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## PARTIAL DIFFERENTIAL EQUATIONS

# Inverse Problem of Determining an Unknown Coefficient in the Beam Vibration Equation

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**Abstract**—We consider the direct initial–boundary value problem for the equation of transverse vibrations of a homogeneous beam freely supported at the ends and study the inverse problem of determining the time-dependent beam stiffness coefficient. With the help of the eigenvalues and eigenfunctions of the beam vibration operator, the problems are reduced to integral equations. The Schauder contraction principle is applied to these equations, and theorems on the existence and uniqueness of solutions are proved.

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## INTRODUCTION

Beams are widely used in the construction of buildings, bridges, overpasses, and other structures. Most of the bridges currently under construction are girder bridges. This type of structure is the main one in the construction of short crossings. Beams used in industrial buildings mainly work in static bending, but when any equipment (machines, compressors, piston engines, etc.) is installed on them, they also experience dynamic loads of periodic nature. Under such loads, the beams also perform transverse vibrations [1, 2].

Inverse problems of mathematical physics have been studied for many classes of differential equations. Inverse problems related to the simplest hyperbolic type equation were explored in the monograph [3]. For the solutions of inverse dynamic problems, methods for proving local existence and uniqueness theorems and uniqueness and conditional stability theorems, as well as numerical approaches to finding the solutions, were considered in the papers [4–13] and elsewhere.

Over the past few years, there has been growing interest in the study of linear and nonlinear initial–boundary value problems for the beam vibration equation [14–17]. The initial–boundary problem for the equation of forced vibrations of a cantilevered beam was studied in [18]. Some initial–boundary value problems and the Cauchy problem were studied for the inhomogeneous beam vibration equation in [19–21], where solutions in the form of series were constructed and uniqueness, existence, and stability theorems for the solutions of these problems were proved. An analytical solution of the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain was found in [22] for any kind of boundary conditions.

Inverse problems of finding the right-hand side (vibration source) and initial conditions for the beam vibration equation were studied in [23]. The present paper considers the inverse problem of determining a time-dependent coefficient in the beam transverse vibration equation. This coefficient represents the beam stiffness from the physical viewpoint.

## 1. STATEMENT OF THE PROBLEM

Consider a freely supported homogeneous beam of length $l$ with constant cross-section. Its forced transverse bending vibrations under the action of an external force $G(x,t)$ are described by the fourth-order equation

$$\rho S u_{tt} + EJ u_{xxxx} + Q(t)u = G(x,t),$$

where $\rho$ is the beam density, $S$ is the cross-section area, $E$ is the modulus of elasticity of the beam material, and $J$ is the moment of inertia of the cross-section about the horizontal axis; the entire length of the beam is supported by an elastic base with stiffness coefficient $Q(t)$.

Dividing by $\rho S$, we write this equation in the form

$$u_{tt} + a^2 u_{xxxx} + q(t)u = f(x,t), \tag{1}$$

where $a^2 = EJ/\rho S$, $q(t) = Q(t)/\rho S$, and $f(x,t) = G(x,t)/\rho S$. We consider Eq. (1) in the rectangular domain $D = \{(x,t) : 0 < x < l,\ 0 < t < T\}$, where $[0,T]$ is the time interval, and $l$ is the beam length, with the initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in [0,l], \tag{2}$$

and the boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \quad 0 \le t \le T. \tag{3}$$

In the direct problem, for given numbers $a$, $l$, and $T$ and sufficiently smooth functions $q(t)$, $f(x,t)$, and $\varphi(x)$, $\psi(x)$, it is required to find a function $u(x,t) \in C^{4,2}(D)$ satisfying Eq. (1) for $(x,t) \in D$ and conditions (2) and (3).

