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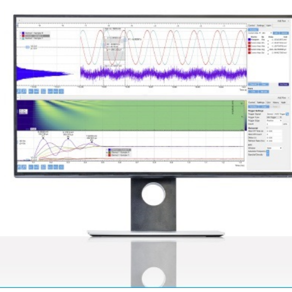

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Optimal Quadrature Formulas with Derivatives in a Periodic Space

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Abstract. The paper is devoted to investigation of optimal formulas for approximate integration with derivatives in the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$ of periodic functions. Here the extremal function for quadrature formulas of Hermite type is obtained. Applying this extremal function the square of the norm for the error functional of the considered Hermite type quadrature formulas is calculated. It is obtained the system of linear algebraic equations minimizing the norm of the error functional by coefficients of the quadrature formulas. Furthermore, here, Ivo Babuška's theorem on zeros of extremal functions for quadrature formulas is generalized.

1. INTRODUCTION

In this paper, the results of integration of periodic functions from [1, 2, 3] are generalized to the case of quadrature formulas that are analogs of Hermite's quadrature formulas.

Recall that, a function $f(x)$ is called periodic if $f(x)=f(x+\beta)$ fulfills for any integer $\beta\in\mathbb{Z}$ and all x. We give the definitions of the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$ of periodic functions.

$\widetilde{L_2^{(m)}}(0,1)$, $m\geq 1$ is the Hilbert space of periodic functions $f(x), -\infty<x<\infty$, differing by a constant, m derivative (in the generalized sense) is square integrable. In this space the inner product is defined as

$$(f,g)_m=\int_0^1 \frac{d^m f}{dx^m}\frac{d^m g}{dx^m}\,\mathrm{d}x.$$

The norm in the space $\widetilde{L_2^{(m)}}(0,1)$ corresponding to the inner product has the form

$$\|\varphi|\widetilde{L_2^{(m)}}\|^2=(\varphi,\varphi)$$

or

$$\|\varphi|\widetilde{L_2^{(m)}}\|^2=\int_0^1\left(\varphi^{(m)}(x)\right)^2\mathrm{d}x.$$

Let φ from the space $\widetilde{L_2^{(m)}}(0,1)$. Integral by segment of the function φ, we going to approximately replace by linear combination of the values of φ and its derivatives $\varphi^{(\alpha)}(x)$ at the nodes $x_k\in[0,1]$, $k=1,2,...,N$, $N\geq m$ and N is a integer number.

Throughout what follows, by a quadrature formula with a derivatives we mean the following approximate equality

$$\int_0^1\varphi(x)\mathrm{d}x\cong\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}\varphi^{(\alpha)}(x_k) \tag{1}$$

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or

$$\int_{-\infty}^{\infty} \varepsilon_{[0,1]}(x)\varphi(x)\mathrm{d}x \cong \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}\varphi^{(\alpha)}(x_k),$$

where $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0,1]$, $t \le m-1$. Here x_k are the nodes and $C_1^{(\alpha)}, C_2^{(\alpha)}, \ldots, C_N^{(\alpha)}$ are the coefficients of the quadrature formula.

The error of the quadrature formula is the difference

$$\begin{aligned}(\ell,\varphi) &= \int_0^1 \varphi(x)\mathrm{d}x - \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}\varphi^{(\alpha)}(x_k) \\ &= \int_{-\infty}^{\infty}\left[\left(\varepsilon_{[0,1]}(x) - \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}(-1)^{\alpha}\delta^{(\alpha)}(x-x_k)\right) * \phi_0(x)\right]\varphi(x)\mathrm{d}x,\end{aligned} \tag{2}$$

where $\delta(x)$ is the Dirac delta-function, $\phi_0(x) = \sum\limits_{\beta=-\infty}^{\infty}\delta(x-\beta)$,

$$\ell(x) = \left(\varepsilon_{[0,1]}(x) - \sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}(-1)^{\alpha}\delta^{(\alpha)}(x-x_k)\right) * \phi_0(x). \tag{3}$$

The equality (2) defines on functions φ additive and homogeneous periodic functional ℓ which is called the error functional of quadrature formulas (1). In the periodic case the error functional of quadrature formulas is defined by the formula (3). The conjugate space $\widetilde{L_2^{(m)*}}(0,1)$ consists of all periodic functionals that are orthogonal to one, i.e.

$$(\ell,1) = 0 \tag{4}$$

Since any periodic algebraic polynomial is a constant. This condition does not depend on m. The orthogonality condition for $\ell(x)$ with one yields the equality

$$\sum_{k=1}^{N} C_k^{(0)} = 1 \tag{5}$$

which we will always consider fulfilled.

