(x)$ in the integral is replaced by a quadratic function. Thus, Filon obtained a quadrature formula with coefficients depending on ω. By modifying the Filon method for constructing quadrature formulas for the approximate integration of strongly oscillating functions for various classes of functions, many special effective methods were obtained, such as the Filon type method, Levin type methods, the Clenshaw–Curtis–Filon method, the modified Clenshaw–Curtis method, various generalizations of quadrature rules, Gauss–Laguerre quadratures (see, for example, [6–13] for more details, [14–16]). In the work of M.D. Ramazanov and Kh.M. Shadimetov [17], a weighted optimal cubature formula in the Sobolev space of multivariable periodic functions was constructed. From this result in the one-variablel case, when the weight function is $p(x)=\exp(\tau ix)$ (where $x\in[0,2\pi]$ and τ is an integer number), it follows the Babuška's result [18], i.e., an optimal quadrature formula is obtained for the approximate calculation of Fourier coefficients.

In the article [19], authors considered optimal and optimal in order quadrature and cubature formulas, in which the integrals of rapidly oscillating functions with piecewise continuous first derivatives bounded by a constant are calculated in one-dimensional and multidimensional cases. The optimal error for these integrals are estimates of the error of numerical integration and upper bounds in the case where strong fluctuations of subinteger functions are indicated. The work [20] discusses spherical Radon transforms, which are used in synthetic aperture radars, sonars, tomography and medical imaging. Spherical Radon transforms map a function over a family of spheres into integrals.

In [21], it is considered a variant of the regularization method for systems with oscillatory-type matrices, which significantly reduces the conventionality of the Laplace integral transform problem, compared to the classical Tikhonov scheme, which proposes a method for actually constructing special quadratures, leading to solving problems associated with the oscillation matrix. Further, in the work [22] the numerical calculation of integrals of strongly oscillating functions was considered

$$\int_0^1 f(x)\sin\left(\frac{\omega}{x^r}\right)dx,\quad \int_0^1 f(x)\cos\left(\frac{\omega}{x^r}\right)dx, \tag{1}$$

where ω and r are real positive numbers, $f(x)$ is not an oscillating function. There, the authors proposed a method for approximate calculation of the integrals (1) for a sufficiently large value of the oscillating parameter r, based on Haar wavelets and hybrid functions. In [23], an algorithm was developed in the form of the Mathematika program for calculating integrals (1). For the approximate calculation of integrals of the form $\int_a^b f(x)e^{i\omega g(x)}dx$, the research [16] presented a collocation method.

In addition, the construction of optimal quadrature formulas for the approximate calculation of integrals of rapidly oscillating functions in various Hilbert and Sobolev spaces is reflected in the works [24–30]. Furthermore, in the works [24, 25] the constructed optimal quadrature formulas for numerical integration of Fourier integrals were applied to reconstruct computed tomography images.

The resent papers [31] and [32] are dedicated to a variational method for the construction of optimal quadrature formulas in the sense of Sard in the Hilbert space $\widetilde{W}_2^{(m,m-1)}$ of complex-valued and periodic functions. There, the coefficients of the optimal quadrature formula are found separately in the case ωh is integer and non-integer cases. In addition, using the constructed optimal quadrature formula, the numerical results of exponentially weighted integrals of certain functions in the case $m=2$ is presented.

The numerical results show that the order of convergence of the optimal quadrature formula for $m=2$ is $O\left(\left(\frac{1}{N+|\omega|}\right)^2\right)$ in the space $\widetilde{W}_2^{(2,1)}$.

