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UDC 51

INITIAL-BOUNDARY VALUE PROBLEM FOR THE INTEGRO-DIFFERENTIAL EQUATION OF BEAM VIBRATION

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Abstract. *This paper considers the initial boundary value problem for the integro-differential equation of vibration of a finite-length beam. From a physical standpoint, the boundary conditions stipulate that the beam is fixed at one end and free at the other. Subsequently, the vibration operator of a homogeneous beam is treated using the method of separation of variables for eigenvalues and eigenfunctions, which reduces the problem to a Fredholm integral equation of the second kind. The method of successive approximations is employed to derive a solution to these equations, and the existence theorem is subsequently demonstrated.*

Keywords: *integro-differential, kernel, beam vibration equation, Fredholm integral equation, eigenvalues and eigenfunctions.*

НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ КОЛЕБАНИЙ БАЛКИ

Аннотация. *В данной работе рассматривается начальная краевая задача для интегро-дифференциального уравнения колебаний балки конечной длины. С физической точки зрения граничные условия описывают, что балка закреплена с одной стороны и свободна с другой. Затем, используя метод разделения переменных для оператора вибрации однородной балки, с помощью собственных значений и собственных функций задача сводится к интегральным уравнениям Фредгольма второго рода. К этим уравнениям применяется метод последовательных приближений и доказывается теорема существования решения.*

Ключевые слова: *интегро-дифференциальное, ядро, уравнение колебаний балки, интегральное уравнение Фредгольма, собственные значения и собственные функции.*

INTEGRO-DIFFERENSIAL BALKA TEBRANISH MASALASI UCHUN BOSHLANG‘ICH-CHEGARAVIY MASALA

Annotatsiya. *Ushbu ishda chekli uzunlikdagi balka tebranishlarining integro-differensial tenglamasi uchun boshlang‘ich chegaraviy masala ko‘rib chiqiladi. Fizik nuqtayi nazardan, chegara shartlari balkaning bir tomonida mustahkamlangan, ikkinchisi esa erkin ekanligini tavsiflaydi. So‘ngra, bir xil balkaning tebranish operatori uchun o‘zgaruvchilarni ajratish usulidan foydalanib, masala xos qiymatlari va xos funksiyalaridan foydalangan holda ikkinchi turdagi Fredgolm integral tenglamalariga keltiriladi. Bu tenglamalarga ketma-ket yaqinlashish usuli qo‘llaniladi va yechimning mavjudligi teoremasi isbotlanadi.*

Kalit so‘zlar: *integro-differensial, yadro, balka tebranish tenglamasi, Fredgolm integral tenglamasi, xos qiymatlar va xos funksiyalar.*

Introduction and statement of the problem. A beam is a linear structural element with a length that is significantly greater than its width and height. It is employed in load-bearing structures with diverse bearing conditions, primarily for its ability to withstand bending forces. They are manufactured with a variety of cross-sections, including round, square (beam), box beam, T-beam, I-beam, channel, and others.

In practice, a horizontally positioned beam typically bears the vertical transverse weight load. However, in some cases, the effect of probable horizontal transverse forces must also be considered, such as in the context of wind load or an earthquake. The loaded beam, in turn, exerts a force on the supports, which may be columns, hangers, walls, or other beams (bars). The load is then transmitted further and ultimately, in the majority of cases, is taken up by the structural elements that are working in compression – the supports.

The case of truss construction, in which the rods rest on a horizontal beam, can be distinguished from the aforementioned examples and examined separately.

The fourth-order partial differential equation is derived from numerous problems associated with rod, beam, and plate vibrations, which have significant applications in the field of structural mechanics [1, 282 p], [2, 298 p.], [3, 143 p.].

