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On an optimal interpolation formula in the space $W_{2,\sigma}^{(1,0)}$

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$W_{2,\sigma}^{(1,0)}$ fazosida optimal interpolyatsion formula
Biz $\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta \cdot \varphi(x_\beta)$ interpolyatsion formulani tahlil qilamiz. So'ngra bu interpolyatsion formulaning xatoligini baholaymiz. ℓ xatolik funksionalining ekstremal funksiyasini topamiz. Shuningdek, $\frac{d^2}{dx^2} - \sigma^2$ differensial operatorning diskret analogi $D(h\beta)$ ni quramiz. Va nihoyat, interpolyatsion formula optimal koeffitsientlarining aniq ko'rinishi topildi.
Kalit so'zlar: Gilbert fazosi; xatolik funksionali; ekstremal funksiya; optimal interpolyatsion formula.

Оптимальная интерполяционная формула в пространстве $W_{2,\sigma}^{(1,0)}$
Рассмотрим интерполяционную формулу $\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta \cdot \varphi(x_\beta)$. Затем оценим погрешность этой интерполяционной формулы. Найдем экстремальную функцию функционала погрешности ℓ. Также, строим дискретный аналог $D(h\beta)$ дифференциального оператора $\frac{d^2}{dx^2} - \sigma^2$. Наконец, мы находим явный вид оптимальных коэффициентов интерполяционной формулы.
Ключевые слова: Гильбертово пространство; функционал погрешности; экстремальная функция; оптимальная интерполяционная формула.

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Introduction

Many works are devoted to the theory of splines and its applications. The first spline functions were constructed from pieces of cubic polynomials. After that, this construction was modified, the degree of polynomials increased, but the idea of their constructions remains permanently. The next essential step in the theory of splines was Holladay's result [10], connecting Schoenberg's cubic splines with the solution of the variational problem on minimum of square of a function norm from the space $L_2^{(2)}$. Further, the Holladay result was generalized by Carl de Boor [8]. Further, a large number of papers appeared, where, depending on specific requirements, the variational functional was modified (see, for example, [1, 2, 3, 4, 5, 7, 14, 18], and for more review see [11]).

The present paper is devoted to a variational method. Here we construct an optimal interpolation formula.

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We consider the following interpolation formula

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta \cdot \varphi(x_\beta) \tag{1}$$

Here C_β and x_β ($\in [0,1]$) are *the coefficients* and *the nodes* of the interpolation formula (1), respectively.

We suppose that functions φ belong to the Hilbert space

$$W_{2,\sigma}^{(1,0)} = \{\varphi : [0,1] \to \mathbb{R} \mid \varphi \text{ is abs. cont. and } \varphi' \in L_2(0,1)\},$$

equipped with the norm

$$\|\varphi\|_{W_{2,\sigma}^{(1,0)}} = \left\{ \int_0^1 (\varphi'(x) + \sigma\varphi(x))^2 dx \right\}^{1/2}, \tag{2}$$

were $\sigma \in \mathbb{R}$ and $\sigma \neq 0$. The inner product of functions φ and ψ in the space $W_{2,\sigma}^{(1,0)}$ is defined as

$$\langle \varphi, \psi \rangle_{W_{2,\sigma}^{(1,0)}} = \int_0^1 (\varphi'(x) + \sigma\varphi(x))(\psi'(x) + \sigma\psi(x))dx.$$

The error of the interpolation formula (1) is the following difference

$$(\ell, \varphi) = \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z)\varphi(x_\beta),$$

which is the value of a functional ℓ at a function φ. The functional ℓ is defined as

$$\ell(z) = \delta(x - z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x - x_\beta) \tag{3}$$

and it is called *the error functional*. Here δ is the Dirac delta-function.

According to the Riesz theorem any linear continuous functional ℓ in a Hilbert space is represented in the form of a inner product. Therefore, in our case, for any function φ from $W_{2,\sigma}^{(1,0)}$ space, we have

$$(\ell, \varphi) = \langle \varphi, \psi_\ell \rangle_{W_{2,\sigma}^{(1,0)}} \tag{4}$$

and

$$\|\ell\|_{W_{2,\sigma}^{(1,0)*}} = \|\psi_\ell\|_{W_{2,\sigma}^{(1,0)}}.$$

Here, ψ_ℓ is the extremal function for the functional ℓ.

It should be noted that according to the Cauchy-Schwarz inequality the absolute value of the error of the formula (1) is estimated by the norm of the error functional as follows

$$|(\ell, \varphi)| \leq \|\ell\|_{W_{2,\sigma}^{(1,0)*}} \cdot \|\varphi\|_{W_{2,\sigma}^{(1,0)}}.$$

It should be noted that the function ψ_ℓ satisfying the equality in the last inequality is called *the extremal function* for the functional ℓ [15].