**Inverse problem.** Find the coefficient $q(t)$ if the following additional information about the solution of the direct problem (1)–(3) is available:

$$g(t) = \int_0^l u(x,t)h(x)\,dx, \quad t \in [0,T], \tag{4}$$

where the functions $g(t)$ and $h(x)$ are given and the function $h(x)$ satisfies the conditions

$$h(x) \in C^4(0,l), \quad h(0) = h(l) = h''(0) = h''(l) = 0. \tag{5}$$

## 2. STUDY OF THE DIRECT PROBLEM

Let us transpose the term $q(t)u$ in Eq. (1) to the right-hand side and introduce the notation $F(x,t) = f(x,t) - q(t)u$. Then the solution of this equation with the initial conditions (2) and the boundary conditions (3) satisfies the relation [20]

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left( \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \sin(\mu_n x) \\ &\quad + \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t F_n(s) \sin\big(\omega_n (t-s)\big)\,ds\, \sin(\mu_n x), \end{aligned} \tag{6}$$

where $\omega_n = a{\mu_n}^2$, $\mu_n = \pi n/l$, $\lambda_n = -{\mu_n}^4 = -(\pi n/l)^4$, and

$$\begin{aligned} \varphi_n &= \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \sin(\mu_n x)\,dx, \\ \psi_n &= \sqrt{\frac{2}{l}} \int_0^l \psi(x) \sin(\mu_n x)\,dx, \\ F_n(t) &= \sqrt{\frac{2}{l}} \int_0^l F(x,t) \sin(\mu_n x)\,dx. \end{aligned} \tag{7}$$

Substituting the expression $f(x,t)-q(t)u(x,t)$ for $F(x,t)$, we write the representation (6) in the form of the integral equation

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left(\varphi_n\cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\sin(\mu_n x) \\ &+\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int\limits_0^t\int\limits_0^l f(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x) \\ &-\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int\limits_0^t q(s)\int\limits_0^l u(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x). \end{aligned} \tag{8}$$

Let us study the properties of the solution of Eq. (8). To this end, we use the successive approximation method and represent the solution of this equation in the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t); \tag{9}$$

here

$$\begin{aligned} u_0(x,t) &= \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left(\varphi_n\cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\sin(\mu_n x) \\ &\qquad+\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int\limits_0^t f_n\sin\big(\omega_n(t-s)\big)\,ds\,\sin(\mu_n x), \\ f_n(t) &= \sqrt{\frac{2}{l}}\int\limits_0^l f(\xi,t)\sin(\mu_n\xi)\,d\xi, \\ u_n(x,t) &= -\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int\limits_0^t q(s)\int\limits_0^l u_{n-1}(\xi,s)\sin(\omega_n(t-s)) \\ &\qquad\times\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x),\quad n=1,2,\ldots \end{aligned} \tag{10}$$

Estimating $u_n$ in the domain $D$, we obtain

$$\begin{aligned} |u_0| &\le \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left[|\varphi_n|+\frac{1}{a}\left(\frac{l}{\pi}\right)^2\frac{|\psi_n|}{n^2}\right]+\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{1}{a}\left(\frac{l}{\pi}\right)^4\frac{|f_n|t}{n^4} \\ &\le C_1^0\sum_{n=1}^{\infty}|\varphi_n|+C_2^0\sum_{n=1}^{\infty}\frac{|\psi_n|}{n^2}+C_3^0\sum_{n=1}^{\infty}\frac{|f_n|t}{n^4}, \\ |u_1| &\le C_1^1\sum_{n=1}^{\infty}\frac{q_0}{n^4}\left[C_1^0\sum_{n=1}^{\infty}|\varphi_n|t+C_2^0\sum_{n=1}^{\infty}\frac{|\psi_n|t}{n^2}+C_3^0\sum_{n=1}^{\infty}\frac{|f_n|}{n^4}\frac{t^2}{2!}\right], \\ &\ldots, \\ |u_k| &\le C_1^k\sum_{n=1}^{\infty}\frac{q_0^k}{n^{4k}}\left[C_1^0\sum_{n=1}^{\infty}|\varphi_n|\frac{t^k}{k!}+C_2^0\sum_{n=1}^{\infty}\frac{|\psi_n|}{n^2}\frac{t^k}{k!}+C_3^0\sum_{n=1}^{\infty}\frac{|f_n|}{n^4}\frac{t^{2k}}{2k!}\right], \end{aligned}$$

where $\max\limits_{0<t<T}|q(t)|=q_0$. Then for the series (9) we have the estimate