The unknown parameters of the quadrature formulas (1) are the nodes x_k and the coefficients $C_k^{(0)}, C_k^{(1)}, \ldots, C_k^{(t)}$, $k = 1,2,\ldots,N$.

A formula for the error of which for a given number of nodes N is the smallest norm in the space $\widetilde{L_2^{(m)*}}(0,1)$ of periodic functionals is called an optimal quadrature formula.

The main results of this work are: Finding the extremal function of a Hermite type quadrature formula, computation the norm of the error functional for the Hermite-type quadrature formula, obtaining a system of algebraic equations for finding the optimal coefficient of quadrature formulas. Proof of general from of a theorem I.Babuška. Similar issues are considered in works (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27])

2. EXTREMAL FUNCTION OF THE HERMITE-TYPE QUADRATURE FORMULA

Thus, we consider the problem of constructing quadrature formulas with derivatives in statement of functional, then to find explicit form for the norm of the error functional $\ell(x)$ in the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$, we will use the notation of an extremal function of this functional introduced by Sobolev [1].

Definition 1. *Function* $\psi_\ell(x)$ *from the space* $\widetilde{L_2^{(m)}}(0,1)$ *is called the extremal function of the given error functional* $\ell(x)$, *if the following holds*

$$(\ell,\psi_\ell)=\int\limits_0^1 \ell(x)\psi_\ell(x)\mathtt{d}x=\|\ell|\widetilde{L_2^{(m)*}}\|\cdot\|\psi_\ell|\widetilde{L_2^{(m)}}\|.$$

It is known, that by definition the space $\widetilde{L_2^{(m)}}(0,1)$ is a Hilbert space, the inner product in it is given by the formula

$$(\varphi,\psi)_m=\int\limits_0^1 \varphi^{(m)}(x)\psi^{(m)}(x)\mathtt{d}x. \tag{6}$$

By the Riesz theorem, our functional $\ell(x)$ is represented the as follows

$$(\ell,\varphi)=(\psi_\ell,\varphi)_m$$

for any $\varphi(x)$ from $\widetilde{L_2^{(m)}}(0,1)$. Here $\psi_\ell(x)$ from $\widetilde{L_2^{(m)}}(0,1)$ is uniquely defined by the functional $\ell(x)$ and it is an extremal for it.

Besides, $\psi_\ell(x)$ is a Riesz element and satisfies the equality

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|=\|\psi_\ell|\widetilde{L_2^{(m)}}(0,1)\|.$$

The function $\psi_\ell(x)$ can be expressed in terms of $\ell(x)$ by solving some differential equation. Really, integrating by parts the expression on the right-hand side of the formula (6) and using the periodicity of functions $\varphi(x)$ and $\psi_\ell(x)$, after some simplifications, we obtain

$$(\ell,\varphi)=(-1)^m\int_0^1 \frac{\mathtt{d}^{2m}\psi_\ell(x)}{\mathtt{d}x^{2m}}\varphi(x)\mathtt{d}x.$$

This immediately implies that the function $\psi_\ell(x)$ is a generalized solution of the equation

$$\frac{\mathtt{d}^{2m}\psi_\ell(x)}{\mathtt{d}x^{2m}}=(-1)^m\ell(x). \tag{7}$$

Now we find an explicit expression for the extremal function of the error functional $\ell(x)$ defined by the formula (3). This will allow us to calculate the norm of the error functional $\ell(x)$ in the conjugate space $\widetilde{L_2^{(m)*}}(0,1)$, i.e. $\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|$.

The following is true.

Theorem 1. *The extremal function of the error functional* $\ell(x)$ *defined by formula (3) in the space* $\widetilde{L_2^{(m)}}(0,1)$ *is all solutions of equation (7) are expressed in the form*

$$\psi_\ell(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-i)^\alpha\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}+d_0. \tag{8}$$

Here $C_k^{(\alpha)}$ *are coefficients of the quadrature formula in the form (1),* d_0 *is some constant,* $i^2=-1$.