Consequently, **the problem** of constructing the optimal interpolation formula in the space $W_{2,\sigma}^{(1,0)}$ is finding the following quantity for fixed nodes x_β:

$$\left\| \mathring{\ell} \right\|_{W_{2,\sigma}^{(1,0)}} = \inf_{C_\beta} \sup_{\|\varphi\|_{W_{2,\sigma}^{(1,0)}}=1} |(\ell, \varphi)|.$$

The coefficients satisfying the last equality are called *the optimal coefficients* and they are denoted as $\mathring{C}_\beta$ (if exist). The interpolation formula (1) with coefficients $\mathring{C}_\beta$ is called *the optimal interpolation formula* and $\mathring{\ell}$ is the error functional corresponding to the optimal interpolation formula (1).

The main aim of the present paper is to construct the optimal interpolation formula (1) in $W_{2,\sigma}^{(1,0)}$ space and to find explicit formulas for optimal coefficients. First such a problem was stated and studied by S.L. Sobolev in [16], where the extremal function of the interpolation formula was found in the Sobolev space $W_2^{(m)}$.

The rest of the paper is organized as follows. In Section 2 the extremal function which corresponds to the error functional ℓ is found and with its help representation of the norm of the error functional (3) is calculated. Further, in order to find the minimum of $\|\ell\|^2$ by coefficients C_β the system of linear equations is obtained for the coefficients of optimal interpolation formulas (1) in the space $W_{2,\sigma}^{(1,0)}$; in Section 3 an algorithm for solution of the system is given; in Section 4 the discrete analog $D(h\beta)$ of the differential operator is constructed; in Section 5 the algorithm for finding of coefficients of optimal interpolation formulas (1) is given; Section 6 is devoted to calculation of optimal coefficients using the algorithm which is given in Section 5.

The extremal function and the norm of the error functional

Using the integration by parts for the inner product $\langle \varphi, \psi_\ell \rangle_{W_{2,\sigma}^{(1,0)}}$ we have

$$\begin{aligned}\langle \varphi, \psi_\ell \rangle_{W_{2,\sigma}^{(1,0)}} &= \int_0^1 (\varphi'(x) + \sigma\varphi(x))(\psi_\ell'(x) + \sigma\psi_\ell(x))dx \\ &= \varphi(x)(\psi_\ell'(x) + \sigma\psi_\ell(x))\Big|_0^1 - \int_0^1 (\psi_\ell''(x) - \sigma^2\psi_\ell(x))\varphi(x)dx.\end{aligned}$$

Putting this result to the right-hand side of (4) we get

$$(\ell, \varphi) = \varphi(x)(\psi_\ell'(x) + \sigma\psi_\ell(x))\Big|_0^1 - \int_0^1 (\psi_\ell''(x) - \sigma^2\psi_\ell(x))\varphi(x)dx.$$

Hence for ψ_ℓ we come to the following differential equation

$$\psi_\ell''(x) - \sigma^2\psi_\ell(x) = -\ell(x) \tag{5}$$

with the boundary conditions

$$\psi_\ell'(1) + \sigma\psi_\ell(1) = 0, \quad \psi_\ell'(0) + \sigma\psi_\ell(0) = 0. \tag{6}$$

Now we solve equation (5) with the conditions (6). It is known that the general solution of a non-homogeneous differential equation is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the equation.

Therefore, we first consider the following homogeneous equation corresponding to equation (5)

$$\psi_\ell''(x) - \sigma^2\psi_\ell(x) = 0. \tag{7}$$

The characteristic equation for (7) is $k^2 - \sigma^2 = 0$ and it has the roots $k = \pm\sigma$. Hence the general solution for equation (7) is $d_1 e^{\sigma x} + d_2 e^{-\sigma x}$, where d_1 and d_2 are real numbers.

It is easy to check that a particular solution of equation (5) is

$$-\ell(x) * G(x),$$

where $G(x)$ is a fundamental solution of the operator $\frac{d^2}{dx^2} - \sigma^2$ and it has the form

$$G(x) = \frac{\operatorname{sgn}(x)}{4\sigma}\left(e^{\sigma x} - e^{-\sigma x}\right). \tag{8}$$

Then we get the following general solution of equation (5)

$$\psi_\ell(x) = -\ell(x) * G(x) + d_1 e^{\sigma x} + d_2 e^{-\sigma x}. \tag{9}$$

Hence, using conditions (6) we come to equations

$$\left(\ell, e^{-\sigma x}\right) = 0 \text{ and } d_1 = 0. \tag{10}$$

The first equation in (10) means exactness of the approximation formula (1) for the function $e^{-\sigma x}$, i.e.

$$\sum_{\beta=0}^{N} C_\beta e^{-\sigma x_\beta} = e^{-\sigma x}. \tag{11}$$

Now, taking (10) into account and denoting $d = d_2$, from (9) we have the following

$$\psi_\ell(x) = -\ell(x) * G(x) + de^{-\sigma x}.$$

Thus, the following result has been proved.