In recent years, there has been a notable increase in interest in the study of direct and inverse problems associated with the equation of beam vibrations. This is evidenced by a growing body of literature on the subject, as evidenced by references [4–16]. In [11], the inverse problem of determining the time-dependent coefficient in the transverse vibration equation of a beam is considered. From a physical perspective, this coefficient represents the stiffness of the beam. In [15], we address a direct problem pertaining to the vibration equation of an infinite beam, specifically the Cauchy problem. Additionally, we examine an inverse problem, namely determining the time-dependent coefficient associated with the lower-order term in the aforementioned equation. The direct problem of nonlocal in time and inverse problems with integral overriding conditions for the beam vibration equation are addressed in [16]. In this paper we consider the initial boundary value problem for the integro-differential equation of vibration of a finite-length beam.

Let us consider the integro-differential equation of beam vibration

$$w_{tt} + a^2 w_{xxxx} = \int_0^t k(t) w(x, t-\tau) d\tau, \#(1)$$

in the domain

$$G = \{(x,t):\ 0 < x < l,\ \ 0 < t \le T\}, \#(2)$$

with initial

$$w(x,0) = \varphi(x), \qquad w_t(x,0) = \psi(x), \qquad x \in [0,l], \#(3)$$

and boundary conditions

$$w(0,t) = w_x(0,t) = w_{xx}(l,t) = w_{xxx}(l,t) = 0, \qquad t \in [0,T], \#(4)$$

where l is the length of the beam, T is the time interval.

Problem. *It is required to determine the function* $w(x,t) \in C_{x,t}^{4,2}(G)$, *satisfying the equalities (1) – (4) given numbers* a, l, T *and sufficiently smooth functions* $k(t)$, $\varphi(x)$, $\psi(x)$.

Solution of the initial-boundary value problem. Let us introduce in equation (1) the notation $F(x,t) = \int_0^t k(t) w(x, t-\tau)$. Then equation (1), get the following form:

$$w_{tt} + a^2 w_{xxxx} = F(x,t). \#(5)$$

To solve the equation (5) with initial (2) and boundary conditions (3), we use the method of separation of variables by representing $w(x,t) = X(x)T(t)$ and obtain the spectral problem with respect to $X(x)$. This problem is fully investigated in [17].

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t) Y_n(x) + \sum_{n=1}^{\infty} v_n(t) Y_n(x)\,. \#(6)$$

$\lambda_n = -d_n^4$ is eigenvalues of the spectral problem, where

$$d_n = \frac{\pi}{k}\left(n - \frac{1}{2} + (1)^n \theta_n\right), \qquad \theta_n = O\left(\frac{1}{n^2}\right),$$

$$X_n(x) = \begin{cases} a_n ch\, d_n\left(x - \frac{l}{2}\right) + b_n \sin d_n\left(x - \frac{l}{2}\right), & n = 2k-1, \\ c_n sh\, d_n\left(x - \frac{l}{2}\right) + f_n \cos d_n\left(x - \frac{l}{2}\right), & n = 2k, \end{cases} \#(7)$$

is eigenfunction system, where

$$a_n = sh^{-1}\left(\frac{d_n l}{2}\right), \ \ b_n = \cos^{-1}\left(\frac{d_n l}{2}\right), \ \ c_n = -ch^{-1}\left(\frac{d_n l}{2}\right), \ \ f_n = \sin^{-1}\left(\frac{d_n l}{2}\right).$$

Normalizing the system of functions (7), we gets

$$Y_n(x) = \frac{X(x)}{\left|\left|X(x)\right|\right|}, \quad \left|\left|X(x)\right|\right| = \begin{cases} \sqrt{l}\, ctg\left(\frac{d_n l}{2}\right), & n = 2k-1, \\ \sqrt{l}\, tg\left(\frac{d_n l}{2}\right), & n = 2k. \end{cases} \#(8)$$

Note that system (9) is orthonormal and full in the space $L_2[0,l]$ and forms an orthonormal basis therein.