In the present work, for numerical calculation of Fourier integrals we consider the problem of construction of the following type quadrature formulas with a derivative

$$\int\limits_0^1 e^{2\pi i\omega x} f(x)dx \cong \sum_{\beta=0}^{N} d_0[\beta] f(h\beta) + \sum_{\beta=0}^{N} d_1[\beta]\, f'(h\beta), \tag{2}$$

where $d_0[\beta]$ and $d_1[\beta]$ are the coefficients, $h=\frac{1}{N}$, N is a natural number, $i^2=-1$, and ω is an arbitrary real number. The function f belongs to the space $W_2^{(2,1)}(0,1)$, where

$$W_2^{(2,1)}(0,1) = \{\varphi : [0,1] \to \mathbb{R} |\ \varphi' \text{ is abs. cont. and } \varphi'' \in L_2(0,1)\}$$

is a Hilbert space and here the inner product of two functions f and g is defined as follows

$$\langle f, \psi\rangle = \int\limits_0^1 \left(f''(x) + f'(x)\right)\ \left(\bar{g}''(x) + \bar{g}'(x)\right) dx, \tag{3}$$

where $\bar{g}$ is the complex conjugate function of the function g. The norm of the function f is correspondingly given by the formula $\left|\left|f|W_2^{(2,1)}(0,1)\right|\right| = \langle f, f\rangle^{1/2}$. Note that the coefficients $d_0[\beta]$, $d_1\,[\beta]$ depend on ω and N.

The quadrature formula (2) is associated with the following functional in the conjugate space

$$\ell_\omega^N\,(x) = \chi_{[0,1]}(x) e^{2\pi i\omega x} - \sum_{\beta=0}^{N} d_0[\beta]\delta(x-h\beta) + \sum_{\beta=0}^{N} d_1[\beta]\delta'(x-h\beta), \tag{4}$$

where $\chi_{[0,1]}(x)$ is the characteristic function of the interval [0,1], $\delta(x)$ is the Dirac delta-function.

The following difference between the integral and the quadrature sum

$$(\ell_\omega^N, f) = \int\limits_0^1 e^{2\pi\omega i x} f(x)dx - \sum_{\beta=0}^{N} d_0[\beta]\, f\,(h\beta) - \sum_{\beta=0}^{N} d_1[\beta] f'(h\beta) \tag{5}$$

is *the error* of the quadrature formula (2).

For the absolute value of the error of the formula (2), according to the Cauchy–Schwarz inequality, we have the following estimation

$$\left|(\ell_\omega^N, f)\right| \le \left|\left|f|W_2^{(2,1)}(0,1)\right|\right|\ \left|\left|\ell_\omega^N|W_2^{(2,1)*}(0,1)\right|\right|.$$

This means that the absolute value of the error (5) of the quadrature formula (2) is estimated by the norm

$$\left|\left|\ell_\omega^N|W_2^{(2,1)*}(0,1)\right|\right| = \sup_{\left|\left|\varphi|W_2^{(2,1)}(0,1)\right|\right|=1} \left|\left(\ell_\omega^N, f\right)\right|$$

of the error functional (4). Our optimization problem is to minimize the norm of the error functional of the quadrature formula (2) by coefficients $d_1\,[\beta]$ for the given $d_0[\beta]$, i.e., find the coefficients $\mathring{d}_1[\beta]$ satisfying the following equality

$$\left|\left|\mathring{\ell}_\omega^N|W_2^{(2,1)*}(0,1)\right|\right| = \inf_{d_1[\beta]} \left|\left|\ell_\omega^N|W_2^{(2,1)*}(0,1)\right|\right|.$$

For coefficients $d_0[\beta]$, $\beta = 0, 1, ..., N$ we take the coefficients

$$d_0[\beta] = \begin{cases} h\left(\frac{K_{\omega,1}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1} - \frac{1}{2\pi i\omega h}\right), & \text{for} \quad \beta = 0, \\ hK_{\omega,1}e^{2\pi i\omega h\beta}, & \text{for} \quad \beta = \overline{1, N-1}, \\ h\left(\frac{K_{\omega,1}e^{2\pi i\omega}}{1-e^{2\pi i\omega h}} + \frac{e^{2\pi i\omega}}{2\pi i\omega h}\right), & \text{for} \quad \beta = N, \end{cases}$$

of the following optimal quadrature formula in the space $L_2^{(1)}(0,1)$ from [24]

$$\int_0^1 e^{2\pi i\omega x} f(x)dx \cong \sum_{\beta=0}^{N} d_0[\beta]f(h\beta).$$

Thus, in order to construct an optimal quadrature formula of the form (3) in the sense of Sard in the space $W_2^{(2,1)}(0,1)$, we need to solve the following problems.