Theorem 1. *The solution of the boundary value problem (5)-(6) has the following form*

$$\psi_\ell(x) = -\ell(x) * G(x) + de^{-\sigma x}, \tag{12}$$

where $G(x)$ is defined by (8). Furthermore, ψ_ℓ is the extremal function for the error functional ℓ.

By the Riesz theorem for the square of the norm of the error functional the following equality holds

$$\|\ell\|_{W_{2,\sigma}^{(1,0)*}} = (\ell, \psi_\ell) = \|\ell\|_{W_{2,\sigma}^{(1,0)*}} \cdot \|\psi_\ell\|_{W_{2,\sigma}^{(1,0)}}.$$

Hence, using (12) and (3) for the norm of the error functional we get the following expression

$$\|\ell\|^2_{W_{2,\sigma}^{(1,0)*}} = 2\sum_{\beta=0}^{N} C_\beta G(z - x_\beta) - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G(x_\beta - x_\gamma). \tag{13}$$

Thus, using the last equality we can get an upper bound for the error of the interpolation formulas (1). Further, in order to obtain the optimal interpolation formula of the form (1) we should find the minimum of the expression (13) by coefficients C_β under the condition (10).

For finding the point of the conditional minimum of the expression (13) under the condition (10) we apply the Lagrange method.

Consider the function

$$\Psi(\mathbf{C}, \lambda) = \|\ell\|^2_{W_{2,\sigma}^{(1,0)*}} - 2\lambda(\ell, e^{-\sigma x}),$$

where $\mathbf{C} = (C_0, C_1, ..., C_N)$ and λ is a Lagrange multiplier.

Equating to 0 the partial derivatives of the function Ψ by C_β $(\beta = \overline{0, N})$ and λ, we get the following system of linear equations

$$G(z - x_\beta) - \sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma) - \lambda \cdot e^{-\sigma x_\beta} = 0, \quad \beta = 0, 1, ..., N, \tag{14}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cdot e^{-\sigma x_\gamma} = e^{-\sigma z}, \tag{15}$$

where $G(x)$ is defined by equality (8).

We have obtained the linear system of $N + 2$ unknowns with $N + 2$ equations.

Next, we solve the system (14)-(15).

An algorithm for solution of the system (14)-(15)

Below mainly is used the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [15, 17]. For completeness we give some definitions about functions of discrete argument.

Assume that the nodes x_β are equal spaced, i.e. $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$

Definition 1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 2. *The inner product* of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right-hand side converges absolutely.

Definition 3. *The convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

Now we turn on to our problem.

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$, $x_\beta = h\beta, h = \frac{1}{N}$. We rewrite the system (14)-(15) in the convolution form

$$C_\beta * G(h\beta) + \lambda e^{-\sigma h\beta} = G(z - h\beta),\ \beta = 1, 2, \ldots, N, \tag{16}$$

$$\sum_{\beta=0}^{N} C_\beta \cdot e^{-\sigma h\beta} = e^{-\sigma z}. \tag{17}$$

We have the following

Problem 1. *Find the discrete function C_β and λ which satisfy the system (16)-(17).*

It should be noted that if we solve Problem 1 we get the optimal coefficients $\mathring{C}_\beta$, $\beta = 0, 1, ..., N$. We do not solve the system (16)-(17) by a direct method. Instead we give an algorithm which is used a discrete analog of the differential operator $\frac{d^2}{dx^2} - \sigma^2$. This algorithm allows us to get the explicit formulas for coefficients of the optimal interpolation formula (1).

Further we explain the algorithm.

We consider the following two functions

$$v(h\beta) = G(h\beta) * C_\beta \tag{18}$$

and

$$u(h\beta) = v(h\beta) + \lambda \cdot e^{-\sigma h\beta}. \tag{19}$$

Now we should express the coefficients C_β by the function $u(h\beta)$. For this we use the discrete analog $D(h\beta)$ of the operator $\frac{d^2}{dx^2} - \sigma^2$. The discrete argument function $D(h\beta)$ satisfies the following equation

$$D(h\beta) * G(h\beta) = \delta_d(h\beta), \tag{20}$$

where $\delta_d(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e. $\delta_d(h\beta)$ is the discrete delta-function.

