ISSN 1066-369X, Russian Mathematics, 2024, Vol. 68, No. 9, pp. 18–28. © Allerton Press, Inc., 2024.
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Inverse Problem for a Fourth-Order Differential Equation with the Fractional Caputo Operator

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Received September 8, 2023; revised April 23, 2024; accepted June 26, 2024

Abstract—In this paper we consider an initial-boundary value problem (direct problem) for a fourth order equation with the fractional Caputo derivative. Two inverse problems of determining the right-hand side of the equation by a given solution to the direct problem at some point are studied. The unknown of the first problem is a one-dimensional function depending on a spatial variable, while in the second problem a function depending on a time variable is found. Using eigenvalues and eigenfunctions, a solution to the direct problem is found in the form of Fourier series. Sufficient conditions are established for the given functions, under which the solution to this problem is classical. Using the results obtained for the direct problem and applying the method of integral equations, we study the inverse problems. Thus, the uniqueness and existence theorems of the direct and inverse problems are proved.

Keywords: initial-boundary value problem, inverse problem, fractional Caputo derivative, Mittag–Leffler function, eigenfunction, eigenvalue, uniqueness, existence

DOI: 10.3103/S1066369X24700567

INTRODUCTION

Extension of the field of integer derivatives led to the well-known mathematical field of fractional calculus. Since most dynamic phenomena evolve continuously, it becomes natural to extend the concept of integer-order derivatives to fractional-order derivatives. This theory has an ancient history and critical applications in various fields [1, 2]. The list of applications of fractional differential equations is growing rapidly in many areas such as the study of creep or relaxation of viscoelastic materials, control problems, plasma physics, and models of diffusion processes [3]. Delay differential equations occupy an important place in various fields of practical applications based on real life. These equations are also used to model systems with time delays, such as in control systems, high-speed processing, power systems, and communications [4]. Fractional differential equations with delay are used for more accurate modeling of dynamic systems, as well as natural phenomena [5].

At present, the theory of inverse problems for equations of mathematical physics has been studied quite widely. Inverse problems associated with classical second-order partial differential equations are investigated in [6]. The methods of proving local theorems of existence and uniqueness of solutions to inverse dynamic problems, theorems of uniqueness and conditional stability, as well as numerical approaches to solving the problems are considered, e.g., in works [7–14].

Until recently, integration and differentiation of fractional orders have received little attention, despite the wide range of possible applications. For example, fractional calculus is used in models of viscoelastic bodies, continuous media with memory, transformation of temperature and humidity in atmospheric layers, in diffusion equations and in other areas. Direct and inverse problems for fractional diffusion equations have been studied, for example, in works [15, 16]. In work [15] Ashurov and Mukhiddinova investigated the inverse problem of determining the right-hand side of the subdiffusion equation with the Riemann–Liouville fractional derivative, and in [16] Durdiev et al. considered a two-dimensional inverse problem for the fractional diffusion equation. In works [17–19] a number of interesting features of fractional subdiffusion equations are presented, indicating a certain similarity of these equations with second-order parabolic differential equations. In work [20] we proved the local existence and global uniqueness theorems and obtained an estimate of the stability of the solution to the problem of determining the reaction coefficient in the diffusion equation with a fractional time derivative. It should be noted that in works [21, 22] Agraval investigated initial-boundary value problems for a fourth-order fractional diffusion-wave equation.

In works [23—27] the inverse problems of finding space-dependent and time-dependent lower terms of the diffusion equation with the generalized fractional Riemann—Liouville derivative are investigated using the expansion in eigenfunctions of a non-self-adjoint spectral problem. The main results of these studies include existence and uniqueness theorems, as well as an estimate of the stability of the solution. In papers [28—31] inverse problems of determining the coefficients and the kernel in these equations were considered.

In the current paper, we consider a direct initial-boundary value problem of a fourth-order equation with the Caputo fractional derivative of order α, $1 < \alpha < 2$, and for this equation, we investigate the inverse problems for determining the right-hand side. It should be noted that when $\alpha \to 2-0$, the equation under consideration turns into an equation describing the bending transverse vibrations of a homogeneous beam under the action of an external force. Over the past few years, there has been increased interest in the study of linear and nonlinear initial-boundary value problems for the beam vibration equation [32—35]. Inverse problems of finding the stiffness coefficient of the basement and the right-hand side for the beam vibration equation are considered in works [36—38].

