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Optimal Formulas for Calculating Linear Functionals

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Abstract—The main result of this work are the construction of composite optimal quadrature formulas in the Sobolev space of periodic functions. Here we find the optimal coefficients of the composite quadrature formulas at $t = 0, t = 1, t = 2$ for any $m \geq 1$. In addition, the norm of the error functionals of the constructed optimal quadrature formulas are calculated, i.e. the norms of optimal error functionals of composite quadrature formulas in the conjugate Sobolev space of periodic functions are obtained.

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1. INTRODUCTION

The construction of optimal formulas for the approximate calculation of linear functionals of a wider class of problems is an urgent problem in computational mathematics. A very characteristic example here is the theory of formulas for approximate integration: quadrature and cubature formulas. A compound quadrature formula is an approximate formula that expresses the value of the integral $\int_0^1 \varphi(x)dx$ as a linear combination of the values of the integrand and its derivatives at points called nodes

$$\int_0^1 \varphi(x)dx \cong \sum_{k=1}^{N} \sum_{\alpha=0}^{t} C_k^{(\alpha)} \varphi^{(\alpha)}(x_k), \tag{1}$$

or

$$\int_{-\infty}^{\infty} \varepsilon_{[0,1]}(x)\varphi(x)dx \cong \sum_{k=1}^{N} \sum_{\alpha=0}^{t} C_k^{(\alpha)} \varphi^{(\alpha)}(x_k),$$

where $\varepsilon_{[0,1]}(x)$ is the indicator (the characteristic function) of the interval $[0, 1]$, $t \leq m-1$. The points x_k are called nodes of the quadrature formula, and the numbers $C_1^{(\alpha)}, C_2^{(\alpha)}, ..., C_N^{(\alpha)}$ are called its coefficients, $\varphi(x) \in \widetilde{L_2^{(m)}}(0, 1)$.

Recall that a function $f(x)$ is called 1-periodic if for any integer $\beta \in \mathbb{Z}$ and $f(x) = f(x + \beta)$ for all x. We give the definition of the Sobolev space $\widetilde{L_2^{(m)}}(0, 1)$ of 1-periodic functions.

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The space $\widetilde{L_2^{(m)}}(0,1)$ is the Hilbert space of 1-periodic functions $f(x)$, $-\infty < x < \infty$, that are square integrable with mth order derivative (in generalized sense), equipped with the inner product

$$(f,g) = \int\limits_0^1 \frac{d^m f}{dx^m}\frac{d^m g}{dx^m}dx.$$

The norm in the space $\widetilde{L_2^{(m)}}(0,1)$ is defined by the formula

$$||\varphi|\widetilde{L_2^{(m)}}||^2 = (\varphi,\varphi) = ||\varphi|\widetilde{L_2^{(m)}}||^2 = \int\limits_0^1 \left(\frac{d^m}{dx^m}\varphi(x)\right)^2 dx.$$

The error of the quadrature formula is the difference

$$(\ell,\varphi) = \int\limits_{-\infty}^{\infty} \ell(x)\varphi(x)dx = \int\limits_0^1 \varphi(x)dx - \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}\varphi^{(\alpha)}(x_k) \tag{2}$$

where $\delta(x)$ is the Dirac delta-function, $\phi_0(x) = \sum\limits_{\beta}\delta(x-\beta)$,

$$\ell(x) = \left(\varepsilon_{[0,1]}(x) - \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}(-1)^{\alpha}\delta^{(\alpha)}(x-x_k)\right) * \phi_0(x). \tag{3}$$

Equality (2) is defined on functions $\varphi(x)$ by an additive and homogeneous periodic functional ℓ, which is called the error functional of the quadrature formula (1). In the periodic case, the error functional of quadrature formulas is determined by formula (3). The conjugate space $\widetilde{L_2^{(m)*}}(0,1)$ consist of all periodic functionals that are orthogonal to unity

$$(\ell, 1) = 0. \tag{4}$$

Due to the fact that any periodic polynomial is constant. This condition does not depend on m.