$$|u(x,t)|\le\sum_{k=1}^{\infty}C_1^k\sum_{n=1}^{\infty}\frac{q_0^k}{n^{4k}}\left(C_1^0\sum_{n=1}^{\infty}|\varphi_n|\frac{t^k}{k!}+C_2^0\sum_{n=1}^{\infty}\frac{|\psi_n|}{n^2}\frac{t^k}{k!}+C_3^0\sum_{n=1}^{\infty}\frac{|f_n|}{n^4}\frac{t^{2k}}{2k!}\right); \tag{11}$$

here and in the following, $C_i^0$ and $C_1^k$ are positive constants.

Thus, the following assertion holds.

**Lemma 1.** *The estimate* (11) *holds for any* $(x,t)\in D$.

The formal term-by-term differentiation of the integral equation (8) gives the relations

$$\begin{aligned} u_{tt} = &-\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\big(a\mu_n^4\varphi_n\cos(\omega_n t)+a\mu_n^2\psi_n\sin(\omega_n t)\big)\sin(\mu_n x)\\ &-\frac{2}{l}\sum_{n=1}^{\infty}\frac{\mu_n^2}{a}\int_0^t\int_0^l f(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x)\\ &+\frac{2}{l}\sum_{n=1}^{\infty}\frac{\mu_n^2}{a}\int_0^t q(s)\int_0^l u(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x), \end{aligned} \tag{12}$$

$$\begin{aligned} u_{xxxx} = &\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^4\left(\varphi_n\cos(\omega_n t)+\frac{\psi_n}{\omega_n}\sin(\omega_n t)\right)\sin(\mu_n x)\\ &+\frac{2}{l}\sum_{n=1}^{\infty}\frac{\mu_n^2}{a}\int_0^t\int_0^l f(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x)\\ &-\frac{2}{l}\sum_{n=1}^{\infty}\frac{\mu_n^2}{a}\int_0^t q(s)\int_0^l u(\xi,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n\xi)\,d\xi\,ds\,\sin(\mu_n x). \end{aligned} \tag{13}$$

**Lemma 2.** *Under the conditions*

$$\begin{gathered} \varphi(x)\in C^5[0,l],\quad \varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=\varphi^{(4)}(0)=\varphi^{(4)}(l)=0,\\ \psi(x)\in C^3[0,l],\quad \psi(0)=\psi(l)=\psi''(0)=\psi''(l)=0,\\ f(x,t)\in C(\overline{D})\cap C_x^3(\overline{D}),\quad f(0,t)=f(l,t)=f''_{xx}(0,t)=f''_{xx}(l,t)=0,\quad 0\le t\le T, \end{gathered}$$

*one has the relations*

$$\varphi_n=\frac{1}{\mu_n^5}\varphi_n^{(5)},\quad \psi_n=-\frac{1}{\mu_n^3}\psi_n''',\quad f_n(t)=-\frac{1}{\mu_n^3}f_n'''(t), \tag{14}$$

*where*

$$\varphi_n^{(5)}=\sqrt{\frac{2}{l}}\int_0^l\varphi^{(5)}(x)\cos(\mu_n x)\,dx,\quad \psi_n'''=\sqrt{\frac{2}{l}}\int_0^l\psi'''(x)\cos(\mu_n x)\,dx,$$

$$f_n'''(t)=\sqrt{\frac{2}{l}}\int_0^l f_{xxx}(x,t)\cos(\mu_n x)\,dx,$$

*with the estimates*

$$\sum_{n=1}^{\infty}|\varphi_n^{(5)}|^2\le\|\varphi^{(5)}\|_{L_2[0,l]},\quad \sum_{n=1}^{\infty}|\psi_n'''|^2\le\|\psi'''\|_{L_2[0,l]}. \tag{15}$$

Integrating by parts five times in the integrals for $\varphi_n$ and three times in the integrals for $\psi_n$ and $f_n(t)$ (see definitions (7) and (10)) and taking into account the assumptions of Lemma 2, we obtain relation (14). Inequality (15) is the Bessel inequality for the coefficients of the Fourier expansions of the functions $\varphi_n^{(5)}$ and $\psi_n'''$ in the cosine system $\{\sqrt{2/l}\,cos(\mu_n x)\}$ on the interval $[0,l]$.