1. PROBLEM FORMULATION

Let us consider a fourth-order equation with a fractional derivative

$$ {}^{C}D_t^{\alpha}u(x,t) + a^2 u_{xxxx} = F(x,t), \tag{1}$$

where ${}^{C}D_t^{\alpha}u$ is the Caputo fractional derivative with respect to the variable t of the function $u(x,t)$ of order $\alpha \in (1,2]$ ([39], p. 90):

$$ {}^{C}D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}u_{\tau\tau}(x,\tau)d\tau, \quad {}^{c}D_t^{2}u(x,t) \equiv \frac{\partial^2}{\partial t^2}u(x,t). \tag{2}$$

Equation (1) is considered in a rectangular area

$$\Sigma = \{(x,t) \mid 0 < x < l, \ \ 0 < t < T\},$$

where l and T are given positive numbers, with the initial

$$u(x,0) = \varphi(x), \ \ u_t(x,0) = \psi(x), \ \ x \in [0,l], \tag{3}$$

and boundary

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = 0, \ \ t \in [0,T], \tag{4}$$

conditions.

In the direct problem, for given numbers a, l, and T and sufficiently smooth functions $f(x,t)$, $\varphi(x)$, and $\psi(x)$, we need to find a function

$$u(x,t) \in C^1\left(\bar{\Sigma}\right) \cap C^{(4,\alpha)}\left(\Sigma\right), \tag{5}$$

satisfying Eq. (1) at $(x,t) \in \Sigma$ and conditions (3) and (4). Here, $C^{(4,\alpha)}\left(\Sigma\right)$ is a class of functions that are continuous in Σ and four times continuously differentiable with respect to x in the domain Σ, for which there exists a continuous derivative ${}^{C}D_t^{\alpha}$.

Let us define the class of functions $C_\gamma^n[0,T]$ ([39], p. 199):

$$C_\gamma^n[0,T] = \{v(t)\colon t^\gamma v^{(n)}(t) \in C[0,T]\},$$

where $n \in \mathbb{N}$, $0 \le \gamma < 1$.

Our work is aimed at studying the following problems.

Inverse problem 1. Let $F(x,t) = f(x)g(t)$ and let $g(t)$ be a known function. Find a pair of functions $\{u(x,t), f(x)\} \in C^1(\overline{\Sigma}) \cap C^{(4,\alpha)}(\Sigma) \times C[0,l]$ satisfying conditions (1)–(4) and, in addition, the condition

$$u(x,t_0) = h(x), \quad 0 \le x \le l, \quad 0 < t_0 \le T, \tag{6}$$

where $h(x) \in C^1[0,l]$ and $t_0 \in (0,T]$ is a given number.

Inverse problem 2. Let $F(x,t) = f(x)g(t)$ and let $f(x)$ be a known function. Find a pair of functions $\{u(x,t), g(t)\} \in C^1(\overline{\Sigma}) \cap C^{(4,\alpha)}(\Sigma) \times AC[0,T]$ satisfying conditions (1)–(4) and the condition

$$u(x_0,t) = q(t), \quad 0 < x_0 < l, \quad 0 \le t \le T, \tag{7}$$

where $q(t) \in C^2[0,T]$ and $x_0 \in (0,l)$ is a given number.

2. STUDY OF THE SOLUTION TO THE DIRECT PROBLEM

To solve Eq. (1) with the initial (3) and boundary (4) conditions, we use the method of separation of variables. Putting $u(x,t) = X(x)T(t)$, we obtain the spectral problem for $X(x)$:

$$\begin{gathered} X^{(IV)} + \lambda X = 0, \\ X(0) = X(l) = X'(0) = X'(l) = 0, \end{gathered} \tag{8}$$

and the problem for $T(t)$:

$$\begin{gathered} {}^C D_t^\alpha T(t) - a^2 \lambda T(t) = F_n(t), \\ T(0) = \varphi_n, \quad T'(0) = \psi_n, \end{gathered} \tag{9}$$

where

$$\varphi_n = \int_0^l \varphi(x) Y_n(x) dx, \quad \psi_n = \int_0^l \psi(x) Y_n(x) dx,$$