The condition (4) can be expressed as

$$\sum_{k=1}^{N} C_k^{(0)} = 1. \tag{5}$$

The unknown parameters of the quadrature formula (1) are nodes x_k and coefficients $C_k^{(0)}$, $C_k^{(1)}, ..., C_k^{(t)}$, $k = 1, 2, ..., N$.

An optimal quadrature formula is the formula which has the minimum error for a given number N of nodes, i.e., the norm of the error functional for the optimal quadrature formula has the smallest value in the space $\widetilde{L_2^{(m)*}}(0,1)$ of periodic functions.

In this paper, the results of the numerical integration of periodic functions obtained by functional analysis methods from [1–4] are generalized to the case of composite quadrature formulas. The work [5] is devoted to weighted non-standard quadrature formulas of Gaussian type, based on the values of some linear differential operators at some nodes.

There are several methods for constructing optimal quadrature formulas in the sense of Sard, for example, the spline method [6–8], the ϕ-function method (see, for example, [9–14]) and the Sobolev method [1–4]. In different spaces, based on these methods, the Sard problem has been studied by many authors (see, for example, [15–19] and the references therein).

The spline method: Schoenberg [20] showed the connection between optimal quadrature formulas in the sense of Sard and natural splines.

The main results of this work are the construction of composite optimal quadrature formulas of the form (1) in the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$. Here we explicitly find the optimal coefficients of the composite quadrature formulas $t=0, t=1, t=2$ for any $m \geq 1$. In addition, the norms of the error functionals of the constructed optimal quadrature formulas are calculated, i.e., the norms of optimal error functionals of composite quadrature formulas are found in the conjugate space $\widetilde{L_2^{(m)*}}(0,1)$. Similar issues are considered in [21–24].

2. MINIMIZATION OF THE NORM OF THE ERROR FUNCTIONAL FOR COMPOSITE QUADRATURE FORMULAS

A composite quadrature formula with an error functional $\ell(x)$ defined by formula (3), considered on the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$, can be characterized in two ways.

On the one hand, it is determined by coefficients $C_k^{(\alpha)},\ k=1,2,...,N;\ \alpha=0,1,...,t$, subject to the condition $(\ell(x),1)=0$, and, on the other hand, an extremal function $\psi_\ell(x)$ of a quadrature formula with derivatives, which is obtained as a solution to the equation

$$\frac{d^{2m}\psi_\ell(x)}{dx^{2m}} = (-1)^m \ell(x), \tag{6}$$

and can be written as

$$\psi_\ell(x) = -\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)} i^\alpha \sum_{\beta\neq 0} \frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}} + d_0. \tag{7}$$

The norm of the error functional $\ell(x)$ for the quadrature formula with derivatives of the form (1) is expressed as a bilinear form of the coefficients of this quadrature formula and the values of the extremal function specified by formula (7).

Indeed, since the space $\widetilde{L_2^{(m)}}(0,1)$ is a Hilbert space, the norm of the error functional $\ell(x)$ and the extremal function $\psi_\ell(x)$ are related by the relation

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = \int_0^1 \left(\frac{d^m\psi_\ell}{dx^m}\right)^2 dx = (-1)^m \int_0^1 \left(\frac{d^{2m}\psi_\ell}{dx^{2m}}\right)^2 \psi_\ell(x)dx,$$

where $\psi_\ell(x)$ is the extremal function of the quadrature formula with derivatives.