Then for the optimal coefficients C_β we have

$$C_\beta = D(h\beta) * u(h\beta). \tag{21}$$

Thus, if we find the function $u(h\beta)$ then the optimal coefficients $\mathring{C}_\beta$ will be found from equality (21). In order to calculate the convolution in (21) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (16) we get that $u(h\beta) = G(z - h\beta)$ for $h\beta \in [0, 1]$. Now we find the representation of the function $u(h\beta)$ for $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0, 1]$ then we get

$$C_\beta = D(h\beta) * u(h\beta) = 0\ ,\ h\beta \notin [0, 1]. \tag{22}$$

Further, we calculate the convolution $v(h\beta) = G(h\beta) * C_\beta$ for $h\beta \notin [0, 1]$.

Suppose $\beta \leq 0$ then taking into account equalities (8) and (15), we have

$$\begin{aligned} v(h\beta) &= C_\beta * G(h\beta) = \sum_{\gamma=-\infty}^{\infty} C_\gamma \cdot G(h\beta - h\gamma) \\ &= \sum_{\gamma=0}^{N} C_\gamma \cdot \frac{\operatorname{sgn}(h\beta - h\gamma)}{4\sigma} \left(e^{\sigma(h\beta - h\gamma)} - e^{-\sigma(h\beta - h\gamma)}\right) \\ &= -\frac{1}{4\sigma} \cdot e^{\sigma h\beta} \cdot e^{-\sigma z} + e^{-\sigma h\beta} \cdot T, \end{aligned}$$

where

$$T = \frac{1}{4\sigma} \cdot \sum_{\gamma=0}^{N} C_\gamma \cdot e^{\sigma h \gamma}.$$

So, for $\beta \leq 0$ we get

$$v(h\beta) = -\frac{e^{\sigma h\beta} \cdot e^{-\sigma z}}{4\sigma} + T \cdot e^{-\sigma h\beta}. \tag{23}$$

Similarly, in the case $\beta \geq N$ for the convolution $v(h\beta) = G(h\beta) * C_\beta(z)$ we obtain

$$v(h\beta) = \frac{e^{\sigma h\beta} \cdot e^{-\sigma z}}{4\sigma} - T \cdot e^{-\sigma h\beta}. \tag{24}$$

From (23) and (24) equalities,

$$v(h\beta) = \begin{cases} -\frac{e^{\sigma h\beta} \cdot e^{-\sigma z}}{4\sigma} + T \cdot e^{-\sigma h\beta}, & \beta \leq 0, \\ \frac{e^{\sigma h\beta} \cdot e^{-\sigma z}}{4\sigma} - T \cdot e^{-\sigma h\beta}, & \beta \geq N. \end{cases} \tag{25}$$

We introduce the following denotations

$$a^- = T + \lambda, \tag{26}$$

$$a^+ = \lambda - T. \tag{27}$$

Then taking into account (16), (19), (25),(26),(27) we have the following problem.

Problem 2. *Find the solution of the equation*

$$D(h\beta) * u(h\beta) = 0, \; h\beta \notin [0, 1],$$

which has the form

$$u(h\beta) = \begin{cases} -\frac{e^{\sigma(h\beta - z)}}{4\sigma} + a^- e^{-\sigma h\beta}, & \beta \leq 0, \\ G(z - h\beta), & 0 \leq \beta \leq N, \\ \frac{e^{\sigma(h\beta - z)}}{4\sigma} + a^+ e^{-\sigma h\beta}, & \beta \geq N. \end{cases} . \tag{28}$$

Here a^- and a^+ are unknowns. If we find a^- and a^+ then from (26) and (27) we have

$$\lambda = \frac{1}{2}(a^+ + a^-) \tag{29}$$

and

$$T = \frac{1}{2}(a^- - a^+). \tag{30}$$

Unknowns a^- and a^+ we will find from equation (22), using the discrete argument function $D(h\beta)$.

If the function $D(h\beta)$ is known, we can find the explicit form of the function $u(h\beta)$ and we have the optimal coefficients. Therefore, Problem 2 and Problem 1 will be solved, respectively. In the next section we will construct the discrete analog $D(h\beta)$ of the differential operator $\frac{d^2}{dx^2} - \sigma^2$.

Construction of the discrete analog of the differential operator $\frac{d^2}{dx^2} - \sigma^2$

Let us examine the following equation

$$D(h\beta) * G(h\beta) = \delta_d(h\beta), \tag{31}$$

where

$$G(h\beta) = \frac{\text{sgn}(h\beta)}{4\sigma}(e^{\sigma h\beta} - e^{-\sigma h\beta}), \tag{32}$$

where $\delta_d(h\beta)$ is the discrete delta-function.