$$F_n(t) = \int_0^l F(x,t) Y_n(x) dx,$$

$$Y_n(x) = \frac{X_n(x)}{\|X_n(x)\|}$$

is the orthonormal basis,

$$X_n(x) = \begin{cases} \dfrac{\sinh d_n\left(x - \frac{l}{2}\right)}{\cosh\left(\frac{d_n l}{2}\right)} - \dfrac{\sin d_n\left(x - \frac{l}{2}\right)}{\cos\left(\frac{d_n l}{2}\right)}, & n = 2k - 1; \\[2ex] \dfrac{\cosh d_n\left(x - \frac{l}{2}\right)}{\sinh\left(\frac{d_n l}{2}\right)} + \dfrac{\cos d_n\left(x - \frac{l}{2}\right)}{\sin\left(\frac{d_n l}{2}\right)}, & n = 2k, \end{cases}$$

$$\|X_n(x)\| = \sqrt{l}\left|\tan\frac{d_n l}{2}\right|$$

is defined as the solution to the spectral problem (8), which was investigated in work [40]. The solution to problem (9) was presented in ([39], p. 232). Based on this, we can write down the solution to the problem (1)–(4) in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) Y_n(x), \tag{10}$$

$$T_n(t) = \int_0^l u(x,t)Y_n(x)dx, \tag{11}$$

where

$$T_n(t) = \varphi_n E_\alpha\left[a^2\lambda_n t^\alpha\right] + \psi_n t E_{\alpha,2}\left[a^2\lambda_n t^\alpha\right] + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left[a^2\lambda_n (t-s)^\alpha\right] F_n(s)ds, \tag{12}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{13}$$

Here, $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function. Let us proceed to establishing the asymptotic properties of the function $E_{\alpha,\beta}(z)$ for large absolute values of the argument z.

Lemma 1 ([41], p. 136). *Let* $0<\alpha<2$, $\beta\in\mathbb{R}$ *be real constants, and let* μ *be a fixed number from the interval* $(\pi\alpha/2, \min\{\pi,\pi\alpha\})$. *Then the estimate holds*:

$$\left|E_{\alpha,\beta}(z)\right| \le \frac{M}{1+|z|}, \quad \mu \le |\arg z| \le \pi,$$

where M *is a constant independent of* z.

The values $\lambda_n = -d_n^4$ are eigenvalues of the spectral problem (8), where

$$d_n \approx \frac{\pi}{2l}(2n-1).$$

Because the system $Y_n(x)$ is complete in the space $L_2(0,l)$, we can prove the uniqueness of the solution to the problem (1)–(5). Indeed, suppose that there exist different functions $u_1(x,t)$ and $u_2(x,t)$ that are solutions to this problem. Then their difference $u(x,t) = u_1(x,t) - u_2(x,t)$ is a solution to the homogeneous problem (1)–(5), where $\varphi(x)\equiv 0$, $\psi(x)\equiv 0$, and $F(x,t)\equiv 0$. Hence, $\varphi_n \equiv 0$, $\psi_n\equiv 0$, and $F_n(t)\equiv 0$, and from (12) we get $T_n(t)\equiv 0$, which, because of (11), is equivalent to the equality

$$\int_0^l u(x,t)Y_n(x)dx = 0.$$

Because the system $Y_n(x)$ is complete in the space $L_2(0, l)$, the function $u(x,t)=0$ almost everywhere in $[0, l]$ and for any $t\in[0, T]$. Because due to condition (5) u is continuous in $\overline{\Sigma}$, $u(x,t)\equiv 0$ in $\overline{\Sigma}$. Thus, the uniqueness of the solution to the problem (1)–(5) is proven.

Now, we move on to the study of the solution to the direct problem. For this purpose, we find

$${}^C D_t^\alpha\left[T_n(t)\right] = {}^C D_t^\alpha\left(\varphi_n E_\alpha\left[a^2\lambda_n t^\alpha\right] + \psi_n t E_{\alpha,2}\left[a^2\lambda_n t^\alpha\right] + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left[a^2\lambda_n (t-s)^\alpha\right] F_n(s)ds\right),$$

$$1<\alpha<2.$$

For convenience, we introduce the notation

$$I_1 := {}^C D_t^\alpha\left(\varphi_n E_\alpha\left[a^2\lambda_n t^\alpha\right]\right) = \varphi_n {}^C D_t^\alpha\left(E_\alpha\left[a^2\lambda_n t^\alpha\right]\right), \tag{14}$$

$$I_2 := {}^C D_t^\alpha\left(\psi_n t E_{\alpha,2}\left[a^2\lambda_n t^\alpha\right]\right) = \psi_n {}^C D_t^\alpha\left(t E_{\alpha,2}\left[a^2\lambda_n t^\alpha\right]\right), \tag{15}$$

$$I_3 := {}^C D_t^\alpha\left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left[a^2\lambda_n (t-s)^\alpha\right] F_n(s)ds\right).$$

