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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Problem of Determining the Reaction Coefficient in a Fractional Diffusion Equation

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Abstract—For a fractional diffusion equation with reaction coefficient depending only on the first two components of the spatial variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and on time $t \geq 0$, we consider the inverse problem of determining this coefficient under the assumption that the initial value at $t = 0$ is known for the solution of the equation and the boundary value at $x_3 = 0$ is given as an additional condition. This inverse problem is reduced to equivalent integral equations, and we apply the contraction mapping principle to prove the existence of solutions of these equations. Local existence and global uniqueness theorems are proved. We also obtain a stability estimate for the solution of the inverse problem.

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INTRODUCTION

Currently, fractional differential equations are of considerable interest both in mathematics itself and in applied fields. These equations are used in modeling many physical and chemical processes, in particular, mass transfer processes in media with fractal properties (see, e.g., [1–6]). In the papers [7–9], a number of interesting features of fractional subdiffusion equations are presented, indicating a certain similarity of these equations to second-order parabolic differential equations.

Direct problems for fractional diffusion equations such as initial and initial–boundary value problems were studied in detail in [1–4] (see also references therein). Compared with direct problems, there are only few results on inverse problems for fractional equations. The inverse problems of determining the coefficient depending only on spatial variables for fractional partial differential equations were investigated in [10, 11]. In the present paper, the desired function depends not only on the spatial variables but also on the time variable. We also note that the inverse problems of determining the source function for equations with fractional integro-differentiation operators were studied in [12–14].

Inverse problems for classical differential heat equations are fairly well studied. In the literature, linear source determination problems and nonlinear coefficient inverse problems with various types of overdetermination conditions are most often encountered (see, e.g., [15–19] and references therein). In these papers, the unique solvability of the problems and the stability of the solutions, as well as the construction of a numerical solution of such problems, are studied. Memory recovery problems for second-order parabolic integro-differential equations with an integral term of convolution type were considered in the papers [20–23]. It was proved in [24] that if the convolution kernel in these equations is chosen in the form of a special Mittag-Leffler function, then the equations are equivalent to the anomalous diffusion equations.

The main results of this paper comprise local existence and global uniqueness theorems as well as a stability estimate for the solution of the problem of determining the reaction coefficient in a time-fractional diffusion equation.

Statement of the problem. Consider the fractional diffusion equation

$$\begin{gathered}({}^C\mathcal{D}_t^\alpha u)(x,t) - \Delta_x u(x,t) + q(x',t)u(x,t) = f(x,t),\\ x = (x_1, x_2, x_3) = (x', x_3), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+,\end{gathered} \tag{1}$$

with the condition

$$u|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^3, \tag{2}$$

where Δ_x is the Laplace operator with respect to the variables x_1, x_2, x_3, $\mathbb{R}_+ = \{t : t > 0\}$, ${}^C\mathcal{D}_t^\alpha$ is the regularized fractional derivative with respect to t (the Gerasimov–Caputo derivative), $0 < \alpha < 1$, i.e.,

$$({}^C\mathcal{D}_t^\alpha u)(x,t) := \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{u_\tau(x,\tau)\,d\tau}{(t-\tau)^\alpha},$$

and $f(x,t)$ and $\varphi(x)$ are given sufficiently smooth functions. The function q in Eq. (1) will be called the *reaction coefficient*; we assume that it is sufficiently smooth as well.

Inverse problem. Find the function $q(x',t)$, $x' \in \mathbb{R}^2$, $t \in \mathbb{R}_+$,—the reaction coefficient in Eq. (1)—if the solution of the Cauchy problem (1), (2) satisfies the condition

$$u|_{x_3=0} = g(x',t), \quad x' \in \mathbb{R}^2, \quad t \in \mathbb{R}_+, \tag{3}$$

where $g(x',t)$ is a given sufficiently smooth function.

A function $u(x,t)$ will be called a *classical solution* of the Cauchy problem (1), (2) if

(a) It is twice continuously differentiable with respect to x for each $t > 0$.

(b) For each $x \in \mathbb{R}^3$, it is continuous in t on $[0,T]$, and the fractional integral

$$(I_{0+}^\alpha u)(x,t) := \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{u(x,\tau)\,d\tau}{(t-\tau)^{1-\alpha}}$$

is continuously differentiable with respect to $t \in \mathbb{R}_+$.

(c) It satisfies Eq. (1) and condition (2).

Let $u(x,t)$ be a classical solution of the Cauchy problem (1), (2) and, as was mentioned above, let $f(x,t)$, $\varphi(x)$, and $g(x',t)$ be sufficiently smooth functions. Let us transform the inverse problem (1)–(3). To this end, we denote the second derivative of the function $u(x,t)$ with respect to the variable x_3 by $v(x,t)$; i.e., $v(x,t) := u_{x_3x_3}(x,t)$. Differentiating relations (1) and (2) twice with respect to x_3, we arrive at the problem

$$({}^C\mathcal{D}_t^\alpha v)(x,t) - \Delta_x v(x,t) + q(x',t)v(x,t) = f_{x_3x_3}(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}_+, \tag{4}$$

$$v|_{t=0} = \varphi_{x_3x_3}(x), \quad x \in \mathbb{R}^3, \tag{5}$$

i.e., to a problem of the form (1), (2).