The following holds.

Now, we should calculate $F[G(x)]$, for this we use the following equation

$$(\delta''(x)-\sigma^2\delta(x))*G(x)=\delta(x).$$

From here, taking into account $F[\delta^{\alpha}(x)]=(-2\pi ip)^{\alpha}$, we have

$$F[G](p)=\frac{1}{(2\pi ip)^2-\sigma^2}. \tag{40}$$

Putting (40) in (39) and after some calculation we get

$$F[\overset{\sqcap}{G}](p)=-\frac{h}{4\pi^2}\sum_{\beta}\frac{1}{(\beta-h(p+\frac{\sigma i}{2\pi}))(\beta-h(p-\frac{\sigma i}{2\pi}))}. \tag{41}$$

From (35), taking into account(41), we have

$$F[\overset{\sqcap}{D}](p)=\left[-\frac{h}{4\pi^2}\sum_{\beta}\frac{1}{(\beta-h(p+\frac{\sigma i}{2\pi}))(\beta-h(p-\frac{\sigma i}{2\pi}))}\right]^{-1}.$$

The function $F[\overset{\sqcap}{D}](p)$ can be described as a Fourier series

$$F[\overset{\sqcap}{D}](p)=\sum_{\beta=-\infty}^{\infty}\hat{D}(h\beta)e^{2\pi iph\beta}. \tag{42}$$

Here $\hat{D}(h\beta)$ is the Fourier coefficients of $F[\overset{\sqcap}{D}](p)$ and

$$\hat{D}(h\beta)=\int_0^{h^{-1}}F[\overset{\sqcap}{D}](p)e^{-2\pi iph\beta}dp. \tag{43}$$

Applying the inverse Fourier transform to the equality (42) to the formula, we get the harrow-shaped function

$$\overset{\sqcap}{D}(x)=\sum_{\beta=-\infty}^{\infty}\hat{D}(h\beta)\delta(x-h\beta).$$

Thus, by the definition of harrow-shaped function $\hat{D}(h\beta)$ is the desired function of the discrete argument $D(h\beta)$ or a discrete analogue of the differential operator $\frac{d^2}{dx^2}-\sigma^2$. To find the function $\overset{\sqcap}{D}(x)$, the calculation of the integral (43) is impractical, we find it in the following way. First we calculate an infinite series in (42). We denote

$$S=\sum_{\beta=-\infty}^{\infty}\frac{1}{(\beta-h(p+\frac{\sigma i}{2\pi}))(\beta-h(p-\frac{\sigma i}{2\pi}))}.$$

To calculate the infinite series S we use the following well-known formula from the theory of residues, if function f has the poles $z_1,z_2,\dots,z_n$, then

$$\sum_{\beta=-\infty}^{\infty}f(\beta)=-\sum_{z_1,z_2,\dots,z_n}\operatorname{res}(\pi\cot\pi z\cdot f(z)).$$

We consider the function

$$f(z)=\frac{1}{(z-h(p+\frac{\sigma i}{2\pi}))(z-h(p-\frac{\sigma i}{2\pi}))},$$

here $z_1=h(p+\frac{\sigma i}{2\pi})$ and $z_2=h(p-\frac{\sigma i}{2\pi})$ are poles of order 1. Then

$$S=\sum_{\beta=-\infty}^{\infty}f(\beta)=-\sum_{z_1,z_2}\operatorname{res}(\pi\cot\pi z\cdot f(z)). \tag{44}$$

Theorem 2. *The discrete analog of the differential operator* $\frac{d^2}{dx^2} - \sigma^2$ *satisfying the equation (31) has the form*

$$D(h\beta) = \frac{2\sigma}{1 - e^{2\sigma h}} \begin{cases} 0, & |\beta| \geq 2, \\ -e^{\sigma h}, & |\beta| = 1, \\ 1 + e^{2\sigma h}, & \beta = 0. \end{cases} \tag{33}$$

Proof. It is known from the theory of generalized functions and the Fourier transform, we use a harrow-shaped function instead of $D(h\beta)$. We consider the following equation

$$\overset{\sqcap}{D}(x) * \overset{\sqcap}{G}(x) = \delta(x), \tag{34}$$

where $\overset{\sqcap}{D}(x) = \sum_{\beta=-\infty}^{\infty} D(h\beta)\delta(x - h\beta)$ and $\overset{\sqcap}{G}(x) = \sum_{\beta=-\infty}^{\infty} G(h\beta)\delta(x - h\beta)$ are harrow-shaped functions corresponding to the discrete argument functions $D(h\beta)$ and $G(h\beta)$ [15].