Proof. We find periodic solution of equation (7). For this, we use the following known formulas of the Fourier transform [1]

$$
\begin{aligned}
F[\varphi(x)] &= \int_{-\infty}^{\infty} e^{2\pi ipx}\varphi(x)\mathrm{d}x,\\
F^{-1}[\varphi(x)] &= \int_{-\infty}^{\infty} e^{-2\pi ipx}\varphi(x)\mathrm{d}x,\\
F[\delta(x-x_k)] &= \int_{-\infty}^{\infty} e^{2\pi ipx}\delta(x-x_k)\mathrm{d}x = e^{2\pi ipx_k},\\
F[\delta(x)] &= \int_{-\infty}^{\infty} e^{2\pi ipx}\delta(x)\mathrm{d}x = 1,\\
F[1(x)] &= \delta(p),\\
F[\phi_0(x)] &= \phi_0(p),\\
F\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\right] &= (2\pi ip)^{2m},\\
F^{-1}F[\varphi(x)] &= FF^{-1}[\varphi(x)] = \varphi(x),\\
F\left[\delta^{(\alpha)}(x-x_k)\right] &= \int e^{2\pi ipx}\delta^{(\alpha)}(x-x_k)\mathrm{d}x = (-1)^{\alpha}(2\pi ip)^{\alpha}e^{2\pi ipx_k}.
\end{aligned}
$$

Using the formula (3) the equation (7) is reduced to the form

$$
\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\psi_\ell(x) = (-1)^m\left[\varepsilon_{[0,1]}(x) - \sum_{k=1}^{N}\sum_{\alpha=0}^{t}(-1)^{\alpha}C_k^{(\alpha)}\delta^{(\alpha)}(x-x_k)\right]*\phi_0(x). \tag{9}
$$

Applying the Fourier transform to both sides (9), we obtain

$$
F\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\psi_\ell(x)\right] = (-1)^m F\left[\left(\varepsilon_{[0,1]}(x) - \sum_{k=1}^{N}\sum_{\alpha=0}^{t}(-1)^{\alpha}C_k^{(\alpha)}\delta^{(\alpha)}(x-x_k)\right)*\phi_0(x)\right]. \tag{10}
$$

In the formula (10), since the Fourier transform is a linear we apply the Fourier transform to each term separately. Using the Fourier transform of the derivatives, we have

$$
F\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\psi_\ell(x)\right] = (2\pi ip)^{2m}F\left[\psi_\ell(x)\right]. \tag{11}
$$

Calculating the convolution and using the fundamentalness of the region $\Omega_0 = [0,1]$, we obtain

$$
\varepsilon_{\Omega_0}(x)*\phi_0(x) = \sum_{\beta}\varepsilon_{[0,1]}(x-\beta) = 1.
$$

From this it is easy to see that

$$
F\left[\varepsilon_{\Omega_0}(x)*\phi_0(x)\right] = \delta(p). \tag{12}
$$

Now we calculate the Fourier transform of the convolution in (10) with $\alpha = 0$

$$
F\left[\sum_{k=1}^{N}\mathring{C}_k^{(0)}\delta(x-x_k)*\phi_0(x)\right] = \sum_{k=1}^{N}\mathring{C}_k^{(0)}F[\delta(x-x_k)]F[\phi_0(x)]
$$

$$= \sum_{k=1}^{N} \mathring{C}_k^{(0)} e^{2\pi p x_k} \sum_{\beta} \delta(p-\beta) = \sum_{k=1}^{N} \mathring{C}_k^{(0)} e^{2\pi i p x_k} \delta(p) + \sum_{k=1}^{N} \mathring{C}_k^{(0)} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(x-p).$$

Using the property of the delta function, we have

$$e^{2\pi i p x_k} \delta(p) = \delta(p)$$

and the formula (5), we have

$$\sum_{k=1}^{N} \mathring{C}_k^{(0)} e^{2\pi i p x_k} \delta(p) = \delta(p).$$

Then

$$F\left[\sum_{k=1}^{N} \mathring{C}_k^{(0)} \delta(x-x_k) * \phi_0(x)\right] = \delta(p) + \sum_{k=1}^{N} \mathring{C}_k^{(0)} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(x-p). \tag{13}$$