To find the additional condition for the function $v(x,t)$, we note that the third term of the Laplacian in Eq. (1) is equal to $v(x,t)$. Setting $x_3 = 0$ in Eq. (1) and using relation (3), we obtain

$$\begin{gathered} v|_{x_3=0} = ({}^C\mathcal{D}_t^\alpha g)(x',t) - \Delta_{x'} g(x',t) + q(x',t)g(x',t) - f(x',0,t), \\ x' \in \mathbb{R}^2, \quad t \in \mathbb{R}_+. \end{gathered} \tag{6}$$

Under the matching condition $\varphi(x',0) = g(x',0)$, we can readily derive relations (1)–(3) from (4)–(6).

For the given functions $q(x',t)$, $f(x,t)$, and $\varphi(x)$ and number $\alpha \in (0,1)$, the problem of determining the solution of the Cauchy problem (4) and (5) will be called the *direct problem*.

By $\Phi_T := \{(x,t) : x \in \mathbb{R}^3,\ 0 < t < T\}$ we denote a layer of thickness T, where $T > 0$ is a fixed number, which can be arbitrary.

Let $C^{\alpha,m}(\Phi_T)$ be the class of functions m times continuously differentiable with respect to $x \in \mathbb{R}^3$ and continuous in t for which the fractional integral I_{0+}^α of order α is continuously differentiable with respect to t on $[0,T]$. Let l be a noninteger positive number, $l \in \mathbb{R}_+ \setminus \mathbb{N}$, and let $n \in \mathbb{N}$. By $C([0,T], H^l(\mathbb{R}^n))$ we denote the class of continuous functions defined on the interval $[0,T]$ and ranging in $H^l(\mathbb{R}^n)$, where $H^l(\mathbb{R}^n)$ is the space of functions $\varphi:\mathbb{R}^n\to\mathbb{R}$ that have continuous partial derivatives of order $\le [l]$ (where $[\cdot]$ is the integer part of a number) and for which the number

$$|\varphi|^l=\sum_{([l])}\sup_{\substack{|x^1-x^2|\le\rho_0\\ x^1,x^2\in\mathbb{R}^n}}\frac{\left|D_x^{[l]}\varphi(x^1)-D_x^{[l]}\varphi(x^2)\right|}{|x^1-x^2|^\alpha}+\sum_{j=0}^{[l]}\sum_{(j)}\sup_{x\in\mathbb{R}^n}\left|D_x^j\varphi(x)\right|$$

is finite [25, p. 15–16], where ρ_0 is a given positive number (which can be chosen arbitrarily), $\alpha=l-[l]$, and $\sum_{(j)}$ is the sum over all multiindices of length j; in particular, $\sum_{([l])}$ is the sum over all multiindices of length $[l]$. The norm in $H^l(\mathbb{R}^n)$ of the value of a function $\phi(t,x)\in C([0,T],H^l(\mathbb{R}^n))$ for a given $t\in[0,T]$ will be denoted by $|\phi|^l(t)$. The same notation will also be used for functions depending only on the variable x. The norm of a function $\phi(t,x)\in C([0,T],H^l(\mathbb{R}^n))$ is defined by the formula

$$\|\phi\|^l:=\max_{t\in[0,T]}|\phi|^l(t).$$

In what follows, we will consider the spaces $C([0,T],H^\alpha(\mathbb{R}^3))$, $C([0,T],H^{2+\alpha}(\mathbb{R}^3))$, and $C([0,T],H^\alpha(\mathbb{R}^2))$, where $\alpha\in(0,1)$.

1. STUDYING THE DIRECT PROBLEM (4), (5)

Eidelman and Kochubei [9] found a representation of the solution using the fundamental solution of the Cauchy problem

$$\begin{gathered}({}^C\mathcal{D}_t^\alpha u)(x,t)-Bu(x,t)=F(x,t),\quad x\in\mathbb{R}^n,\quad t\in(0,T],\\ u|_{t=0}=u_0(x),\quad x\in\mathbb{R}^n,\end{gathered}$$

where

$$B:=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j}+c(x)$$

is a uniformly elliptic second-order differential operator with bounded continuous real coefficients. For $B\equiv\Delta$, where Δ is the n-dimensional Laplacian, for any bounded continuous function $u_0(x)$ (locally Hölder continuous if $n>1$) and any bounded function $F(x,t)$ continuous in both variables x and t and locally Hölder continuous in x, this solution has the form