Problem 1. Obtain an analytical expression for the norm of the error functional $\ell_\omega^N(x)$ of the quadrature formula (2) in the space $W_2^{(2,1)*}(0,1)$.

Problem 2. Determine the coefficients $\mathring{d}_1[\beta]$ that give the minimum to the norm of the error functional of a quadrature formula of the form (2) in the space $W_2^{(2,1)*}(0,1)$.

Coefficients $\mathring{d}_1[\beta]$ are called *optimal coefficients* of the quadrature formula with derivative (3).

The rest of the paper is organized as follows. In the second section, the analytical representation of the square of the norm of the error function is found. The system of the Wienner–Hopf type was obtained for optimal coefficients. In the third section, the analytical representations of optimal coefficients are found.

2. NORM OF THE ERROR FUNCTIONAL

First, we solve Problem 1. To calculate the norm of the error functional $\ell_\omega^N(x)$ in the space $W_2^{(2,1)}(0,1)$, we use the concept of an extremal function. Function $U_\ell^\omega(x)$ is called *an extremal function* for the functional $\ell_\omega^N(x)$ if the following equality holds [33]

$$(\ell_\omega^N, U_\ell^\omega) = ||\ell_\omega^N|| \ \ ||U_\ell^\omega||. \tag{6}$$

Since the space $W_2^{(2,1)}(0,1)$ is a Hilbert space then the extremal function $U_\ell^\omega(x)$ in this space is found using the Riesz theorem [34], which states the general form of a linear functional on a Hilbert space. Then for the functional $\ell_\omega^N(x)$ and for any $f \in W_2^{(2,1)}(0,1)$ there is a function $U_\ell^\omega(x) \in W_2^{(2,1)}(0,1)$ for which the following equality is valid

$$\left(\ell_\omega^N, f\right) = \langle U_\ell^\omega, f\rangle, \tag{7}$$

where

$$\langle U_\ell^\omega, f\rangle = \int_0^1 (\overline{U}_\ell^{\omega''}(x) + \overline{U}_\ell^{\omega'}(x))(f''(x) + f'(x))dx \tag{8}$$

is the inner product in the space $W_2^{(2,1)}(0,1)$.

In [35], it was proven that the extremal function $U_\ell^\omega(x)$ is the solution to the following boundary value problem

$$\frac{d^4U_\ell^\omega(x)}{dx^4} - \frac{d^2U_\ell^\omega(x)}{dx^2} = \overline{\ell_\omega^N}(x), \tag{9}$$

$$\left(\frac{d^3U_\ell^\omega(x)}{dx^3} - \frac{dU_\ell^\omega(x)}{dx}\right)\Big|_{x=0}^{x=1} = 0, \tag{10}$$

$$\left(\frac{d^2U_\ell^\omega(x)}{dx^2}+\frac{dU_\ell^\omega(x)}{dx}\right)\Big|_{x=0}^{x=1}=0. \tag{11}$$

And for the extremal function $U_\ell^\omega(x)$ the following theorem was proved.

Theorem 1. *The solution to the boundary value problem (9)−(11) is the extremal function $U_\ell^\omega(x)$, which has the form $U_\ell^\omega(x)=\overline{\ell_\omega^N}(x)*G_2(x)+P_1e^{-x}+P_0$, where*

$$G_2(x)=\frac{\mathrm{sgn}x}{2}\left(\frac{e^x-e^{-x}}{2}-x\right),\quad \mathrm{sgn}x=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0,\end{cases} \tag{12}$$

is a fundamental solution of a fourth order differential operator, i.e., solution to the equation

$$\left(\frac{d^4}{dx^4}-\frac{d^2}{dx^2}\right)G_2\left(x\right)=\delta\left(x\right),$$

P_0 is a constant, $\delta(x)$ is the well-known Dirac delta-function, and $\overline{\ell_\omega^N}(x)$ is the complex conjugate of $\ell_\omega^N(x)$.