Now we proceed calculation of the following Fourier transform of convolution in (10) for $1 \leq \alpha \leq t$. Since the Fourier transform is a linear, we obtain

$$F\left[\sum_{k=1}^{N} \sum_{\alpha=1}^{t} (-1)^{\alpha} C_k^{(\alpha)} \delta^{(\alpha)}(x-x_k) * \phi_0(x)\right] = \sum_{k=1}^{N} \sum_{\alpha=1}^{t} (-1)^{\alpha} C_k^{(\alpha)} F[\delta^{(\alpha)}(x-x_k)] F[\phi_0(x)]$$

$$= \sum_{k=1}^{N} \sum_{\alpha=1}^{t} (-1)^{\alpha} C_k^{(\alpha)} (-1)^{\alpha} (2\pi i p)^{\alpha} e^{2\pi i p x_k} \phi_0(p) = \sum_{k=1}^{N} \sum_{\alpha=1}^{t} C_k^{(\alpha)} (2\pi i p)^{\alpha} e^{2\pi i p x_k} \sum_{\beta} \delta(p-\beta)$$

$$= \sum_{k=1}^{N} \sum_{\alpha=1}^{t} C_k^{(\alpha)} \left((2\pi i p)^{\alpha} e^{2\pi i p x_k} \delta(p) + (2\pi i p)^{\alpha} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(p-\beta) \right).$$

Since $(2\pi i p)^{\alpha} e^{2\pi i p x_k} \delta(p) = 0$, for $\alpha \geq 1$ we have

$$F\left[\sum_{k=1}^{N} \sum_{\alpha=1}^{t} (-1)^{\alpha} C_k^{(\alpha)} \delta^{(\alpha)}(x-x_k) * \phi_0(x)\right] = \sum_{k=1}^{N} \sum_{\alpha=1}^{t} C_k^{(\alpha)} (2\pi i p)^{\alpha} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(p-\beta). \tag{14}$$

By virtue of (11), (12), (13) and (14) the formula (10) takes the following form

$$(2\pi i p)^{2m} F[\psi_{\ell}(x)]$$

$$= (-1)^m \left[\delta(p) - \delta(p) - \sum_{k=1}^{N} \mathring{C}_k e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(p-\beta) - \sum_{k=1}^{N} \sum_{\alpha=1}^{t} C_k^{\alpha} (2\pi i p)^{\alpha} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(p-\beta)\right]$$

$$= (-1)^{m+1} \sum_{k=1}^{N} \sum_{\alpha=1}^{t} C_k^{\alpha} (2\pi i p)^{\alpha} e^{2\pi i p x_k} \sum_{\beta \neq 0} \delta(p-\beta). \tag{15}$$

This shows that the right-hand side of (15) is equal to zero in the vicinity of the origin.

Therefore, we can divide both sides of (15) by $(2\pi ip)^{2m}$.

This division will be ambiguous, [1] the function $F[\psi_\ell(x)]$ is determined from equation (15) up to a term of the form

$$\mathtt{d}_0\delta(p)+\sum_{\alpha=1}^{2m-1}\mathtt{d}_\alpha D^{(\alpha)}(p),$$

i.e. a linear combination of $\delta(p)$ and $D^{(\alpha)}(p)$. However, the function $F[\psi_\ell(x)]$ must be harrow-shaped, then all terms except $\mathtt{d}_0\delta(p)$ must be omitted. From here we get

$$F[\psi_\ell(x)]=(-1)^{m+1}\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(2\pi ip)^\alpha e^{2\pi ipx_k}\sum_{\beta\neq 0}\frac{\delta(p-\beta)}{(2\pi ip)^{2m}}+\mathtt{d}_0\delta(p). \tag{16}$$

It is not difficult to see that

$$(2\pi ip)^\alpha e^{2\pi ipx_k}\sum_{\beta\neq 0}\frac{\delta(p-\beta)}{(2\pi ip)^{2m}}=\sum_{\beta\neq 0}(2\pi i\beta)^\alpha e^{2\pi i\beta x_k}\frac{\delta(p-\beta)}{(2\pi i\beta)^{2m}}. \tag{17}$$

By virtue of (17) equality (16) takes the form

$$F[\psi_\ell(x)]=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}\sum_{\beta\neq 0}\frac{(2\pi i\beta)^\alpha e^{2\pi i\beta x_k}\delta(p-\beta)}{(2\pi\beta)^{2m}}+\mathtt{d}_0\delta(p).$$

Hence, applying the inverse Fourier transform, we get

$$\psi_\ell(x)=\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}\sum_{\beta\neq 0}\frac{(2\pi i\beta)^\alpha e^{-2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m}}+\mathtt{d}_0. \tag{18}$$

If in (18) we substitute $-\beta$ instead of β, then the value of $\psi_\ell(x)$ does not change, then we have

$$\psi_\ell(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-i)^\alpha\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}+\mathtt{d}_0$$

which shows the validity of Theorem 1.