However, $\frac{d^{2m}\psi_\ell(x)}{dx^{2m}} = (-1)^m\ell(x)$, that means

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = (\ell,\psi_\ell) = \int_0^1 \ell(x)\psi_\ell(x)dx. \tag{8}$$

Now, using formulas (3) and (7), we calculate the norm of the error functional for the quadrature formula using (8)

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = (\ell,\psi_\ell) = \int_0^1 \ell(x)\psi_\ell(x)dx$$

$$= \int_0^1 \left(\left(\varepsilon_{[0,1]}(x) - \sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right) * \phi_0(x)\right)$$

$$\times \left(-\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)} i^\alpha \sum_{\beta\neq 0} \frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}} + d_0\right) dx. \tag{9}$$

Taking into account condition (4) the integral (9) is calculated over the interval $[0,1]$ and the nodes x_k belong to the interval $[0,1]$, then (9) takes the following form

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = \int_0^1 \left(\varepsilon_{[0,1]}(x) - \sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right)$$
$$\times\left(-\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)} i^{\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}\right)dx.$$

It is easy to see that the integral of the function $e^{2\pi i\beta(x-x_k)}$ on the interval $[0,1]$ is zero, i.e.,

$$\int_0^1 e^{2\pi i\beta(x-x_k)}dx = 0, \quad \text{for} \quad \beta = \pm 1, \pm 2, \ldots. \tag{10}$$

By virtue of (10), we reduce the norm of the error functional to the form

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = \int_0^1 \left(\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right)$$
$$\times\left(\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)} i^{\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}\right)dx. \tag{11}$$

Using properties of δ function, we have

$$\int_0^1 \delta^{(\alpha')}(x-x_{k'})e^{2\pi i\beta(x-x_k)}dx = (-1)^{\alpha'}(2\pi i\beta)^{\alpha'}e^{2\pi i\beta(x_{k'}-x_k)}.$$

Then, (11) takes the form

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t} C_{k'}^{(\alpha')}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_k^{(\alpha)} i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}$$
$$= \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_{k'}^{(\alpha')}C_k^{(\alpha)} i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}.$$

So we have proved the following theorem.

Theorem 1. *The norm of the error functional (3) for the composite quadrature formula with derivatives in the space* $\widetilde{L_2^{(m)*}}(0,1)$ *is expressed as*

$$||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 = \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_{k'}^{(\alpha')}C_k^{(\alpha)} i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}. \tag{12}$$

3. FINDING THE MINIMUM VALUE OF THE NORM OF THE ERROR FUNCTIONAL FOR THE COMPOSITE QUADRATURE FORMULAS

To find the minimum of the norm (12), we apply the Lagrange method of undetermined multipliers. To do this, we consider the Lagrange function

$$\Psi(C^{(\alpha)},\lambda) = ||\ell|\widetilde{L_2^{(m)*}}(0,1)||^2 + 2\lambda(\ell,1).$$

Using formulas (5) and (12), we reduce the Lagrange function to the form

$$\Psi(C^{(\alpha)},\lambda)=\sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t}C_{k'}^{(\alpha')}C_{k}^{(\alpha)}i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+2\lambda\left(\sum_{k=1}^{N}C_k^{(0)}-1\right).$$

All partial derivatives with respect to $C_k^{(\alpha)}$ and λ, $k=1,2,...,N;$ $\alpha=0,1,2,...,t,$ of the function $\Psi(C^{(\alpha)},\lambda)$ equation to zero, we have

$$\frac{\partial\Psi(C^{(\alpha)},\lambda)}{\partial C_k^{(\alpha)}}=0,\ \ k=1,2,...,N,\ \alpha=0,1,...,t,\qquad \frac{\partial\Psi(C^{(\alpha)},\lambda)}{\partial\lambda}=0.$$

This gives a system of equations

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}C_{k'}^{(\alpha')}i^{\alpha'+\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+\delta[\alpha]\lambda=0, \tag{13}$$

at $k=1,2,...,N,\ \alpha=0,1,...,t,$

$$\sum_{k'=1}^{N}C_{k'}^{(0)}=1 \tag{14}$$

Here $C_{k'}^{(\alpha')}$ are the coefficients of composite quadrature formulas with derivatives, λ is a certain constant, $\delta[\alpha]$ is is defined as follows

$$\delta[\alpha]=\begin{cases}1 & \text{for}\quad \alpha=0,\\ 0 & \text{for}\quad \alpha\neq 0.\end{cases}$$