Now we can calculate the norm of the error functional using the extremal function in the space $W_2^{(2,1)}(0,1)$. For the squared norm of the error functional, we use equalities (7), (8), and Theorem 1. After some simplifications we find the analytical expression

$$\begin{aligned}\left|\left|\ell_\omega^N|W_2^{(2,1)*}(0,1)\right|\right|^2 &= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}d_1\left[\beta\right]\overline{d_1}[\gamma]\frac{\mathrm{sgn}(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{-h\beta+h\gamma}}{2}\right)\\
&+\sum_{\beta=0}^{N}\int_0^1\left(\overline{d_1}\left[\beta\right]e^{2\pi i\omega x}+d_1\left[\beta\right]e^{-2\pi i\omega x}\right)\frac{\mathrm{sgn}(x-h\beta)}{2}\left(\frac{e^{x-h\beta}+e^{-x+h\beta}}{2}-1\right)dx\\
&+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\left(d_1\left[\beta\right]\overline{d_0}[\gamma]+\overline{d_1}\left[\beta\right]d_0\left[\gamma\right]\right)\frac{\mathrm{sgn}(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}+e^{-h\beta+h\gamma}}{2}-1\right)\\
&+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}d_0\left[\beta\right]\overline{d_0}[\gamma]\frac{\mathrm{sgn}(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}-e^{-h\beta+h\gamma}}{2}-(h\beta-h\gamma)\right)\\
&-\sum_{\beta=0}^{N}\int_0^1\left(\overline{d_0}\left[\beta\right]e^{2\pi i\omega x}+\overline{d_0}\left[\beta\right]e^{-2\pi i\omega x}\right)\frac{\mathrm{sgn}(x-h\beta)}{2}\left(\frac{e^{x-h\beta}-e^{-x+h\beta}}{2}-(x-h\beta)\right)dx\\
&+\int_0^1\int_0^1 e^{2\pi i\omega x}e^{-2\pi i\omega y}\frac{\mathrm{sgn}(x-y)}{2}\left(\frac{e^{x-y}-e^{-x+y}}{2}-(x-y)\right)dxdy.\end{aligned} \tag{13}$$

Thus, Problem 1 is solved.

From formula (13) it is clear that the squared norm of the error functional $\ell_\omega^N(x)$ is a multivariable function of the coefficients $d_1\left[\beta\right]$ $\left(\beta=\overline{0,N}\right)$. To find the local minimum point of the squared norm of the error functional (5) under the condition $\left(\ell_\omega^N,e^{-x}\right)=0$, we use the Lagrange method of undetermined multipliers. To do this, consider the function

$$\Psi\left(\overline{d_1},d_1,\lambda_1\right)=||\ell||^2+2\lambda_1\left(\ell_\omega^N,e^{-x}\right).$$

Equating to zero the partial derivatives of the function $\Psi\left(\overline{d_1},d_1,\lambda_1\right)$ by $\overline{d_1}[\beta]$, $\beta=0,1,...,N$, and λ_1, we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N}d_1\left[\gamma\right]\frac{\mathrm{sgn}(h\beta-h\gamma)}{4}\left(e^{h\beta-h\gamma}-e^{-h\beta+h\gamma}\right)+\lambda_1e^{-h\beta}=f_2\left(h\beta\right),\quad \beta=\overline{0,N}, \tag{14}$$

$$\sum_{\gamma=0}^{N} d_1[\gamma]\, e^{-h\gamma} = \mu_\omega, \tag{15}$$

where

$$f_2(h\beta) = \int_0^1 e^{2\pi i\omega x} \frac{\operatorname{sgn}(x-h\beta)}{2}\left(\frac{e^{x-h\beta}+e^{-x+h\beta}}{2}-1\right)dx$$
$$+\sum_{\gamma=0}^{N} d_0[\gamma]\frac{\operatorname{sgn}(h\beta-h\gamma)}{2}\left(\frac{e^{h\beta-h\gamma}+e^{-h\beta+h\gamma}}{2}-1\right), \tag{16}$$

$$\mu_\omega = -\int_0^1 e^{(2\pi i\omega-1)x}dx + \sum_{\gamma=0}^{N} d_0[\gamma]e^{-h\gamma}. \tag{17}$$