If the functions $\varphi(x)$, $\psi(x)$, and $f(x,t)$ satisfy the assumptions of Lemma 2, then, by virtue of the representations (14) and (15), the series (12) and (13) are estimated by the following convergent series:

$$\big|u(x,t)\big| \leq \sum_{k=1}^{\infty} C_1^k \sum_{n=1}^{\infty} \frac{q_0}{n^{4k}} \left( C_1^0 \sum_{n=1}^{\infty} |\varphi_n| \frac{t^k}{k!} + C_2^0 \sum_{n=1}^{\infty} \frac{|\psi_n|}{n^2} \frac{t^k}{k!} + C_3^0 \sum_{n=1}^{\infty} \frac{|f_n|}{n^4} \frac{t^{2k}}{2k!} \right), \tag{16}$$

$$\big|u_{tt}(x,t)\big| \leq C_1 \sum_{n=1}^{\infty} \frac{1}{n} \big(|\varphi_n^{(5)}| + |\psi_n'''|\big) + C_2 \sum_{n=1}^{\infty} \frac{T}{n} \big|f_n'''(t)\big| + C_3 \sum_{n=1}^{\infty} \frac{T}{n} q_0 |u|, \tag{17}$$

$$\big|u_{xxxx}(x,t)\big| \leq \tilde{C}_1 \sum_{n=1}^{\infty} \frac{1}{n} \big(|\varphi_n^{(5)}| + |\psi_n'''|\big) + \tilde{C}_2 \sum_{n=1}^{\infty} \frac{T}{n} \big|f_n'''(t)\big| + \tilde{C}_3 \sum_{n=1}^{\infty} \frac{T}{n} q_0 |u|, \tag{18}$$

where the $\tilde{C}_i$, $i = 1, 2, 3$, are positive constants.

Then the series (16), (17), and (18) converge uniformly in the rectangle $\overline{D}$, and consequently, the function (8) satisfies relations (1)–(3).

## 3. MAIN RESULT AND THE PROOF

Having multiplied both parts of Eq. (1) by $h(x)$ and integrated from 0 to $l$ over $x$, in view of conditions (4) and (5), we obtain

$$g''(t) + a^2 \int_0^l u(x,t) h^{(4)}(x)\, dx + q(t) g(t) = \int_0^l f(x,t) h(x)\, dx.$$

By solving this equation for $q(t)$, we find that

$$q(t) = \frac{1}{g(t)} \int_0^l f(x,t) h(x)\, dx - \frac{g''(t)}{g(t)} - \frac{a^2}{g(t)} \int_0^l u(x,t) h^{(4)}(x)\, dx. \tag{19}$$

Now we substitute the expression (19) for $q(t)$ into Eq. (8) and arrive at an integral equation for $u(x,t)$,

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left( \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \sin(\mu_n x) \\ &+ \frac{2}{l} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t \int_0^l f(x,s) \sin\big(\omega_n(t-s)\big) \sin(\mu_n x)\, dx\, ds \\ &- \frac{2}{l} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t \int_0^l \frac{1}{g(s)} \int_0^l f(\xi,s) h(s) u(y,s) h^{(4)}(y) \sin\big(\omega_n(t-s)\big) \sin(\mu_n x)\, d\xi\, dy\, ds\, \sin(\mu_n x) \\ &- \frac{2a^2}{l} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t \int_0^l \frac{1}{g(s)} \int_0^l u(\xi,s) h^{(4)}(\xi) u(y,s) h^{(4)}(y) \sin\big(\omega_n(t-s)\big) \sin\mu_n(y)\, dy\, d\xi\, ds\, \sin(\mu_n x) \\ &- \frac{2}{l} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t \int_0^l \frac{g''(s)}{g(s)} u(y,s) h^{(4)}(y) \sin\big(\omega_n(t-s)\big) \sin\mu_n(y)\, dy\, ds\, \sin(\mu_n x). \end{aligned}$$