$$u(x,t)=\int\limits_{\mathbb{R}^n}Z(x-\xi,t)u_0(\xi)\,d\xi+\int\limits_0^t\int\limits_{\mathbb{R}^n}Y(x-\xi,t-\tau)F(\xi,\tau)\,d\xi\,d\tau;\tag{7}$$

here

$$\begin{aligned}Z(x,t)&=\pi^{-n/2}|x|^{-n}H_{1,2}^{2,0}\left[\frac14 t^{-\alpha}|x|^2\middle|\begin{matrix}(1,\alpha)\\ \\(n/2,1),(1,1)\end{matrix}\right],\\ Y(x,t)&=\pi^{-n/2}|x|^{-n}t^{\alpha-1}H_{1,2}^{2,0}\left[\frac14 t^{-\alpha}|x|^2\middle|\begin{matrix}(\alpha,\alpha)\\ \\(n/2,1),(1,1)\end{matrix}\right],\end{aligned}$$

where by H we have denoted the Fox H-function [26, p. 2–6]. The function $Y(x,t)$ is actually the Riemann–Liouville derivative of $Z(x,t)$ with respect to t of order $1-\alpha$ [9]. (If $x\ne0$, then $Z(x,t)\to0$ as $t\to0$. In this case, the Riemann–Liouville derivative coincides with the Gerasimov–Caputo derivative, $Y(x,t)=({}^C\mathcal{D}_t^\alpha Z)(x,t)$.)

Introducing the notation $f_{x_3x_3}(x,t)-q(x',t)v(x,t)=:F(x,t)$ in Eq. (4), for the direct problem (4), (5) with $n=3$ in view of the representation (7) we obtain the following integral equation for the function $v(x,t)$:

$$v(x,t)=v_0(x,t)-\int\limits_0^t\int\limits_{\mathbb{R}^3}Y(x-\xi,t-\tau)q(\xi_1,\xi_2,\tau)v(\xi,\tau)\,d\xi\,d\tau,\tag{8}$$

where

$$v_0(x,t) := \int\limits_{\mathbb{R}^3} Z(x-\xi,t)\varphi_{\xi_3\xi_3}(\xi)\,d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau) f_{\xi_3\xi_3}(\xi,\tau)\,d\xi\,d\tau, \tag{9}$$
$$\xi = (\xi_1,\xi_2,\xi_3), \qquad d\xi = d\xi_1\,d\xi_2\,d\xi_3.$$

The following assertion holds.

Lemma. *If* $q(x,t) \in C([0,T], H^\alpha(\mathbb{R}))$, $f(x,t) \in C([0,T], H^{\alpha+2}(\mathbb{R}^3))$, *and* $\varphi(x) \in H^{\alpha+2}(\mathbb{R}^3)$, *then there exists a unique solution of the integral equation* (8) *such that* $v(x,t) \in C^{1-\alpha,2}(\Phi_T)$, *where* $\alpha \in (0,1)$.

Proof. Let us use the successive approximation method and consider the sequence $(v_n(x,t))$ of functions recursively defined by the formulas

$$v_n(x,t) = -\int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau) q(x',\tau) v_{n-1}(\xi,\tau)\,d\xi\,d\tau, \quad n = 1,2,\ldots, \tag{10}$$

where the function $v_0(x,t)$ is given by (9). In what follows, we need estimates for the functions $Z(t,x)$ and $Y(t,x)$ and for some of their derivatives. Let $m = (m_1, m_2, \ldots, m_n)$ be a multiindex of nth order, let $|m| = m_1 + m_2 + \ldots + m_n$ be its length, and let

$$D_x^m u = \frac{\partial^{|m|} u}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_n^{m_n}}, \quad D_x^0 u = u.$$

The following estimates hold for the functions $Z(x,t)$ and $Y(t,x)$ and their derivatives [9]:

(a) If $|x|^2 \ge t^\alpha$, then
$$\left|D_x^m Z(x,t)\right| \le C t^{-\alpha(3+m)/2} e^{-\mu_m t^{-\alpha/(2-\alpha)} |x|^{2/(2-\alpha)}}$$
for $n = 3$ and $|m| \le 3$, and
$$\left|{}^C\mathcal{D}_t^\alpha Z(x,t)\right| \le C t^{-5\alpha/2} e^{-\mu_n t^{-\alpha/(2-\alpha)} |x|^{2/(2-\alpha)}} \tag{11}$$
for $n = 3$.

(b) If $|x|^2 < t^\alpha$, $x \ne 0$, then
$$\left|D_x^m Z(x,t)\right| \le C t^{-\alpha} |x|^{-1-m}, \quad |m| \le 3,$$
for $n = 3$ and $m \ne 0$, and
$$\left|Z(x,t)\right| \le C t^{-\alpha} |x|^{-1}, \quad |m| \le 3, \tag{12}$$
for $n = 3$.