We calculate the integrals (16) and (17) and get

$$f_2(h\beta) = d_0[0]\left[\frac{\cosh(h\beta)}{2}-1\right] - d_0[N]\frac{\cosh(h\beta-1)}{2} + \frac{e^{2\pi i\omega h\beta}-1}{2\pi i\omega}$$
$$+\frac{e^{h\beta}}{4}\left[\frac{e^{2\pi i\omega-1}-2e^{(2\pi i\omega-1)h\beta}+1}{2\pi i\omega-1} + K_{\omega,1}h\,\frac{2e^{(2\pi i\omega-1)h\beta}-e^{(2\pi i\omega-1)h}-e^{2\pi i\omega-1}}{e^{(2\pi i\omega-1)h}-1}\right]$$
$$+\frac{e^{-h\beta}}{4}\left[\frac{e^{2\pi i\omega+1}-2e^{(2\pi i\omega+1)h\beta}+1}{2\pi i\omega+1} + K_{\omega,1}h\,\frac{2e^{(2\pi i\omega+1)h\beta}-e^{(2\pi i\omega+1)h}-e^{2\pi i\omega+1}}{e^{(2\pi i\omega+1)h}-1}\right]$$
$$-K_{\omega,1}h\frac{e^{2\pi i\omega h\beta}-e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1}. \tag{18}$$

$$\mu_\omega = \frac{1-e^{2\pi i\omega-1}}{2\pi i\omega-1} + hK_{\omega,1}\frac{e^{2\pi i\omega-1}-e^{(2\pi i\omega-1)h}}{e^{(2\pi i\omega-1)h}-1} + h\left[\frac{K_{\omega,1}(e^{2\pi i\omega h}-e^{2\pi i\omega-1})}{e^{2\pi i\omega h}-1}+\frac{e^{2\pi i\omega-1}-1}{2\pi i\omega h}\right], \tag{19}$$

where $K_{\omega,1} = \left(\frac{\sin\pi\omega h}{\pi\omega h}\right)^2$. Now we can present the main result of this work.

3. MAIN RESULT

The main result of this work is expressed in the following theorem.

Theorem 2. *The coefficients of optimal quadrature formulas of the form (2) in the sense of Sard in the space $W_2^{(2,1)}(0,1)$ with $\omega \in R\setminus\{0\}$ and $\omega h \notin Z$ have the form*

$$d_0[0] = h\left(\frac{K_{\omega,1}e^{2\pi i\omega h}}{e^{2\pi i\omega h}-1} - \frac{1}{2\pi i\omega h}\right), \quad d_0[\beta] = hK_{\omega,1}e^{2\pi i\omega h\beta}, \quad \beta = \overline{1, N-1},$$
$$d_0[N] = h\left(\frac{K_{\omega,1}e^{2\pi iw}}{1-e^{2\pi i\omega h}} + \frac{e^{2\pi i\omega}}{2\pi i\omega h}\right), \quad d_1[0] = \frac{\mu_\omega}{2} + \frac{f_2(h)-e^{-h}f_2(0)}{\sinh(h)},$$
$$d_1[\beta] = \frac{f_2(h\beta+h)+f_2(h\beta-h)}{\sinh(h)} - 2\coth(h)f_2(h\beta), \quad \beta = \overline{1, N-1},$$
$$d_1[N] = \frac{e\mu_\omega}{2} + \frac{f_2(1-h)-e^h f_2(1)}{\sinh(h)},$$