We denote the solution to system (13) and (14) by $\mathring{C}_k^{(\alpha)},\ \mathring{\lambda}$, which represents a stationary point for the Lagrange function $\Psi(C^{(\alpha)},\lambda)$. Then, we rewrite the system (13) and (14) in the form

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}\mathring{C}_{k'}^{(\alpha')}i^{\alpha'+\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+\delta[\alpha]\mathring{\lambda}=0, \tag{15}$$

$k=1,2,...,N,\ \alpha=0,1,...,t,$

$$\sum_{k'=1}^{N}\mathring{C}_{k'}^{(0)}=1. \tag{16}$$

We introduce the denotation

$$B_{2m}(x)=\sum_{\beta\neq 0}\frac{e^{2\pi i\beta x}}{(2\pi\beta)^{2m}},$$

then

$$B_{2m}(x_{k'}-x_k)=\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m}},\qquad B_{2m-\alpha'-\alpha}(x)=i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta x}}{(2\pi\beta)^{2m-\alpha-\alpha'}}.$$

With this notation, we rewrite the system (15) and (16) in the form

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}\mathring{C}_{k'}^{(\alpha')}B_{2m-\alpha-\alpha'}(x_{k'}-x_k)+\delta[\alpha]\mathring{\lambda}=0, \tag{17}$$

at $k=1,2,...,N,\ \alpha=0,1,...,t,$

$$\sum_{k'=1}^{N}\mathring{C}_{k'}^{(0)}=1. \tag{18}$$

From the theory of the Lagrange method of undetermined multipliers it follows that $\mathring{C}_k^{(\alpha)}$, $k = 1, 2, ..., N$, $\alpha = 0, 1, ..., t$ are the desired values of the optimal coefficients of the quadrature formula (1) with derivatives in the space $\widetilde{L_2^{(m)}}(0,1)$ and they give a conditional minimum to the norm of the error functional for the quadrature formula, subject to the condition (5).

The following holds.

Theorem 2. *Let the error functional ℓ be defined on the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$, i.e., its value for a constant is zero, and be optimal, i.e., among all functionals of the form (3) with a given system of nodes x_k it has the smallest norm in the conjugate space $\widetilde{L_2^{(m)*}}(0,1)$. Then, there is a solution $\psi_\ell(x)$ to equation (6) from the space $\widetilde{L_2^{(m)}}(0,1)$ which the values of its derivatives of order α at the nodes x_k are equal to zero.*

Proof. Let us now consider formula (7) for the general solution of equation (6). Let us choose a constant in it $d_0 = -\mathring{\lambda}$, where $\mathring{\lambda}$ is the value of the Lagrange multiplier we found. With this choice, for the function values $\psi_\ell(x)$ at points x_k we have

$$\psi_\ell^{(\alpha)}(x_k) = -\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} \mathring{C}_{k'}^{(\alpha')} i^{\alpha'+\alpha} \sum_{\beta\neq 0} \frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}} - \delta[\alpha]\mathring{\lambda}.$$

By virtue of equations (15) and (16), we see that $\psi_\ell^{(\alpha)}(x_k) = 0$ for $k = 1, 2, ..., N$ and $\alpha = 0, 1, 2, ..., t$. This proves Theorem 2. □

4. OPTIMAL COMPOSITE QUADRATURE FORMULAS IN THE SOBOLEV SPACE $\widetilde{L_2^{(m)}}(0,1)$

Let $x_k = hk$, $hk \in [0,1]$, $k = 1, 2, ..., N$, $h = \frac{1}{N}$, $N = 1, 2, ...$ In addition, coefficients $\mathring{C}_k^{(\alpha)} = \mathring{C}^{(\alpha)}[k]$.