(c) If $|x|^2 < t^\alpha$, $x \ne 0$, then
$$\left|{}^C\mathcal{D}_t^\alpha Z(x,t)\right| \le C t^{-2\alpha} |x|^{-1}$$
for $n = 3$.

(d) if $|x|^2 \ge t^\alpha$, then
$$\left|D_x^m Y(x,t)\right| \le C t^{-1+\alpha-\alpha(3+m)/2} e^{-\mu_m t^{-\alpha/(2-\alpha)} |x|^{2/(2-\alpha)}}, \quad |m| \le 3, \tag{13}$$
for $n = 3$.

(e) If $|x|^2 < t^\alpha$, $x \neq 0$, $n = 3$, then

$$\big|Y(x,t)\big| \le Ct^{-1-\alpha/2}, \quad \big|D_x Y(x,t)\big| \le Ct^{-\alpha-1},$$

$$\big|D_x^m Y(x,t)\big| \le Ct^{-\alpha-1}|x|^{-1}, \quad |m| = 2, \tag{14}$$

$$\big|D_x^m Y(x,t)\big| \le Ct^{-\alpha-1}|x|^{-2}, \quad |m| = 3. \tag{15}$$

In these estimates, $\mu_0 := (2-\alpha)\alpha^{\alpha/(2-\alpha)}$, for μ_m we can take any positive number less than μ_0, and by C we have denoted a positive constant whose value is, generally speaking, different in different estimates.

The construction of the function $Z(x,t)$ implies the relation

$$\int\limits_{\mathbb{R}^3} Z(\xi,t)\,d\xi = 1; \tag{16}$$

in addition, as shown in [9],

$$\int\limits_{\mathbb{R}^3} Y(\xi,t)\,d\xi = C_0 t^{\alpha-1}, \quad t \in (0,T]; \tag{17}$$

here the constant C_0 depends only on α.

Set $d_0 := \|q\|^\alpha$, $\varphi_0 := |\varphi|^{\alpha+2}$, and $f_0 := \|f\|^{\alpha+2}$. Using definition (10) and relations (16) and (17), we estimate the absolute value of the function $v_n(x,t)$ in the domain Φ_T as follows:

$$\big|v_0(x,t)\big| \le \varphi_0 + C_0 f_0 \frac{T^\alpha}{\alpha} =: \lambda_0,$$

$$\big|v_1(x,t)\big| \le C_0 d_0 \lambda_0 \int\limits_0^t (t-\tau)^{\alpha-1}\,d\tau = C_0 d_0 \lambda_0 \frac{t^\alpha}{\alpha} = \lambda_0 \frac{C_0 d_0 \Gamma(\alpha)}{\Gamma(1+\alpha)} t^\alpha,$$

$$\big|v_2(x,t)\big| \le \lambda_0 \big(C_0 d_0 \Gamma(\alpha)\big)^2 \frac{1}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{\tau^\alpha\,d\tau}{(t-\tau)^{1-\alpha}} = \lambda_0 \frac{\big(C_0 d_0 \Gamma(\alpha)\big)^2}{\Gamma(1+\alpha)} I_{0+}^\alpha t^\alpha,$$

where $I_{0+}^\alpha t^\alpha$ is the fractional Riemann–Liouville integral of the power function t^α and $\Gamma(\cdot)$ is the Euler gamma function. One can readily verify [27, p. 15] that

$$I_{0+}^\alpha t^{n\alpha} = \frac{\Gamma(1+n\alpha)}{\Gamma\big(1+(n+1)\alpha\big)} t^{(1+n)\alpha}, \quad n = 0,1,2,\ldots$$

Using this relation, we continue the estimate of the function $v_2(x,t)$,

$$\big|v_2(x,t)\big| \le \lambda_0 \frac{\big(C_0 d_0 \Gamma(\alpha)\big)^2}{\Gamma(1+\alpha)} I_{0+}^\alpha t^\alpha = \lambda_0 \frac{\big(C_0 d_0 \Gamma(\alpha)\big)^2}{\Gamma(1+2\alpha)} t^{2\alpha}.$$

In a similar way, for arbitrary $n = 0,1,2,\ldots$ we obtain

$$\big|v_n(x,t)\big| \le \lambda_0 \frac{\big(C_0 d_0 \Gamma(\alpha)\big)^n}{\Gamma(1+n\alpha)} t^{n\alpha}.$$

It follows from the above estimates that the series $v(x,t) = \sum_{n=0}^\infty v_n(x,t)$ converges uniformly in the domain Φ_T, because in this domain it can be majorized by the converging number series

$$\lambda_0 \sum_{n=0}^\infty \frac{\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big)^n}{\Gamma(1+n\alpha)}.$$

This implies the following estimate for the solution of the integral equation (8):

$$\big|v(x,t)\big| \le \lambda_0 \sum_{n=0}^{\infty} \frac{\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big)^n}{\Gamma(1+n\alpha)} = \lambda_0 E_\alpha\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big), \quad (x,t) \in \Phi_T, \tag{18}$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function of a nonnegative real argument [27, p. 40–45].