Proof. First, we rewrite the system (14) and (15) in the form of convolution equations

$$d_1[\beta] * G_1(x) + \lambda_1 e^{-h\beta} = f_2(h\beta) \quad \text{for} \quad (h\beta)\in[0,1],$$
$$d_1[\beta] = 0, \quad \text{for} \quad (h\beta)\notin[0,1],$$

where

$$\sum_{\beta=0}^{N} d_1\left[\beta\right] e^{-h\beta} = \mu_\omega, \quad G_1(x) = \frac{\operatorname{sgn} x}{4}\left(e^{x} - e^{-x}\right).$$

We denote

$$U(h\beta) = d_1[\beta] * G_1(x) + \lambda_1 e^{-h\beta}. \tag{20}$$

For $\beta \le 0$, the function $U(h\beta)$ is defined as follows

$$U(h\beta) = -\frac{1}{4}\sum_{\gamma=0}^{N} d_1\left[\gamma\right]\left(e^{h\beta - h\gamma} - e^{-h\beta + h\gamma}\right) + \lambda_1 e^{-h\beta}$$

$$= -\frac{e^{h\beta}}{4}\mu_\omega + \frac{e^{-h\beta}}{4}\sum_{\gamma=0}^{N} d_1\left[\gamma\right] e^{h\gamma} + \lambda_1 e^{-h\beta} = -\frac{e^{h\beta}}{4}\mu_\omega + e^{-h\beta} b^{-}.$$

From here, taking into account (18), we obtain

$$U(h\beta) = -\frac{e^{h\beta}}{4}\mu_\omega + e^{-h\beta} b^{-}, \quad \text{where} \quad b^{-} = \frac{1}{4}\sum_{\gamma=0}^{N} d_1\left[\gamma\right] e^{h\gamma} + \lambda_1. \tag{21}$$

Similarly, for $\beta \ge N$ we get

$$U(h\beta) = \frac{e^{h\beta}}{4}\mu_\omega + e^{-h\beta} b^{+}, \quad \text{where} \quad b^{+} = -\frac{1}{4}\sum_{\gamma=0}^{N} d_1\left[\gamma\right] e^{h\gamma} + \lambda_1. \tag{22}$$

Hence, from equalities (21) and (22) the function $U(h\beta)$ has the form

$$U(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}\mu_\omega + e^{-h\beta} b^{-}, & \beta \le 0, \\ f_2(h\beta), & 0 \le \beta \le N, \\ \frac{e^{h\beta}}{4}\mu_\omega + e^{-h\beta} b^{+}, & \beta \ge N. \end{cases} \tag{23}$$

where $f_2\left(h\beta\right)$ and μ_ω are given by equations (18) and (19).

Now we will determine the unknown coefficients b^{-} and b^{+}. To do this, from equality (24) with $\beta = 0$ and $\beta = N$ we obtain the following relations

$$b^{-} = f_2(0) + \frac{1}{4}\mu_\omega, \quad b^{+} = e\left(f_2(1) - \frac{e}{4}\mu_\omega\right).$$

To find them we the well-known operator (see [35])

$$D_1(h\beta) = \frac{1}{1 - e^{2h}}\begin{cases} 0, & |\beta| \ge 2, \\ -2e^{h}, & |\beta| = 1, \\ 2(1 + e^{2h}), & \beta = 0. \end{cases} \tag{24}$$

This operator satisfies the equality $D_1(h\beta) * G_1(h\beta) = \delta(h\beta)$, where $\delta(h\beta)$ is the discrete delta function

$$\delta(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \ne 0. \end{cases}$$

From this and from the definition of $U(h\beta)$ it follows that

$$d_1[\beta] = D_1(h\beta) * U(h\beta), \quad \beta = 0, 1, ..., N. \tag{25}$$

We calculate the convolution in (25) and, using (23), (24), we obtain

$$d_1[\beta] = D_1(h\beta) * U(h\beta) = \sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) U(h\gamma)$$

$$= \left[\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) f_2(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma) U(-h\gamma) + \sum_{\gamma=1}^{\infty} D_1(1 + h\gamma - h\beta) U(1 + h\gamma)\right]. \quad (26)$$