It is known [15] that the class of harrow-shaped functions and the class of functions of the discrete argument are isomorphic. Therefore, instead of the function $D(h\beta)$, it suffices to investigate the function $\overset{\sqcap}{D}(x)$. Applying the Fourier transform to both sides of equality (34) and noting that

$$F[\varphi(x) * \psi(x)] = F[\varphi] \cdot F[\psi] \ , \ F[\delta(x)] = 1,$$

we have

$$F[\overset{\sqcap}{D}(x)] \cdot F[\overset{\sqcap}{G}(x)] = 1,$$

or

$$F[\overset{\sqcap}{D}(x)] = \frac{1}{F[\overset{\sqcap}{G}(x)]}. \tag{35}$$

Now, we calculate the function $F[\overset{\sqcap}{G}(x)]$ the Fourier transform of the function $\overset{\sqcap}{G}(x)$. The following are known [15],

$$\Phi_0(x) = \sum_{\beta=-\infty}^{\infty} \delta(x - \beta) \ , \ \delta(hx) = h^{-1}\delta(x) \ , \ \sum_{\beta} e^{2\pi i x\beta} = \sum_{\beta} \delta(x - \beta). \tag{36}$$

Using formulas from (36) we get

$$\begin{aligned} \overset{\sqcap}{G}(x) &= \sum_{\beta=-\infty}^{\infty} G(h\beta) \cdot \delta(x - h\beta) \\ &= G(x) \sum_{\beta=-\infty}^{\infty} \delta(x - h\beta) = h^{-1}G(x) \sum_{\beta=-\infty}^{\infty} \delta(h^{-1}x - \beta) \\ &= h^{-1}G(x) \cdot \Phi_0(h^{-1}x). \end{aligned} \tag{37}$$

For the Fourier transform of the function $\Phi_0(h^{-1}x)$ we have

$$\begin{aligned} F[\Phi_0(h^{-1}x)] &= \sum_{\beta=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(h^{-1}x - \beta)e^{2\pi ipx}dx = h \sum_{\beta=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - h\beta)e^{2\pi ipx}dx \\ &= h \sum_{\beta=-\infty}^{\infty} e^{2\pi iph\beta} = h \sum_{\beta=-\infty}^{\infty} \delta(hp - \beta) = h\Phi_0(hp). \end{aligned}$$

Thus,

$$F[\Phi_0(h^{-1}x)] = h\Phi_0(hp). \tag{38}$$

Taking into account (37) and (38) we obtain

$$F[\overset{\sqcap}{G}(x)] = F[h^{-1}G(x)\Phi_0(h^{-1}x)] = h^{-1}F[G(x)] * h\Phi_0(hp) = F[G](p) * \Phi_0(hp). \tag{39}$$

We obtain the following

$$\operatorname*{res}_{z_1=h(p+\frac{\sigma i}{2\pi})}\frac{\pi\cot\pi z}{(z-h(p+\frac{\sigma i}{2\pi}))(z-h(p-\frac{\sigma i}{2\pi}))}=\frac{\pi^2}{h\sigma i}\cot\frac{h(2\pi p+\sigma i)}{2}$$

and

$$\operatorname*{res}_{z_2=h(p-\frac{\sigma i}{2\pi})}\frac{\pi\cot\pi z}{(z-h(p+\frac{\sigma i}{2\pi}))(z-h(p-\frac{\sigma i}{2\pi}))}=-\frac{\pi^2}{h\sigma i}\cot\frac{h(2\pi p-\sigma i)}{2}. \tag{45}$$

Putting the last two equalities in equation (44) we have

$$S=\frac{\pi^2}{h\sigma i}\cdot\frac{\sin h\sigma i}{\sin(\frac{h(2\pi p+\sigma i)}{2})\sin(\frac{h(2\pi p-\sigma i)}{2})}.$$

We denote $\lambda=e^{2\pi iph}$ then we get

$$\sin(\frac{h(2\pi p+\sigma i)}{2})=\frac{\lambda-e^{\sigma h}}{2i\lambda^{\frac{1}{2}}e^{\frac{\sigma h}{2}}}, \tag{46}$$

$$\sin(\frac{h(2\pi p-\sigma i)}{2})=\frac{\lambda e^{\sigma h}-1}{2i\lambda^{\frac{1}{2}}e^{\frac{\sigma h}{2}}}, \tag{47}$$

$$\sin\sigma hi=\frac{e^{-\sigma h}-e^{\sigma h}}{2i}=\frac{1-e^{2\sigma h}}{2ie^{\sigma h}}. \tag{48}$$