Further, note that under the assumptions of the Lemma the function $v_0(x,t)$ belongs to the space $C^2(\mathbb{R}^n)$ for each $t > 0$. To prove this fact, fix $x^0 \in \mathbb{R}^3$ and split the integration domain $\mathbb{R}^3$ in the representation (9) into two sets, $\Omega_1 = \{\xi \in \mathbb{R}^3 : |\xi - x^0| \ge t^\alpha\}$ and $\Omega_2 = \mathbb{R}^3 \backslash \Omega_1$. Then the function $v_0(x,t)$ can be represented in the form of the sum $v_0^1(x,t) + v_0^2(x,t)$ of two terms. (The function $v_0^i(x,t)$ is determined by the right-hand side of relation (9) with $\mathbb{R}^3$ replaced by Ω_i, $i = 1,2$.)

If a point x lies in a small neighborhood of the point x^0 and $\xi \in \Omega_1$, then $|x - \xi|$ is bounded away from zero. Thus, to calculate $\partial^2 v_0^1(x,t)/\partial x_j^2$, we can differentiate in the integrand,

$$v_0^1(x,t) = \int\limits_{\Omega_1} \frac{\partial^2 Z(x-\xi,t)}{\partial x_j^2} \varphi_{\xi_3\xi_3}(\xi)\, d\xi + \int\limits_0^t \int\limits_{\Omega_1} \frac{\partial^2 Y(x-\xi,t-\tau)}{\partial x_j^2} f_{\xi_3\xi_3}(\xi,\tau)\, d\xi\, d\tau,$$

$j = 1,2,3$; i.e., $v_0^1(x,t) \in C^2(\Omega_1)$.

To calculate $\partial^2 v_0^2(x,t)/\partial x_j^2$, note that the estimates (11), (12), (14), and (15) for the functions $Z(x-\xi,t)$ and $Y(x-\xi,t-\tau)$ contain a singularity of the form $|x-\xi|^{-k}$ with exponent $k > 0$. Consequently, the integrals over Ω_2 in the estimates of the functions $\partial v_0^2(x,t)/\partial x_j$ and $\partial^2 v_0^2(x,t)/\partial x_j^2$ will have such singularities. It follows from the theory of Newtonian potential that improper integrals with such singularity converge uniformly with respect to x and define a function continuous in Ω_2 provided that k is less than the dimension of the domain Ω_2, $k < 3$ [28, p. 335]. By virtue of this fact and the local Hölder property of the functions $\varphi_{x_3x_3}$ and $f_{x_3x_3}$ in x, the derivatives $\partial v_0^2(x,t)/\partial x_j$ and $\partial^2 v_0^2(x,t)/\partial x_j^2$ are functions continuous in Ω_2. Thus, $v_0(x,t) \in C^2(\mathbb{R}^n)$.

Since the functions $Z(x-\xi,t)$ and $Y(x-\xi,t-\tau)$ satisfy the homogeneous equation corresponding to Eq. (4), we conclude that $\mathcal{D}^\alpha v_0(x,t) \in C^2(\mathbb{R}^3)$. Consequently, $v_0(x,t) \in C^{1-\alpha,2}(\Phi_T)$. By virtue of definition (10), it can readily be seen that all $v_j(x,t)$ have this property. Then the general theory of integral equations implies the inclusion $v(x,t) \in C^{1-\alpha,2}(\Phi_T)$; i.e., the function $v(x,t)$ is a classical solution of the Cauchy problem (4), (5).

Now by $\tilde{v}(x,t)$ we denote the solution of the original integral equation (8) in which the functions q, $f_{x_3x_3}$, and $\varphi_{x_3x_3}$ have been replaced by perturbed functions $\tilde{q}$, $\tilde{f}_{x_3x_3}$, and $\tilde{\varphi}_{x_3x_3}$, respectively, i.e., the equation

$$\tilde{v}(x,t) = \tilde{v}_0(x,t) - \int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau)\tilde{q}(x',\tau)\tilde{v}(\xi,\tau)\, d\xi\, d\tau, \tag{19}$$

where

$$\tilde{v}_0(x,t) := \int\limits_{\mathbb{R}^3} Z(x-\xi,t)\tilde{\varphi}_{\xi_3\xi_3}(\xi)\, d\xi + \int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau)\tilde{f}_{\xi_3\xi_3}(\xi,\tau)\, d\xi\, d\tau. \tag{20}$$