Using equalities (23) and (26), we obtain

$$d_1[\beta] = \left[\sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma)\left(-\frac{e^{-h\gamma}}{4}\mu_\omega + e^{h\gamma}\left(f_2(0) + \frac{1}{4}\mu_\omega\right)\right) + \sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) f_2(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(1 + h\gamma - h\beta)\left(\frac{e^{h(N+\gamma)}}{4}\mu_\omega + e^{-h(N+\gamma)}\left(e f_2(1) - \frac{e^2}{4}\mu_\omega\right)\right)\right].$$

Now we move on to calculating the optimal coefficients of quadrature formulas of the form (2) in the space $W_2^{(2,1)}(0,1)$.

For $\beta = 0$, $\beta = \overline{1, N-1}$, and $\beta = N$ from equality (26) after some calculations we obtain

$$d_1[0] = \frac{\mu_\omega}{2} + \frac{f_2(h) - e^{-h} f_2(0)}{\sinh(h)},$$

$$d_1(h\beta) = \frac{f_2(h\beta + h) + f_2(h\beta - h)}{\sinh(h)} - 2\coth(h) f_2(h\beta), \quad \beta = \overline{1, N-1},$$

$$d_1[N] = \frac{e\mu_\omega}{2} + \frac{f_2(1-h) - e^{h} f_2(1)}{\sinh(h)}.$$

Theorem 2 is completely proved.

4. CONCLUSIONS

To sum up, this study deals with the construction of an optimal quadrature formula for the approximate calculation of Fourier integrals in the space $W_2^{(2,1)}(0,1)$. The study derives expressions for the squared norm of the error functional of quadrature formulas with the first derivative. By minimizing this norm by coefficients, the study obtains a system for the optimal coefficients. This system is solved using a discrete analogue of a second-order differential operator. Finally, an analytical formulas for the optimal coefficients of quadrature formulas with derivative are derived.