For convenience, we introduce the notation:

$$\Psi(x,t) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left( \varphi_n \cos(\omega_n t) + \frac{\psi_n}{\omega_n} \sin(\omega_n t) \right) \sin(\mu_n x)$$

$$+\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n}\int_0^t\int_0^l f(x,s)\sin\big(\omega_n(t-s)\big)\sin(\mu_n x)\,dx\,ds,$$

$$G_1(x,\xi,y,t,s)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n g(s)}f(\xi,s)h(s)h^{(4)}(y)\sin\big(\omega_n(t-s)\big)\sin\mu_n(y)\sin(\mu_n x),$$

$$G_2(x,y,t,s)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{g''(s)}{\omega_n g(s)}h^{(4)}(y)\sin\big(\omega_n(t-s)\big)\sin\mu_n(y)\sin(\mu_n x),$$

$$G_3(x,\xi,y,t,s)=\frac{2a^2}{l}\sum_{n=1}^{\infty}\frac{1}{\omega_n g(s)}h^{(4)}(\xi)h^{(4)}(y)\sin\big(\omega_n(t-s)\big)\sin\mu_n(y)\sin(\mu_n x)$$

and write Eq. (6) in the more convenient form

$$\begin{aligned}u(x,t)=\Psi(x,t)&-\int_0^t\int_0^l\int_0^l u(y,s)G_1(x,\xi,y,t,s)\,d\xi\,dy\,ds\\&-\int_0^t\int_0^l u(y,s)G_2(x,y,t,s)\,dy\,ds-\int_0^t\int_0^l\int_0^l u(\xi,s)u(y,s)G_3(x,\xi,y,t,s)\,dy\,d\xi\,ds.\end{aligned}\tag{20}$$

Denote the operator taking the function $u(x,t)$ to the right-hand side of Eq. (20) by $A$. Then Eq. (20) is written as the operator equation

$$u=Au.\tag{21}$$

Let

$$\Psi_0=\max_{(x,t)\in D}\big|\Psi(x,t)\big|,$$
$$\lambda_1=\max_{\substack{(x,t)\in D\\ \xi,y\in[0,l]\\ s\in[0,T]}}\big|G_1(x,\xi,y,t,s)\big|,\quad \lambda_2=\max_{\substack{(x,t)\in D\\ y\in[0,l]\\ s\in[0,T]}}\big|G_2(x,y,t,s)\big|,\quad \lambda_3=\max_{\substack{(x,t)\in D\\ \xi,y\in[0,l]\\ s\in[0,T]}}\big|G_3(x,\xi,y,t,s)\big|,\tag{22}$$

let $S_d(0)=\{u:\|u\|\le d\}$, and let $d$ be some positive number.

To prove the existence of a solution of the operator equation (21), we use the Schauder principle [24].

**Lemma 3.** *Let the assumptions of Lemma 2 be satisfied, and let $|g|\ge g_0>0$, where $g_0$ is a known number. Then for all $T$ and $d>\Psi_0$ satisfying the estimate*

$$0<T\le(d-\Psi_0)/M_*,\quad \textit{where}\quad M_*=(\lambda_1+dl\lambda_2+\lambda_3)dl,\tag{23}$$

*where the numbers $\lambda_i$ $(i=1,2,3)$ are defined by relations (22), the operator $A$ is uniformly bounded.*

**Proof.** First, we establish the uniform boundedness of the operator $A$. To this end, we show that there exists a $\rho\in(0,d]$ such that $\|Au\|\le\rho$, where $\|Au\|=\max_{(x,t)\in\overline{D}}|Au|$. For $u\in S_d(0)$ and $(x,t)\in D$, by virtue of (22) we find the estimate $\|Au\|\le\Psi_0+M_*T\equiv\rho$. For $T$ that satisfy the estimate (23), the operator $A$ is uniformly bounded. The proof of the lemma is complete.