Let us find an estimate for the norm of the difference between the solution $v(x,t)$ to Eq. (8) and the solution $\tilde{v}(x,t)$ to Eq. (19). Composing the difference $v - \tilde{v}$ with the help of Eqs. (8) and (19), for this difference we obtain the integral equation

$$v(x,t) - \tilde{v}(x,t) = v_0(x,t) - \tilde{v}_0(x,t) - \int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau)(q(x',\tau) - \tilde{q}(x',\tau))v(\xi,\tau)\, d\xi\, d\tau$$
$$- \int\limits_0^t \int\limits_{\mathbb{R}^3} Y(x-\xi,t-\tau)\tilde{q}(x',\tau)(v(\xi,\tau) - \tilde{v}(\xi,\tau))\, d\xi\, d\tau;$$

from this, we derive the following linear integral inequality for $|v(x,t)-\tilde v(x,t)|$:

$$\begin{aligned}\big|v(x,t)-\tilde v(x,t)\big| \le \big|v_0(x,t)-\tilde v_0(x,t)\big| + \lambda_0 C_0 \frac{T^\alpha}{\alpha} E_\alpha\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big)\|q-\tilde q\|^\alpha \\ + \tilde q_0 \int_0^t \int_{\mathbb{R}^3} Y(x-\xi,t-\tau)\big|v(\xi,\tau)-\tilde v(\xi,\tau)\big|\,d\xi\,d\tau,\end{aligned} \tag{21}$$

where $\tilde q_0 := \|\tilde q\|^\alpha$. Relations (9) and (20) imply the estimate

$$\big|v_0(x,t)-\tilde v_0(x,t)\big| \le \|\varphi_{x_3x_3}-\tilde\varphi_{x_3x_3}\| + C_0\frac{T^\alpha}{\alpha}\|f_{x_3x_3}-\tilde f_{x_3x_3}\|^\alpha.$$

Let $\sigma = \sigma(\alpha, T, d_0, \tilde q_0, \varphi_0, f_0) = \max\{1, \tilde q_0, C_0T^\alpha/\alpha, \lambda_0 C_0 T^\alpha/\alpha E_\alpha(C_0 d_0\Gamma(\alpha)T^\alpha)\}$. Applying the successive approximation method to inequality (21),

$$\big|v(x,t)-\tilde v(x,t)\big|_0 \le \sigma\big(\|\varphi_{x_3x_3}-\tilde\varphi_{x_3x_3}\|^\alpha + \|f_{x_3x_3}-\tilde f_{x_3x_3}\|^\alpha + \|q-\tilde q\|^\alpha\big),$$

$$\big|v(x,t)-\tilde v(x,t)\big|_n \le \tilde q_0 \int_0^t\int_{\mathbb{R}^3} Y(x-\xi,t-\tau)\big|v(\xi,\tau)-\tilde v(\xi,\tau)\big|_{n-1}\,d\xi\,d\tau, \quad n=1,2,\ldots,$$

we arrive at the estimate

$$\big|v(x,t)-\tilde v(x,t)\big| \le \sigma\lambda_0 E_\alpha\big(C_0d_0\Gamma(\alpha)T^\alpha\big)\big(\|\varphi_{x_3x_3}-\tilde\varphi_{x_3x_3}\|^\alpha + \|f_{x_3x_3}-\tilde f_{x_3x_3}\|^\alpha + \|q-\tilde q\|^\alpha\big), \tag{22}$$

which is a stability estimate for the solution of the Cauchy problem (4), (5). The uniqueness of the solution of this problem also follows from the estimate (21).

We will use the estimate (21) in the next section of the paper.

2. STUDY OF THE INVERSE PROBLEM (4)–(6)

Setting $x_3 = 0$ in Eq. (8) and definition (9) and using the additional condition (6), after simple transformations we obtain the following integral equation for the coefficient $q(x',t)$:

$$\begin{aligned}q(x',t) &= q_0(x',t) \\ &- \frac{1}{g(x',t)}\int_0^t\int_{\mathbb{R}^3} Y(x_1-\xi_1, x_2-\xi_2, \xi_3, t-\tau) q(\xi_1,\xi_2,\tau) v(\xi_1,\xi_2,\xi_3,\tau)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau,\end{aligned} \tag{23}$$

where

$$\begin{aligned}q_0(x',t) := \frac{1}{g(x',t)}\Bigg[& f(x',0,t) + \Delta_{x'} g(x',t) - ({}^C\mathcal{D}_t^\alpha g)(x',t) \\ &+ \int_0^t\int_{\mathbb{R}^3} Z(x_1-\xi_1, x_2-\xi_2, \xi_3, t)\varphi_{\xi_3\xi_3}(\xi_1,\xi_2,0)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^3} Y(x-\xi_1, x-\xi_2, \xi_3, t-\tau) f_{\xi_3\xi_3}(\xi_1,\xi_2,\xi_3,\tau)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau\Bigg].\end{aligned}$$

We introduce an operator A defining its action by the right-hand side of Eq. (23); i.e.,

$$\begin{aligned}A[q](x',t) &= q_0(x',t) \\ &- \frac{1}{g(x',t)}\int_0^t\int_{\mathbb{R}^3} Y(x_1-\xi_1, x_2-\xi_2, \xi_3, t-\tau) q(\xi_1,\xi_2,\tau) v(\xi_1,\xi_2,\xi_3,\tau)\,d\xi_1\,d\xi_2\,d\xi_3\,d\tau.\end{aligned} \tag{24}$$