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. C. Li and F. Zeng, *Numerical Methods for Fractional Calculus* (CRC, New York, 2015).
2. D. Baleanu, K. Diethelm, E. Scalesm, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods,* 2nd ed. (World Scientific, Singapore, 2016).
3. T. K. Yuldashev and T. A. Abduvahobov, "Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima," Lobachevskii J. Math. **44**, 4401–4409 (2023).
4. T. K. Yuldashev and B. J. Kadirkulov, "Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions," Lobachevskii J. Math. **42**, 649–662 (2021).
5. L. Filon, "On a quadrature formula for trigonometric integrals," Proc. R. Soc. Edinburgh **49**, 38–47 (1928).
6. N. S. Bakhvalov and L. G. Vasileva, "Evaluation of the integrals of oscillating functions by interpolation at nodes of Gaussian quadratures," Comput. Math. Math. Phys. **8**, 241–249 (1968).
7. E. A. Filinn, "A modification of Filon's method of numerical integration," J. ACM **7**, 181–184 (1960).
8. T. Havie, "Remarks on an expansion for integrals of rapidly oscillation functions," BIT Numer. Math. **13**, 16–29 (1973).
9. A. Iserles and S. P. Nørsett, "On quadrature methods for highly oscillatory integrals and their implementation," BIT Numer. Math. **44**, 755–772 (2004).
10. D. Levin, "Fast integration of rapidly oscillatory functions," J. Comput. Appl. Math. **67**, 95–101 (1996).
11. G. V. Milovanović, "Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures," Comput. Math. Appl. **36** (8), 19–39 (1998).
12. L. F. Shampine, "Efficient Filon method for oscillatory integrals," Appl. Math. Comput. **221**, 691–702 (2013).
13. Z. Xu, G. V. Milovanović, and S. Xiang, "Efficient computation of highly oscillatory integrals with Henkel kernel," Appl. Math. Comput. **261**, 312–322 (2015).
14. A. Iserles and S. P. Nørsett, "Efficient quadrature of highly oscillatory integrals using derivatives," Proc. R. Soc. London, Ser. A **461**, 1383–1399 (2005).
15. G. V. Milovanović and M. P. Stanić, "Numerical integration of highly oscillating functions," in *Analytic Number Theory, Approximation Theory, and Special Functions* (Springer, New York, 2014), pp. 613–649.
16. S. Oliver, "Numerical approximation of highly oscillatory integrals," Ph.D. Dissertation (Univ. of Cambridge Press, Cambridge, 2008).
17. M. D. Ramazanov and Kh. M. Shadimetov, "Weight optimal cubature formulas in a periodic Sobolev space," Dokl. Math. **60**, 217–219 (1999).
18. I. Babuška, E. Vitasek, and M. Prager, *Numerical Processes in Differential Equations* (Wiley, New York, 1966).
19. V. K. Zadiraka, S. S. Mel'nikova, and L. V. Luts, "Optimal quadrature and cubature formulas for computing Fourier transform of finite functions of one class. Case of strong oscillation," Kibern. Sist. Anal. **43**, 731–748 (2007).
20. G. Hwang and S. Moon, "Inversion formulas for quarter-spherical Radon transforms," AIMS Math. **8**, 31258–31267 (2023).
21. A. V. Lebedova and V. M. Ryabov, "Regularization of the procedure for inverting the Laplace transform using quadrature formulas," Mathematics **55**, 414–418 (2022).
22. A. I. Hascelik, "On numerical computation of integrals with integrand of the form $f(x)\sin(\omega/x^r)$ on [0,1]," J. Comput. Appl. Math. **223**, 399–408 (2009).
23. K. T. Shivaram and H. T. Prakasha, "Numerical integration of highly oscillating functions using quadrature method," Global J. Pure Appl. Math. **12**, 2683–2690 (2016).
24. A. R. Hayotov, S. Jeon, C. O. Lee, and Kh. M. Shadimetov, "Optimal quadrature formulas for non-periodic functions in Sobolev space and its application to CT image reconstruction," Filomat **35**, 4177–4195 (2021).
25. A. R. Hayotov, S. Jeon, and Kh. M. Shadimetov, "Application of optimal quadrature formulas for reconstruction of CT images," J. Comput. Appl. Math. **388**, 113313 (2021).
26. A. R. Hayotov and S. S. Babaev, "An optimal quadrature formula for numerical integration of the right Riemann-Liouville fractional Integral," Lobachevskii J. Math. **44**, 4285–4298 (2023).
27. F. A. Nuraliev, "Cubature formulas of Hermite type in the space of periodic functions of two variables," AIP Conf. Proc. **2365**, 020031 (2021).
28. Kh. M. Shadimetov, "Optimal lattice quadrature and cubature formulas," Dokl. Math. **63**, 92–94 (2001).
29. Kh. M. Shadimetov and O. Gulomov, "Optimal quadrature formulas for calculating integrals of rapidly oscillating functions," J. Math. Sci. **277**, 446–457 (2023).
30. Kh. M. Shadimetov, F. A. Nuraliev, and Sh. S. Kuziev, "Optimal quadrature formula of Hermite type in the space of differentiable functions," Int. J. Anal. Appl. **22**, 25 (2024).
31. A. R. Hayotov and U. N. Khayriev, "Construction of an optimal quadrature formula in the Hilbert space of periodic functions," Lobachevskii J. Math. **43**, 3151–3160 (2022).

32. Kh. M. Shadimetov, A. R. Hayotov, and U. N. Khayriev, “Optimal quadrature formulas for approximating strongly oscillating integrals in the Hilbert space $\tilde{W}_2^{(m,m-1)}$ of periodic functions,” J. Comput. Appl. Math. **453**, 116133 (2025). https://doi.org/10.1016/j.cam.2024.116133
33. S. L. Sobolev, *Introduction to the Theory of Cubature Formulas* (Nauka, Moscow, 1974) [in Russian].
34. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Nauka, Moscow, 1981) [in Russian].
35. Kh. M. Shadimetov and A. R. Hayotov, “Optimal quadrature formulas in the sense of Sard in $W_2^{(m,m-1)}$ space,” Colcolo **51**, 211–243 (2014).