Using equalities (46) ,(47) and (48) we have

$$S=-\frac{\pi^2}{h\sigma}\cdot\frac{2\lambda(e^{2\sigma h}-1)}{\lambda^2e^{\sigma h}-\lambda(1+e^{2\sigma h})+e^{\sigma h}}.$$

We put the expression obtained for S in (41) then

$$F[\overset{\sqcap}{G}](p)=\frac{\lambda(e^{2\sigma h}-1)}{2(\lambda^2e^{\sigma h}-(1+e^{2\sigma h})\cdot\lambda+e^{\sigma h})\sigma}.$$

Hence, taking into account (35) we get

$$F[\overset{\sqcap}{D}]=\frac{2(\lambda^2e^{\sigma h}-(1+e^{\sigma h})\lambda+e^{\sigma h})\sigma}{(e^{\sigma h}-1)\lambda}=\frac{2\sigma}{e^{2\sigma h}-1}\cdot(e^{\sigma h}\lambda-(1+e^{2\sigma h})\cdot\lambda^0+\lambda^{-1}e^{\sigma h}).$$

Hence we come

$$D(h\beta)=\frac{2\sigma}{e^{2\sigma h}-1}\begin{cases}0 & |\beta|\geq 2,\\ e^{\sigma h} & |\beta|=1,\\ -(1+e^{2\sigma h}), & \beta=0.\end{cases} \tag{49}$$

Theorem is proved.

□

Theorem 3. *The discreet analog of the differential operator* $\frac{d^2}{dx^2}-\sigma^2$ *satisfies the following equalities* 1. $D(h\beta)*e^{-\sigma h\beta}=0$, 2. $D(h\beta)*e^{\sigma h\beta}=0$.

Proof. By direct calculations we have the following

$$\begin{aligned}D(h\beta)*e^{-\sigma h\beta} &= \sum_{\gamma=-\infty}^{\infty}D(h\gamma)\cdot e^{-\sigma h(\beta-\gamma)}\\ &= e^{-\sigma h\beta}(D(-h)e^{-\sigma h}+D(0)e^0+d(h)e^{\sigma h})\\ &= \frac{2\sigma}{1-e^{2\sigma h}}\cdot e^{-\sigma h\beta}(-e^{-\sigma h}\cdot e^{\sigma h}+1+e^{2\sigma h}-e^{\sigma h}\cdot e^{\sigma h})=0\end{aligned}$$

and

$$\begin{aligned}D(h\beta)*e^{\sigma h\beta} &= \sum_{\gamma=-\infty}^{+\infty}D(h\gamma)\cdot e^{\sigma h(\beta-\gamma)}=e^{\sigma h\beta}\cdot\sum_{\gamma=-\infty}^{+\infty}D(h\gamma)\cdot e^{-\sigma h\gamma}\\ &= e^{\sigma h\beta}\cdot(D(-h)e^{\sigma h}+D(0)e^0+D(h)\cdot e^{-\sigma h})\\ &= \frac{e^{\sigma h\beta}}{1-e^{2\sigma h}}(-2\sigma e^{\sigma h}\cdot e^{\sigma h}+2\sigma(1+e^{2\sigma h}-2\sigma e^{\sigma h}\cdot e^{-\sigma h}))=0.\end{aligned}$$

Theorem is proved.

□

Optimal coefficients in the space $W_{2,\sigma}^{(1,0)}$

In this section for optimal coefficients by direct calculation of (21) we have the following.

Theorem 4. *The coefficients of the optimal approximation formula (1) in $W_{2,\sigma}^{(1,0)}$ space are as follows:*

$$\begin{aligned}\mathring{C}_\beta &= \frac{1}{2(1-e^{2\sigma h})}\Big[-\operatorname{sgn}(z-h\beta-h)\cdot\left(e^{\sigma(z-h\beta)-e^{-\sigma(z-h\beta-2h)}}\right)\\ &+\operatorname{sgn}(z-h\beta+h)\cdot\left(e^{-\sigma(z-h\beta)-e^{\sigma(z-h\beta+2h)}}\right)\\ &+\operatorname{sgn}(z-h\beta)(1+e^{2\sigma h})\cdot\left(e^{\sigma(z-h\beta)}-e^{-\sigma(z-h\beta)}\right)\Big],\ \beta=0,...,N.\end{aligned}$$