Then Eq. (23) can be written in the more concise form

$$q(x',t) = A[q](x',t).$$

Let $q_{00} := \|q_0\|^\alpha$. Fix a number $\rho > 0$ and consider the ball

$$B_T^\alpha(q_0,\rho) := \Big\{ q(x',t) : q(x',t) \in C\big([0,T], H^\alpha(\mathbb{R}^2)\big), \quad \|q - q_0\|^\alpha \le \rho \Big\}, \quad \alpha \in (0,1).$$

Theorem 1. *If* $f(x,t) \in C([0,T], H^{\alpha+2}(\mathbb{R}^3))$, $\varphi(x) \in H^{\alpha+2}(\mathbb{R}^3)$, $g(x',t) \in C([0,T], H^\alpha(\mathbb{R}^2))$, $\|g(x',t)\|^\alpha \ge g_0 > 0$, *and* $g(x',0,0) = \varphi(x',0,0)$, *then there exists a number* $T^* \in (0,T]$ *such that the inverse problem* (1)–(3) *has a unique solution* $q(x',t) \in C([0,T^*], H^\alpha(\mathbb{R}^2))$.

Proof. First, we will prove that for sufficiently small $T > 0$ the operator A takes the ball $B_T^\alpha(q_0,\rho)$ into itself; i.e., the condition $q(x',t) \in B_T^\alpha(q_0,\rho)$ implies that $A[q](x',t) \in B_T^\alpha(q_0,\rho)$. Indeed, for any function $q(x',t) \in C([0,T], H^\alpha(\mathbb{R}^2))$ the function $A[q](x',t)$ calculated by formula (24) belongs to the class $C([0,T], H^\alpha(\mathbb{R}^2))$. Moreover, for the norm of the difference of the functions $A[q]$ and q_0, using the estimate (18), we obtain

$$\big\|A[q] - q_0\big\|^\alpha \le \frac{C_0 d_0 \lambda_0}{\alpha g_0} T^\alpha E_\alpha\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big). \tag{25}$$

Note that the function on the right-hand side in the estimate (25) is monotone increasing with T and that the function $q(x',t)$ lies in the ball $B_T^\alpha(q_0,\rho)$, which implies the inequality $\|q\|^\alpha \le \rho + q_{00}$. Consequently, the estimate (25) remains valid if in this estimate we replace $\|q\|^\alpha$ by the expression $\rho + q_{00}$. Performing these replacements, we arrive at the estimate

$$\big\|A[q] - q_0\big\|^\alpha \le \frac{C_0 \lambda_0 (\rho + q_{00})}{\alpha g_0} T^\alpha E_\alpha\big((\rho + q_{00}) C_0 \Gamma(\alpha) T^\alpha\big).$$

Let T_1 be a positive root of the equation

$$\frac{C_0 \lambda_0 (\rho + q_{00})}{\alpha g_0} T^\alpha E_\alpha\big((\rho + q_{00}) C_0 \Gamma(\alpha) T^\alpha\big) = \rho.$$

Then for $T \in [0, T_1]$ we obviously have the inclusion $A[q](x',t) \in B_T^\alpha(q_0,\rho)$.

Now let us consider two functions $q(x',t)$ and $\tilde q(x',t)$ belonging to the ball $B_T^\alpha(q_0,\rho)$ and estimate the distance between their images $A[q](x',t)$ and $A[\tilde q](x',t)$ in the space $C([0,T], H^\alpha(\mathbb{R}^2))$. The function $\tilde v(x,t)$ corresponding to the coefficient $\tilde q(x',t)$ satisfies the integral equation (19) with the functions $\varphi_{x_3x_3} = \tilde\varphi_{x_3x_3}$ and $f_{x_3x_3} = \tilde f_{x_3x_3}$. Composing the difference $A[q](x',t) - A[\tilde q](x',t)$ with the help of Eqs. (8) and (19) and then estimating the norm of this difference, we obtain

$$\big\|A[q](x',t) - A[\tilde q](x',t)\big\|^\alpha \le \frac{C_0 T^\alpha}{\alpha g_0}\big[\|v\|\|q - \tilde q\|^\alpha + \|q\|^\alpha \|v - \tilde v\|\big].$$

Using inequality (18) and the estimate (22) with $\varphi_{x_3x_3} = \tilde\varphi_{x_3x_3}$ and $f_{x_3x_3} = \tilde f_{x_3x_3}$, we continue the previous inequality in the form

$$\big\|A[q](x',t) - A[\tilde q](x',t)\big\|^\alpha \le \frac{C_0 T^\alpha}{\alpha g_0} \lambda_0 E_\alpha\big(C_0 d_0 \Gamma(\alpha) T^\alpha\big)(1 + \sigma \tilde q_0)\|q - \tilde q\|^\alpha. \tag{26}$$

The functions $q(x',t)$ and $\tilde q(x',t)$ belong to the ball $B_T^\alpha(q_0,\rho)$; therefore, the norm $\|\cdot\|^\alpha$ of each of them does not exceed $\rho + q_{00}$. Note that the function on the right-hand side in inequality (26) with the factor $\|q - \tilde q\|^\alpha$ is monotone increasing with $\|q\|^\alpha$, $\|\tilde q\|^\alpha$, and T.