Then the solution of the stated problem (1) – (4) will be sought in the form of the sum of series (6),

$$w_n(t) = \varphi_n \cos ad_n^2\, t \;\; + \frac{\psi_n}{a\, d_n^2} \sin ad_n^2\, t,$$

where

$$\varphi_n = \int_0^t \varphi(x) Y_n(x) dx\,, \qquad \psi_n = \int_0^l \psi(x) Y_n(x) dx\,, \qquad v_n(t) = \frac{1}{ad_n^2} \int_0^l F_n(s) \sin a\, d_n^2 (t-s) ds,$$

$$F_n(t) = \int_0^l F(x,t) Y_n(x) dx,$$

and $Y_n(x)$ is determining by the formula (8).

Upon inserting $F(x,t)$ into the solution, we obtain:

$$w(x,t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos ad_n^2 t + \frac{\psi_n}{a\, d_n^2} \sin ad_n^2\, t \right) Y_n(x) +$$

$$+ \sum_{n=1}^{\infty} \frac{Y_n(x)}{ad_n^2} \int_0^l \sin ad_n^2\,(t-s) \int_0^l \int_0^s k(s) u(\xi, s-\tau) d\tau d\xi ds\,. \#(9)$$

For the sake of convenience, we introduce the following notation

$$G_0(x,t) = \sum_{n=1}^{\infty} \left(\varphi_n \cos ad_n^2 t + \frac{\psi_n}{a\, d_n^2} \sin ad_n^2\, t \right) Y_n(x),$$

$$K(x, t-s) = \sum_{n=1}^{\infty} \frac{Y_n(x)}{ad_n^2} k(s) \sin ad_n^2\,(t-s).$$

Subsequently, our solution (9) assumes the following form:

$$w(x,t) = G_0(x,t) + \int_0^l \int_0^l \int_0^s K(x,t-s) w(\xi, s-\tau) d\tau d\xi ds.$$

The Fredholm integral equation of the second kind has been obtained. The equation is to be solved using the method of successive approximations for the Fredholm integral equation of the second kind, with the solution presented in the following form:

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x,t), \#(10)$$

where

$$w_0(x,t) = G_0(x,t), \qquad w_k = \int_0^l \int_0^l \int_0^s K(x,t-s) w_{k-1}\,(\xi, s-\tau) d\tau d\xi ds.$$

In estimating the value of u_n in the domain G, we have

$$|w_0| \le g_0,$$

$$|w_1| \le \left| \int_0^l \int_0^l \int_0^s K(x,t-s) w_0\,(\xi, s-\tau) d\tau d\xi ds \right| \le K_0 g_0 \frac{l^3}{2},$$

$$|w_2| \le \left| \int_0^l \int_0^l \int_0^s K(x,t-s) w_1\,(\xi, s-\tau) d\tau d\xi ds \right| \le K_0^2 g_0 \left(\frac{l^3}{2} \right)^2,$$

$$\dots$$

$$|w_k| \le \left| \int_0^l \int_0^l \int_0^s K(x,t-s) w_{k-1}\,(\xi, s-\tau) d\tau d\xi ds \right| \le K_0^k g_0 \left(\frac{l^3}{2} \right)^k,$$

where

$$K_0 = \max_{x\in[0,l], s\in[0,T]} |K(x,t-s)| = \max_{x\in[0,l], s\in[0,T]} \left| \sum_{n=1}^{\infty} \frac{Y_n(x)}{ad_n^2} k(s) \sin ad_n^2\,(t-s) \right| \le$$

$$\le \frac{C_0 l^2 k_0}{a\pi^2 \sqrt{l}} \sum_{n=1}^{\infty} \frac{1}{n^2} \le \frac{2C_0 l^{\frac{3}{2}} k_0}{a\pi^2},$$

$$C_0 = \max\{C_1, 6\},\quad C_1 = \frac{4}{(1-e^{-d_1 l})^2},\quad k_0 = \max_{s\in[0,T]}|k(s)|,\quad g_0 = \max_{x\in[0,l],\ t\in[0,T]}|G_0(x,t)|.$$

Additionally, there are C_i positive constants that are dependent on both l and T.