In this section we prove that all coefficients $C^{(\alpha)}[k] = C^{(\alpha)}$, i.e., they do not depend on k. Then, we explicitly find the optimal coefficients $\mathring{C}^{(\alpha)}[k]$, for $\alpha = 0, 1, 2, ..., t$, $k = 1, 2, ..., N$.

In this case system (17) and (18) takes the form

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} \mathring{C}^{(\alpha')}[k'] B_{2m-\alpha-\alpha'}[k'-k] + \delta[\alpha]\mathring{\lambda} = 0, \tag{19}$$

for $k = 1, 2, ..., N$, $\alpha = 0, 1, ..., t$,

$$\sum_{k'=1}^{N} \mathring{C}^{(0)}[k'] = 1, \tag{20}$$

here $[k] = hk$, $\delta[\alpha] = \begin{cases} 1 \text{ for } \alpha = 0, \\ 0 \text{ for } \alpha \neq 0. \end{cases}$ We write the system of equations (19) and (20) in the form of convolution equations

$$\sum_{\alpha'=0}^{t} B_{2m-\alpha-\alpha'}[k] * \left(\varepsilon_{[0,1]}[\beta]\, \mathring{C}^{(\alpha)}[k]\right) + \delta[\alpha]\mathring{\lambda} = 0, \tag{21}$$

for $\alpha = 0, 1, ..., t$, $[k] = hk \in [0,1]$.

$$\sum_{k'=1}^{N} \mathring{C}^{(0)}[k'] = 1. \tag{22}$$

Here $\varepsilon_{[0,1]}[\beta] = \begin{cases} 1 \text{ for } [\beta] = h\beta \in [0,1], \\ 0 \text{ for } [\beta] \notin [0,1]. \end{cases}$ The following is true.

Theorem 3. *Optimal coefficients $\mathring{C}^{(\alpha)}[k],\ k = 1, 2, ..., N,\ \alpha = 0, 1, 2, ..., t$ of the lattice quadrature formulas with derivatives does not depend on k, i.e., all coefficients are constants $\mathring{C}^{(0)}[k] = \mathring{C}^{(0)}, \mathring{C}^{(1)}[k] = \mathring{C}^{(1)}, ..., \mathring{C}^{(t)}[k] = \mathring{C}^{(t)}$, where $\mathring{C}^{(0)}, \mathring{C}^{(1)}, ..., \mathring{C}^{(t)}$ are constant numbers.*

Here we provide the proof of this theorem for $t = 0, 1, 2$.

Proof. To do this, we use the following known results from [25–27]

$$(-1)^m hD^{(m)}[\beta] * B_{2m}[\beta] = \phi[\beta] - h, \tag{23}$$

$$D^{(m)}[\beta] * [\beta]^n = 0, \quad \text{for} \quad n \leq 2m - 1, \tag{24}$$

where $D^{(m)}[\beta]$ is the discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}}$,

$$B_{2m}[\beta] = \sum_{\gamma \neq 0} \frac{e^{2\pi i\beta h\gamma}}{|2\pi\gamma|^{2m}}, \quad \Phi[\beta] = \sum_{\gamma} \delta[\beta - h^{-1}\gamma], \tag{25}$$

$[\beta] = h\beta,\ h^{-1} = N$ is a natural number, $\delta[\beta - h^{-1}\gamma]$ is the discrete delta-function

$$\delta[\beta - h^{-1}\gamma] = \begin{cases} 1, & [\beta - h^{-1}\gamma] = 0, \\ 0, & [\beta - h^{-1}\gamma] \neq 0, \end{cases} \tag{26}$$

$\beta \in \mathbb{Z}$. It is known that if $g[\beta] = g(h\beta)$ is a 1-periodic discrete argument function, i.e., $g(h\beta + \gamma), \gamma \in \mathbb{Z}$, then the following equality holds

$$g[\beta] = \left(g[\beta]\varepsilon_{[0,1]}[\beta]\right) * \Phi[\beta], \quad \varepsilon_{[0,1]}[\beta] = \begin{cases} 1 & \text{for} \quad [\beta] = h\beta \in [0,1], \\ 0 & \text{for} \quad [\beta] \notin [0,1]. \end{cases} \tag{27}$$

Since the system (21) is the convolutional form of system (15), therefore, in the proof we use these two types of systems.