Consequently, the estimate (26) remains valid if in this estimate (including σ) we replace $\|q\|^\alpha$ and $\|\tilde q\|^\alpha$ by $\rho + q_{00}$. Thus, we have

$$\big\|A[q](x',t) - A[\tilde q](x',t)\big\|^\alpha \le \frac{C_0 T^\alpha}{\alpha g_0} \lambda_0 E_\alpha\big((\rho + q_{00}) C_0 \Gamma(\alpha) T^\alpha\big)\big(1 + \sigma(\rho + q_{00})\big)\|q - \tilde q\|^\alpha.$$

Let T_2 be a positive root of the equation

$$r(T) := \frac{C_0 T^\alpha}{\alpha g_0} \lambda_0 E_\alpha\big((\rho + q_{00}) C_0 \Gamma(\alpha) T^\alpha\big)\big(1 + \sigma(\rho + q_{00})\big) = 1.$$

Then for $T \in [0, T_2]$ the distance between the functions $A[q](x', t)$ and $A[\tilde{q}](x', t)$ in the function space $C([0, T], H^\alpha(\mathbb{R}^2))$ does not exceed the distance between the functions $q(x', t)$ and $\tilde{q}(x', t)$ multiplied by $r(T) < 1$. Consequently, if we take $T^* = \min(T_1, T_2)$, then A will be a contraction operator in the ball $B_T^\alpha(q_0, \rho)$. Therefore, by the Banach theorem, the operator A has a unique fixed point in the ball $B_T^\alpha(q_0, \rho)$; i.e., there exists a unique solution of Eq. (24). The proof of the theorem is complete.

Let T be some positive number. Consider the set $\Omega(\gamma_0)$ ($\gamma_0 > 0$ is some fixed number) of functions (f, φ, g) for which all assumptions of Theorem 1 are satisfied and $\max\{\|f\|^{\alpha+2}, |\varphi|^{\alpha+2}, \|g\|^\alpha\} \le \gamma_0$. By $Q(\gamma_1)$ we denote the class of functions $q(x', t) \in C([0, T], H^\alpha(\mathbb{R}^2))$ satisfying the inequality $\|q\|^\alpha \le \gamma_1$ with some fixed positive number γ_0.

Theorem 2. *Let $(f, \varphi, g) \in \Omega(\gamma_0)$, $(\tilde{f}, \tilde{\varphi}, \tilde{g}) \in \Omega(\gamma_0)$, and $(q, \tilde{q}) \in Q(\gamma_1)$. Then the solution of the inverse problem satisfies the stability estimate*

$$\|q - \tilde{q}\|^\alpha \le c\big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\big), \tag{27}$$

where the constant c depends only on T, α, γ_0, and γ_1.

Proof. Using Eq. (23), we write out the equation for $\tilde{q}(x', t)$ and then compose the difference $q(x', t) - \tilde{q}(x', t)$. Further, estimating this expression and using inequalities (18) and (22), we obtain the inequality

$$|q - \tilde{q}|^\alpha(t) \le c_0\big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\big) + c_1 \int_0^t |q - \tilde{q}|^\alpha(\tau)\, d\tau, \quad t \in [0, T], \tag{28}$$

in which the constants c_0 and c_1 depend on the same constants as c. Based on this, by the Gronwall inequality we obtain the estimate

$$|q - \tilde{q}|^\alpha(t) \le c_0 \exp(c_1 t)\big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\big), \quad t \in [0, T],$$

which, in turn, implies the desired estimate (27) with the constant $c = c_0 \exp(c_1 t)$.

Theorem 2 obviously implies the uniqueness of the solution of the inverse problem.

Theorem 3. *Let the functions $q(x', t)$, $f(x, t)$, $\varphi(x)$, $g(x', t)$ and $\tilde{q}(x', t)$, $\tilde{f}(x, t)$, $\tilde{\varphi}(x)$, $\tilde{g}(x', t)$ satisfy the same conditions as in Theorem 2. If $f = \tilde{f}$, $\varphi = \tilde{\varphi}$, and $g = \tilde{g}$ for $(x, t) \in \Phi_T$, then one has the equality $q(x', t) = \tilde{q}(x', t)$, $x' \in \mathbb{R}^2$, $t \in [0, T]$.*

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