Now calculating the derivatives of $\psi_\ell(x)$ with respect to x we have

$$\psi_\ell'(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-1)^\alpha(i)^{\alpha+1}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha-1}},$$
$$\psi_\ell''(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-1)^\alpha i^{\alpha+2}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha-2}},$$

or

$$\psi_\ell^{(q)}(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-1)^\alpha i^{\alpha+q}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha-q}},\ \ q=1,2,...,t.$$

Here the series converges for $2m-2t>1$, or $m-t>\frac{1}{2}$. Hence, in our case, $t=0,1,...,m-1$.

3. THE NORM OF THE ERROR FUNCTIONAL FOR QUADRATURE FORMULAS WITH DERIVATIVES

The Hermite-type quadrature formulas with the error functional $\ell(x)$ defined by the formula (3), considered on the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$, can be characterized in two ways

In one side, it is determined by the coefficients $C_k^{(\alpha)},\ k=1,2,...,N;\ \alpha=0,1,...,t$, subject to conditions

$$(\ell(x),1)=0,$$

but on the other side, the extremal function $\psi_\ell(x)$ of the quadrature formula with derivatives, which is obtained as a solution to the equation

$$\frac{\mathrm{d}^{2m}\psi_\ell(x)}{\mathrm{d}x^{2m}}=(-1)^m\ell(x)$$

and can be written as

$$\psi_\ell(x)=-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-1)^\alpha i^\alpha\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}+\mathrm{d}_0. \tag{19}$$

The norm of the error functional $\ell(x)$ of the quadrature formula with derivatives of the form (1) is expressed in terms of the bilinear form of the coefficients of this quadrature formula and the values of the extremal function given by the formula (19)

In fact, since the space $\widetilde{L_2^{(m)}}(0,1)$ is a Hilbert space, the square of the norm of the error functional $\ell(x)$ and the extremal function $\psi_\ell(x)$ are related by the relation

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2=\int_0^1\left(\frac{\mathrm{d}^m\psi_\ell}{\mathrm{d}x^m}\right)^2\mathrm{d}x=(-1)^m\int_0^1\frac{\mathrm{d}^{2m}\psi_\ell}{\mathrm{d}x^{2m}}\psi_\ell(x)\mathrm{d}x,$$

where $\psi_\ell(x)$ is the extremal function of the quadrature formula with derivatives of the form (1).

But $\frac{\mathrm{d}^{2m}\psi_\ell(x)}{\mathrm{d}x^{2m}}=(-1)^m\ell(x)$, and hence

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2=(\ell,\psi_\ell)=\int_0^1\ell(x)\psi_\ell(x)\mathrm{d}x. \tag{20}$$

Hence, using formulas (3), (19), we calculate the square of the norm of the error functional for the quadrature formula

$$\begin{aligned}\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2&=(\ell,\psi_\ell)=\int_0^1\ell(x)\psi_\ell(x)\mathrm{d}x\\&=\int_0^1\left(\left(\varepsilon_{[0,1]}(x)-\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right)*\phi_0(x)\right)\\&\quad\times\left(-\sum_{k=1}^{N}\sum_{\alpha=0}^{t}C_k^{(\alpha)}(-i)^\alpha\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}+\mathrm{d}_0\right)\mathrm{d}x.\end{aligned} \tag{21}$$

Taking into account the condition (4), the integral (21) is calculated over the segment $[0,1]$ and the nodes x_k belong to the segment $[0,1]$, then (21) takes the following form

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2 = \int_0^1 \left(\varepsilon_{[0,1]}(x) - \sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right) \times\left(-\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}(-i)^{\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}\right)\mathrm{d}x.$$

It is not difficult to see that the integral of the function $e^{2\pi i\beta(x-x_k)}$, i.e.