Proof. We consider the function $u(h\beta)$:

$$u(h\beta)=\begin{cases}-\frac{e^{\sigma(h\beta-z)}}{4\sigma}+a^-e^{-\sigma h\beta}, & \beta\le 0,\\ G(z-h\beta), & 0\le\beta\le N,\\ \frac{e^{\sigma(h\beta-z)}}{4\sigma}+a^+e^{-\sigma h\beta}, & \beta\le N.\end{cases}$$

From here we can find unknowns a^- and a^+ as follows.
For $\beta=0$ we have

$$-\frac{e^{-\sigma z}}{4\sigma}+a^-=\frac{1}{4\sigma}(e^{\sigma x}-e^{-\sigma x}).$$

Hence

$$a^-=\frac{e^{\sigma z}}{4\sigma}.$$

Similarly, for $\beta=N$, we get

$$\frac{e^{\sigma}e^{-\sigma z}}{4\sigma}+a^+e^{-\sigma}=-\frac{1}{4\sigma}\cdot e^{\sigma z}\cdot e^{-\sigma}+\frac{e^{\sigma}e^{-\sigma z}}{4\sigma}.$$

Whence

$$a^+=-\frac{e^{\sigma z}}{4\sigma}.$$

Thus the function $u(h\beta)$ is fully determined

$$u(h\beta)=\begin{cases}-\frac{e^{\sigma(h\beta-z)}-e^{-\sigma(h\beta-z)}}{4\sigma}, & \beta\le 0,\\ G(z-h\beta), & 0\le\beta\le N,\\ \frac{e^{\sigma(h\beta-z)}-e^{-\sigma(h\beta-z)}}{4\sigma}, & \beta\ge N.\end{cases}$$

Now we calculate the optimal coefficients C_β using the formula (21).

$$C_\beta=D(h\beta)*u(h\beta)=\sum_{\gamma=-\infty}^{\infty}D(h\gamma)\cdot u(h\beta-h\gamma).$$

Hence, using the discrete argument functions $D(h\beta)$ and $u(h\beta)$, for $\beta = 0, 1, ..., N$ we get

$$\begin{aligned} C_\beta &= D(h\beta) * u(h\beta) \\ &= D(h)\Big[u(h\beta+h)+u(h\beta-h)\Big]+D(0)\cdot u(h\beta) \\ &= \frac{-2\sigma}{1-e^{2\sigma h}}\cdot e^{\sigma h}\cdot\Bigg[\frac{\mathrm{sgn}(z-h\beta-h)}{4\sigma}\left(e^{\sigma(z-h\beta-h)-e^{-\sigma(z-h\beta-h)}}\right) \\ &\quad+\frac{\mathrm{sgn}(z-h\beta+h)}{4\sigma}\cdot\left(e^{\sigma(z-h\beta+h)}-e^{-\sigma(z-h\beta+h)}\right) \\ &\quad-(1+e^{2\sigma h})(\frac{\mathrm{sgn}(z-h\beta)(e^{\sigma(z-h\beta)}-e^{-\sigma(z-h\beta)})}{4\sigma})\Bigg]. \end{aligned}$$

Hence we get the statement of the theorem. Theorem 4 is proved.

□

Calculation of the norm of the error functional

Firstly, solving the system (29)-(30), using Theorem 2, taking into account a^- and a^+ we have

$$\lambda = \frac{1}{2}(a^- + a^+) = 0,$$

Then from (14) we obtain

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma) = G(z - x_\beta),\ \ \beta = 0, 1, \ldots, N.$$

Consequently, for the error functional $\mathring{\ell}$ from (13) we get

$$\begin{aligned} \|\mathring{\ell}\|^2_{W_{2,\sigma}^{(1,0)*}} &= 2\sum_{\beta=0}^{N} C_\beta G(z-x_\beta) - \sum_{\gamma=0}^{N}\sum_{\beta=0}^{N} C_\beta C_\gamma G(x_\beta - x_\gamma) \\ &= \sum_{\beta=0}^{N} C_\beta\left(G(z-x_\beta) - \sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma)\right) + \sum_{\beta}^{N} C_\beta G(z-x_\beta). \end{aligned}$$

Thus, we have

$$\|\mathring{\ell}\|^2_{W_{2,\sigma}^{(1,0)*}} = \sum_{\beta=0}^{N} \mathring{C}_\beta G(z - x_\beta).$$

Conclusion

In this work for construction of optimal approximation formula (1) and estimating it's error we have done the following:

- the extremal function was found;
- the discrete analog of the differential operator $\frac{d^2}{dx^2} - \sigma^2$ was constructed;
- the optimal coefficients were found;
- The error functional of the optimal approximation formula (1) was calculated.

Remark. If we get $\sigma = 1$, we have some results of the works [6, 9].

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