Let's move on to the proof of the theorem. To do this, we first consider the system (21) and (22) for $t = 0$. In this case, the system has the form

$$B_{2m}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + \mathring{\lambda} = 0, \tag{28}$$

$$\sum_{k'=1}^{N} \mathring{C}^{(0)}[k'] = 1. \tag{29}$$

Applying the operator $hD^{(m)}[k]$ to both sides of equation (28), we obtain

$$hD^{(m)}[k] * \left(B_{2m}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + \mathring{\lambda}\right) = 0, \quad \text{for} \quad [k] \in [0,1].$$

From here, using formulas (23), (24), and (27), we obtain

$$\mathring{C}^{(0)}[k] = h\sum_{k'=1}^{N} \mathring{C}^{(0)}[k'] = 0, \quad [k] \in [0,1], \quad k = 1, 2, ..., N.$$

By virtue of (29), we finally have

$$\mathring{C}^{(0)}[k] = h, \quad \text{for} \quad k = 1, 2, ..., N. \tag{30}$$

So, when $t = 0$, then the theorem is proven, i.e., the optimal coefficients in this case do not depend on k, $\mathring{C}^{(0)}[k] = \mathring{C}^{(0)} = h$.

Now we move on to finding unknown constant $\mathring{\lambda}$ for $t = 0$. From (28) we get

$$\mathring{\lambda} = -B_{2m}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right).$$

It is known that from (15)

$$\mathring{\lambda} = -\sum_{k'=1}^{N}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(k'h-kh)}}{(2\pi\beta)^{2m}} = -h\sum_{\beta\neq 0}\frac{1}{(2\pi\beta)^{2m}}\sum_{k'=1}^{N}e^{2\pi i\beta h(k'-k)}.$$

However,

$$\sum_{k'=1}^{N}e^{-2\pi i\beta hk}e^{2\pi i\beta hk'} = e^{-2\pi i\beta hk}\sum_{k'=1}^{N}e^{2\pi i\beta hk'} = \begin{cases} N & \text{for} \quad h\beta\in\mathbb{Z}, \\ 0 & \text{for} \quad h\beta\notin\mathbb{Z}. \end{cases}$$

From here, instead of β substituting βh^{-1}, we get

$$\mathring{\lambda} = -\sum_{\beta\neq 0}\frac{h^{2m}}{(2\pi\beta)^{2m}}. \tag{31}$$

So we have found a solution to system (28) and (29), i.e., the solution of this system is determined by formula (30) and (31).

Let $t=1$, then the system (21) and (22) takes the form

$$B_{2m}[k]*\left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-1}[k]*\left(\mathring{C}^{(1)}[k]\varepsilon_{[0,1]}[k]\right) + \mathring{\lambda} = 0, \tag{32}$$

$$B_{2m-1}[k]*\left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-2}[k]*\left(\mathring{C}^{(1)}[k]\varepsilon_{[0,1]}[k]\right) = 0, \tag{33}$$

$$\sum_{k'=1}^{N}\mathring{C}^{(0)}[k'] = 1. \tag{34}$$

In this case, for $x_k = kh$ the system (15) and (16) takes the form

$$\sum_{k'=1}^{N}\mathring{C}^{(0)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m}} + i\sum_{k'=1}^{N}\mathring{C}^{(1)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-1}} + \mathring{\lambda} = 0, \tag{35}$$

$$i\sum_{k'=1}^{N}\mathring{C}^{(0)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-1}} - \sum_{k'=1}^{N}\mathring{C}^{(1)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-2}} = 0, \tag{36}$$

$$\sum_{k'=1}^{N}\mathring{C}^{(0)}[k'] = 1. \tag{37}$$

So, $\mathring{C}^{(0)}_{k'} = h$, then it is obvious that

$$\sum_{k'=1}^{N}h\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-1}} = \sum_{\beta\neq 0}\frac{h^{2m-1}}{(2\pi\beta)^{2m-1}} = 0.$$