$$\int_0^1 e^{2\pi i\beta(x-x_k)}\mathrm{d}x = 0,\ \ \beta = \pm 1, \pm 2, \dots. \tag{22}$$

By virtue of (22), the square of the norm of the error functional is reduced to the form

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2 = \int_0^1 \left(\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} C_{k'}^{(\alpha')}(-1)^{\alpha'}\delta^{(\alpha')}(x-x_{k'})\right) \times\left(\sum_{k=1}^{N}\sum_{\alpha=0}^{t} C_k^{(\alpha)}(-i)^{\alpha}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x-x_k)}}{(2\pi\beta)^{2m-\alpha}}\right)\mathrm{d}x. \tag{23}$$

Using the property of δ-function, we have

$$\int_0^1 \delta^{(\alpha')}(x-x_{k'})e^{2\pi i\beta(x-x_k)}\mathrm{d}x = (-1)^{\alpha'}(2\pi i\beta)^{\alpha'}e^{2\pi i\beta(x_{k'}-x_k)}.$$

Then equality (23) takes the form

$$\begin{aligned}\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2 &= \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t} C_{k'}^{(\alpha')}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_k^{(\alpha)}(-1)^{\alpha}i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}\\ &= \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_{k'}^{(\alpha')}C_k^{(\alpha)}(-1)^{\alpha}i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}.\end{aligned}$$

So, we have proved the following.

Theorem 2. *The squared norm of the error functional (3) of a quadrature formula with derivatives in the space* $\widetilde{L_2^{(m)*}}(0,1)$ *is written in the form*

$$\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2 = \sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t} C_{k'}^{(\alpha')}C_k^{(\alpha)}(-1)^{\alpha}i^{\alpha+\alpha'}\sum_{\beta\neq 0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}. \tag{24}$$

4. FINDING THE MINIMUM NORM OF THE ERROR FUNCTIONAL FOR QUADRATURE FORMULAS WITH DERIVATIVES

To find the minimum of square of the norm (24), we apply the method of indefinite Lagrange multipliers. To do this, we consider the Lagrange function

$$\Psi(C^{(\alpha)},\lambda)=\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2+2\lambda(\ell,1).$$

Using the formulas (5) and (24), the Lagrange function is reduced to the form

$$\Psi(C^{(\alpha)},\lambda)=\sum_{k'=1}^{N}\sum_{\alpha'=1}^{t}\sum_{k=1}^{N}\sum_{\alpha=1}^{t}C_{k'}^{(\alpha')}C_{k}^{(\alpha)}(-1)^{\alpha}i^{\alpha+\alpha'}\sum_{\beta\neq0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+2\lambda\left(\sum_{k=1}^{N}C_k^{(0)}-1\right).$$

Equating to zero all partial derivatives with respect to $C_k^{(\alpha)}$ and λ, $k=1,2,...,N$, $\alpha=0,1,2,...,t$, from the function $\Psi(C^{(\alpha)},\lambda)$, we have

$$\frac{\partial\Psi(C^{(\alpha)},\lambda}{\partial C_k^{(\alpha)}}=0,\ \ k=1,2,...,N,\ \alpha=0,1,...,t,$$
$$\frac{\partial\Psi(C^{(\alpha)},\lambda}{\partial\lambda}=0.$$

These gives the system of equations

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}C_{k'}^{(\alpha')}(-1)^{\alpha'}i^{\alpha'+\alpha}\sum_{\beta\neq0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+\delta[\alpha]\lambda=0, \tag{25}$$

at $k=1,2,...,N,\ \alpha=0,1,...,t,$

$$\sum_{k'=1}^{N}C_{k'}^{(0)}=1. \tag{26}$$

Here $C_{k'}^{(\alpha')}$ are the coefficients of quadrature formulas with derivatives, λ is a constant $\delta[\alpha]$ is equal to one for $\alpha=0$ and is equal otherwise, i.e.

$$\delta[\alpha]=\begin{cases}1 \text{ if } \alpha=0,\\ 0 \text{ if } \alpha\neq0.\end{cases}$$

The solution of the system (25) and (26) will be denoted by $\mathring{C}_k^{(\alpha)}$, $\mathring{\lambda}$, which represent a stationary point for the Lagrange function $\Psi(C^{(\alpha)},\lambda)$.