In order for the series to converge, it is necessary that $K_0\frac{l^3}{2} < 1$. This leads to the conclusion that the condition for l is as follows:

$$l < \left(\frac{a\pi^2}{C_0 k_0}\right)^{\frac{2}{9}}.$$

Subsequently, the series (10) is demonstrated to satisfy the following estimate:

$$|w(x,t)| \le g_0 \sum_{k=1}^{\infty}\left(K_0\frac{l^3}{2}\right)^k . \#(11)$$

This proves the following:

Lemma 1. *For any $(x,t) \in G$ and for all l that satisfy the following condition:*

$$l < \left(\frac{a\pi^2}{C_0 k_0}\right)^{\frac{2}{9}},$$

estimate (11) is true.

The formal term-by-term differentiation of the integral equation (9) yields the following result:

$$w_{tt}(x,t) = \sum_{n=1}^{\infty} -(\varphi_n a^2 d_n^4 \cos a d_n^2 t + \psi_n a d_n^2 \sin a d_n^2\, t)\, Y_n(x) +$$

$$-\sum_{n=1}^{\infty} Y_n(x) a d_n^2 \int_0^l \sin a d_n^2\,(t-s) \int_0^l \int_0^s k(s) u(\xi, s-\tau) d\tau d\xi ds\, , \#(12)$$

$$w_{xxxx}(x,t) = \sum_{n=1}^{\infty} d_n^4\left(\varphi_n \cos a d_n^2 t + \frac{\psi_n}{a\, d_n^2}\sin a d_n^2\, t\right) Y_n(x) +$$

$$+\sum_{n=1}^{\infty} \frac{d_n^2 Y_n(x)}{a}\int_0^l \sin a d_n^2\,(t-s)\int_0^l\int_0^l k(s)u(\xi, s-\tau)d\tau d\xi ds\, . \#(13)$$

Lemma 2. *If the functions $\varphi(x)$, $\psi(x)$ satisfy the specified conditions,*

$$\varphi(x) \in C^5[0,l],\ \varphi^{VI}(x) \in L_2[0,l],\ \varphi(0) = \varphi'(0) = \varphi''(l) = \varphi'''(l) = \varphi^{IV}(0) = \varphi^{V}(0) = 0,$$
$$\varphi(x) \in C^3[0,l],\ \varphi''''(x) \in L_2[0,l],\ \psi(0) = \psi'(0) = \psi''(l) = \psi'''(l),$$

then the following representations hold:

$$\varphi_n = \frac{\varphi_n^{VI}}{d_n^6},\qquad \psi_n = \frac{\psi_n''''}{d_n^4}, \#(14)$$

where

$$\varphi_n^{VI} = \begin{cases} \frac{1}{||X_n||}\int_0^l \varphi^{VI}(x)\left(a_n ch d_n\left(x-\frac{l}{2}\right) + b_n\, sin\, d_n\left(x - \frac{l}{2}\right)\right)dx, & n = 2k-1, \\ \frac{1}{||X_n||}\int_0^l \varphi^{VI}(x)\left(c_n sh d_n\left(x-\frac{l}{2}\right) - f_n\, cos\, d_n\left(x - \frac{l}{2}\right)\right)dx, & n = 2k, \end{cases}$$

$$\psi_n'''' = \int_0^l \psi_n''''(x) Y_n(x) dx.$$

By integrating by parts, the integrals over φ_n six times and over ψ_n four times, while taking into account the conditions set forth in Lemma 2, we arrive at the result (14). In accordance with the conclusions of Lemma 2, the series (9), (12), and (13) are majorized by the convergent numerical series

$$C^* \sum_{n=1}^{\infty}\frac{1}{n^2}(|\varphi_n^{VI}| + |\psi_n''''|),$$

it has been demonstrated that the series (9) converges uniformly. Therefore, the sum of the series satisfies the conditions stated in Problems (1) – (4).

The following theorem is thus derived:

Theorem 1. *If the functions φ_n, ψ_n satisfy the conditions set forth in Lemma 2 and $k(t) \in C[0,T]$, then there exists a unique solution to the problem (1) – (4). This solution is determined by the sum of the series (9).*

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