Due to the above, equality (36) takes the form

$$\sum_{k'=1}^{N}\mathring{C}^{(1)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-2}} = 0. \tag{38}$$

It follows from (33) that

$$B_{2m-2}[k]*\left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) = 0. \tag{39}$$

Hence, applying the operator $hD^{(m-1)}[k]$ to both sides of equation (38), we have

$$\mathring{C}^{(1)}[k] - h\sum_{k'=1}^{N}\mathring{C}^{(1)}[k'] = 0.$$

It follows from this that $\mathring{C}^{(1)}[k]$ does not depend on k, i.e., $\mathring{C}^{(1)}[k] = \mathring{C}^{(1)}$. Then, equation (38) takes the form

$$\sum_{k'=1}^{N}\mathring{C}^{(1)}_{k'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-2}} = \mathring{C}^{(1)}\sum_{k'=1}^{N}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta h(k'-k)}}{(2\pi\beta)^{2m-2}} = \mathring{C}^{(1)}\sum_{\beta\neq 0}\frac{h^{2m-3}}{(2\pi\beta)^{2m-2}}.$$

On the other hand, by virtue of (38)

$$\mathring{C}^{(1)}\sum_{\beta\neq 0}\frac{h^{2m-3}}{(2\pi\beta)^{2m-2}} = 0,$$

from here it immediately follows that $\mathring{C}^{(1)} = \mathring{C}^{(1)}[k] = 0$ at $k = 1, 2, ..., N$.

From (35) it follows that $\mathring{\lambda} = -\sum\limits_{\beta\neq 0}\frac{h^{2m}}{(2\pi\beta)^{2m}}$. So the solution to system (32), (33), an (34) is determined by the formulas

$$\mathring{C}^{(0)} = h, \quad \mathring{C}^{(1)} = 0, \quad \lambda = -\sum_{\beta\neq 0}\frac{h^{2m}}{(2\pi\beta)^{2m}}.$$

Let $t = 2$, then the system (21) and (22) has the form

$$B_{2m}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-1}[k] * \left(\mathring{C}^{(1)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-2}[k] * \left(\mathring{C}^{(2)}[k]\varepsilon_{[0,1]}[k]\right) + \mathring{\lambda} = 0, \tag{40}$$

$$B_{2m-1}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-2}[k] * \left(\mathring{C}^{(1)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-3}[k] * \left(\mathring{C}^{(2)}[k]\varepsilon_{[0,1]}[k]\right) = 0, \tag{41}$$

$$B_{2m-2}[k] * \left(\mathring{C}^{(0)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-3}[k] * \left(\mathring{C}^{(1)}[k]\varepsilon_{[0,1]}[k]\right) + B_{2m-4}[k] * \left(\mathring{C}^{(2)}[k]\varepsilon_{[0,1]}[k]\right) = 0, \tag{42}$$

$$\sum_{k'=1}^{N}\mathring{C}^{(0)}[k'] = 1.$$

To find the unknowns, $\mathring{C}^{(2)}[k]$ consider equations (42) as $\mathring{C}^{(0)}[k] = h,\ \mathring{C}^{(1)}[k] = 0$ at $k = 1, 2, ..., N$, then this equation takes the form

$$B_{2m-4}[k] * \left(\mathring{C}^{(2)}[k]\varepsilon_{[0,1]}[k]\right) = -\sum_{\beta\neq 0}\frac{h^{2m-2}}{(2\pi\beta)^{2m-2}}. \tag{43}$$

Applying the operator $hD^{(m-2)}[k]$ to both sides of equation (43) and taking into account (23) and (24), we obtain

$$\mathring{C}^{(2)}[k] - h\sum_{k'=1}^{N}\mathring{C}^{(2)}[k'] = 0. \tag{44}$$

It immediately follows that $\mathring{C}^{(2)}[k] = \mathring{C}^{(2)},\ k = 1, 2, ..., N$.