**Lemma 4.** *The operator $A$ is equicontinuous.*

**Proof.** Recall the definition of an equicontinuous operator. An operator $A$ is said to be *equicontinuous* if for each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that the inequality

$$\|Au_1-Au_2\|\le\varepsilon\tag{24}$$

holds for all $u_1, u_2 \in S_d(0)$ with $\|u_1 - u_2\| \le \delta$. Consider the difference

$$\begin{aligned} Au_1 - Au_2 = &- \int_0^t \int_0^l \big(u_1(x,s) - u_2(x,s)\big) G_1(x,\xi,t,s)\, ds\, dx \\ &- \int_0^t \int_0^l \int_0^l \big(u_1(\xi,s)u_1(x,s) - u_2(\xi,s)u_2(x,s)\big) G_2\, dx\, d\xi\, ds \\ &- \int_0^t \int_0^l \big(u_1(x,s) - u_2(x,s)\big) G_3(x,t,s)\, dx\, ds \end{aligned}$$

and introduce the notation $u_1 - u_2 =: \tilde{u}$. Performing obvious estimates, we obtain

$$\|Au_1 - Au_2\| \le (\lambda_1 lT + 2\lambda_2 l^2 Td + \lambda_3 lT)\|\tilde{u}\| \equiv M^*\|\tilde{u}\|,$$

whence $\|Au_1 - Au_2\| \le M^*\|u_1 - u_2\| \le M^*\delta$. Consequently, if we take $\delta_0 = \varepsilon/M^*$, then inequality (24) will hold for $\delta \in (0, \delta_0]$; i.e., the operator $A$ is equicontinuous. Then the operator $A$ is completely continuous on $S_d$, and it has at least one fixed point on $S_d$ by the Schauder principle [25]. The proof of the lemma is complete.

Thus, Lemmas 3 and 4 imply the following assertion on the existence of a solution of the operator equation (21).

**Theorem 1.** *Let the assumptions of Lemmas 2 and 3 be satisfied, and let relation (5) hold. Then Eq. (21) has a solution $u(x,t) \in C^{4,2}(D)$ for $T$ satisfying the estimate (23).*

Let us prove the uniqueness of this solution.

**Theorem 2.** *For all $u \in S_d(0)$ and $|g(t)| \ge g_0 > 0$ and for*

$$T < \frac{g_0}{2C_0 d(g_0 + Ha^2 l)}, \tag{25}$$

*where $C_0 = l^2/3a\pi$ and $H = \max_{0<x<l} \|h(x)\|$, the operator equation (21) has a unique solution in the class $C^{4,2}(D)$.*

**Proof.** Let problem (1)–(4) have two solutions $u_1$, $u_2$, $u_1 \ne u_2$, and $q_1$, $q_2$, $q_1 \ne q_2$. We denote their differences by $\tilde{u} = u_1 - u_2$ and $\tilde{q} = q_1 - q_2$. For the difference $\tilde{u}$, we obtain the problem

$$\begin{gathered} \tilde{u}_{tt} + a^2 \tilde{u}_{xxxx} = -q_1\tilde{u} - \tilde{q}u_2, \\ \tilde{u}|_{t=0} = 0, \quad \tilde{u}_t|_{t=0} = 0, \\ \tilde{u}(0,t) = \tilde{u}_{xx}(0,t) = \tilde{u}(l,t) = \tilde{u}_{xx}(l,t) = 0. \end{gathered}$$

The solution of this problem is written as

$$\tilde{u}(x,t) = \frac{2}{l} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^t \int_0^l (-q_1\tilde{u} - \tilde{q}u_2) \sin(\mu_n x) \sin\big(\omega(t-s)\big)\, dx\, ds\, \sin\left(\frac{\pi n}{l} x\right).$$

For $\tilde{q}$, the representation (19) implies the estimate $\|\tilde{q}\| \le Ha^2 l \|\tilde{u}\|/g_0$, whence we have

$$\|\tilde{u}\| \le 2C_0 T \left(d + H\frac{a^2}{g_0} ld\right)\|\tilde{u}\|.$$

Hence for $T$ that satisfy estimate (25) we obtain $u_1 = u_2$. The proof of the theorem is complete.

Based on the function $u(x,t) \in S_d(0)$ thus found and using formula (19), one can find the unknown coefficient $q(t)$—the solution of the Inverse problem.

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