Then the system (25) and (26) can be rewritten as

$$\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t}\mathring{C}_{k'}^{(\alpha')}i^{\alpha'+\alpha}\sum_{\beta\neq0}\frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}}+\delta[\alpha]\mathring{\lambda}=0, \tag{27}$$

$k=1,2,...,N,\ \alpha=0,1,...,t,$

$$\sum_{k'=1}^{N}\mathring{C}_{k'}^{(0)}=1. \tag{28}$$

Let us denote by

$$B_{2m}(x) = \sum_{\beta\neq 0} \frac{e^{2\pi i\beta x}}{(2\pi\beta)^{2m}},$$

then

$$B_{2m}(x_{k'} - x_k) = \sum_{\beta\neq 0} \frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m}},$$

$$B_{2m-\alpha'-\alpha}(x) = i^{\alpha+\alpha'} \sum_{\beta\neq 0} \frac{e^{2\pi i\beta x}}{(2\pi\beta)^{2m-\alpha-\alpha'}}.$$

With this notation, the system (27), (28) can be rewritten as

$$\begin{aligned}
&\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} \mathring{C}_{k'}^{(\alpha')}(-1)^{\alpha'}B_{2m-\alpha-\alpha'}(x_{k'}-x_k)+\delta[\alpha]\mathring{\lambda}=0,\\
&\text{at}\quad k=1,2,...,N,\ \alpha=0,1,...,t,\\
&\sum_{k'=1}^{N}\mathring{C}_{k'}^{(0)}=1.
\end{aligned}$$

From the theory of the method of indefinite Lagrange multipliers it follows that $\mathring{C}_k^{(\alpha)}$, $k = 1,2,...,N$, $\alpha = 0,1,...,t$ are the sought values of the optimal coefficients of the quadrature formula (1) with derivatives in the space $\widetilde{L_2^{(m)}}(0,1)$ and they give the conditional minimum to the square of the norm $\|\ell|\widetilde{L_2^{(m)*}}(0,1)\|^2$ of the error functional of this quadrature formula under the condition (5).

The following holds.

Theorem 3. *Let the error functional* (ℓ,φ) *be defined on the Sobolev space* $\widetilde{L_2^{(m)}}(0,1)$*, i.e. its value for a constant is equal to zero and optimal, i.e. among all functionals of the form (3) with a given system of nodes* x_k*, it has the smallest norm in the dual space* $\widetilde{L_2^{(m)*}}(0,1)$*. Then there exists a solution* $\psi_\ell(x)$ *of equation (7) and its derivatives of order* α *reduce to zero at the points* x_k *and belongs to* $\widetilde{L_2^{(m)}}(0,1)$*.*

Proof. Consider now the formula (8) for the general solution of equation (7). Let us choose the constant $d_0 = -\mathring{\lambda}$ in it, where $\mathring{\lambda}$ is the value we found for the Lagrange multiplier. With this choice, for the values of the function $\psi_\ell(x)$ and its derivatives of order α at the points x_k, we have

$$\psi_\ell^{(\alpha)}(x_k) = -\sum_{k'=1}^{N}\sum_{\alpha'=0}^{t} \mathring{C}_{k'}^{(\alpha')}(-1)^{\alpha'} i^{\alpha'+\alpha} \sum_{\beta\neq 0} \frac{e^{2\pi i\beta(x_{k'}-x_k)}}{(2\pi\beta)^{2m-\alpha-\alpha'}} - \delta[\alpha]\mathring{\lambda}.$$

By virtue of the equations (27) and (28), we see that $\psi_\ell^{(\alpha)}(x_k) = 0$ for $k = 1,2,...,N$ and $\alpha = 0,1,2,...,t$.
This proves Theorem 3.

It should be noted that a similar theorem in the case $\alpha = 0$ was proved by I. Babušhka (see [1]).

CONCLUSION

Thus, in the present manuscript we studied the problem on construction of Hermite type optimal quadrature formulas in the Sobolev space $\widetilde{L_2^{(m)}}(0,1)$ of periodic functions. Here the main results are formulated in three theorems. In the first theorem we obtained the extremal function of the considered quadrature formulas. In the second theorem we obtained the square of the norm for the error functional of Hermite type quadrature formulas. The last theorem is devoted to generalization of Ivo Babuška's theorem on zeros of extremal functions.

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