From (15) and (43) we have

$$-\mathring{C}^{(2)}\sum_{\beta\neq 0}\frac{h^{2m-5}}{(2\pi\beta)^{2m-4}}=-\sum_{\beta\neq 0}\frac{h^{2m-2}}{(2\pi\beta)^{2m-2}}.$$

From here

$$\mathring{C}^{(2)}[k]=\mathring{C}^{(2)}=h^3\frac{B_{2m-2}}{B_{2m-4}},\quad k=1,2,...,N.$$

Substituting the obtained coefficients

$$\mathring{C}^{(2)}[k]=h,\ \mathring{C}^{(1)}[k]=0,\quad \mathring{C}^{(2)}[k]=h^3\frac{B_{2m-2}}{B_{2m-4}}, \tag{45}$$

to (40) and using (15) in the case $\alpha=0$, we find

$$\mathring{\lambda}=h^{2m}\sum_{\beta\neq 0}\left(\frac{1}{(2\pi\beta)^{2m-2}}-\frac{1}{(2\pi\beta)^{2m}}\right).$$

It is easy to see that coefficients (45) also satisfy equations (41).

So we have proven the following theorems.

Theorem 4. *Among quadrature formulas of the form (1) for $t=0$ and $t=1$ the following quadrature formula is optimal in the space $\widetilde{L_2^{(m)}}(0,1)$ for any $m\geq 1$*

$$\int\limits_0^1\varphi(x)dx\cong h\sum_{\beta=0}^{N}\varphi(h\beta). \tag{46}$$

Theorem 5. *Among the quadrature formulas of the form (1) with $t=2$ the following quadrature formula*

$$\int\limits_0^1\varphi(x)dx\cong h\sum_{\beta=0}^{N}\varphi(h\beta)+h^3\frac{B_{2m-2}}{B_{2m-4}}\sum_{\beta=0}^{N}\varphi''(h\beta), \tag{47}$$

is optimal in the space $\widetilde{L_2^{(m)}}(0,1)$ for any $m\geq 1$.

5. NORM OF THE ERROR FUNCTIONAL FOR COMPOSITE OPTIMAL QUADRATURE FORMULAS

The following theorems are true.

Theorem 6. *The norm of the error functional for optimal quadrature formulas of the form (1) at $t=0,1$ has the form*

$$||\mathring{\ell}|\widetilde{L_2^{(m)*}}(0,1)||^2=\frac{h^{2m}|B_{2m}|}{(2m)!}.$$

Theorem 7. *The norm of the error functional for composite quadrature formulas of the form (1) at $t=2$ has the form*

$$||\mathring{\ell}|\widetilde{L_2^{(m)*}}(0,1)||^2=\frac{h^{2m}}{(2m)!}\left(|B_{2m}|-\frac{|B_{2m-2}|^2}{|B_{2m-4}|}\frac{(2m-1)2m}{(2m-2)(2m-3)}\right).$$

The proofs of these theorems follow from Theorems 1, 4, and 5.

6. CONCLUSIONS

Thus, this work is devoted to the construction of composite optimal quadrature formulas of the form (1) in the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$. Here we obtained the following main results:

—To obtain an upper bound for composite quadrature formulas in the Sobolev space of periodic functions, we found an extremal function of the form (7);

—The expression for the norm of the error functional (3) was obtained analytically;

—The optimal coefficients of composite quadrature formulas are found for $t = 0, t = 1$ and $t = 2$ for any $m \geq 1$;

—The norm of optimal error functionals for composite quadrature formulas at $t = 0, t = 1$ and $t = 2$ for any $m \geq 1$ are calculated.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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