Let us start by calculating the fractional derivative in (14). To this end, according to (13), we find

$$ {}^{C}D_t^{\alpha} E_{\alpha}\left[a^2\lambda_n t^{\alpha}\right] = {}^{C}D_t^{\alpha}\sum_{k=0}^{\infty}\frac{\left(a^2\lambda_n t^{\alpha}\right)^k}{\Gamma(\alpha k+1)} = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{\partial^2}{\partial\tau^2}\sum_{k=0}^{\infty}\frac{\left(a^2\lambda_n \tau^{\alpha}\right)^k}{\Gamma(\alpha k+1)} $$
$$ = a^2\lambda_n E_{\alpha}\left[a^2\lambda_n t^{\alpha}\right]. $$

This leads to

Proposition 1. *For* $1<\alpha<2$ *and* $\lambda>0$ *the relation holds*:

$$ {}^{C}D_t^{\alpha} E_{\alpha}[\lambda t^{\alpha}] = \lambda E_{\alpha}[\lambda t^{\alpha}], \quad 0<t\le T. $$

Moving on to (15), we have

$$ {}^{C}D_t^{\alpha} t E_{\alpha,2}\left[a^2\lambda_n t^{\alpha}\right] = {}^{C}D_t^{\alpha}\sum_{k=0}^{\infty}\frac{\left(a^2\lambda_n\right)^k t^{\alpha k+1}}{\Gamma(\alpha k+2)} $$
$$ = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{\partial^2}{\partial\tau^2}\sum_{k=0}^{\infty}\frac{\left(a^2\lambda_n\right)^k \tau^{\alpha k+1}}{\Gamma(\alpha k+2)}d\tau = a^2\lambda_n t E_{\alpha,2}\left[a^2\lambda_n t^{\alpha}\right]. $$

This implies the following proposition

Proposition 2. *For* $1<\alpha<2,\ \beta>0$, *and* $\lambda>0$ *the relation holds*:

$$ {}^{C}D_t^{\alpha} t^{\beta} E_{\alpha,\alpha}[\lambda t^{\alpha}] = \lambda t^{\beta} E_{\alpha,\alpha}[\lambda t^{\alpha}], \quad 0<t\le T. $$

Proposition 3 ([27])**.** *For* $1<\alpha<2$ *and* $\lambda>0$, *if* $f(t)\in AC[0,T]$, *then the equality holds*:

$$ {}^{C}D_t^{\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda(t-s)^{\alpha}\right]f(s)ds = f(t) - \lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda(t-s)^{\alpha}\right]f(s)ds. $$

Let us show that series (10) is the solution to the problem (1)–(5). To do this, we first make sure that the sum of series (10) is a continuous function in a closed domain $\overline{\Sigma}$. Evaluating the expression for $T_n(t)$, we find

$$ |T_n(t)| \le \left|\varphi_n E_{\alpha}[a^2\lambda_n t^{\alpha}]\right| + \left|\psi_n t E_{\alpha,2}\left[a^2\lambda_n t^{\alpha}\right]\right| + \left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[a^2\lambda_n (t-s)^{\alpha}\right]F_n(s)ds\right|. $$

Further, according to Lemma 1, we obtain

$$ |T_n(t)| \le \widetilde{C}_1\left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right), $$

where $F_n(t_m) = \max\limits_{0\le t\le T}|F_n(t)|$ and $\widetilde{C}_1 = \widetilde{C}_1(\alpha,T)$; below, $\widetilde{C}_i$ depends only on α and T.

Now, let us find an estimate for the function ${}^{c}D_t^{\alpha}T_n(t)$ in a similar manner:

$$ \left|{}^{C}D_t^{\alpha}T_n(t)\right| \le \left|a^2\lambda_n T_n(t) + F_n(t)\right| \le \widetilde{C}_2\lambda_n\left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right). $$

Lemma 2. *For any* $t\in[0,T]$ *the estimates hold*:

$$ |T_n(t)| \le \widetilde{C}_1\left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right), \tag{16} $$

$$ \left|{}^{C}D_t^{\alpha}T_n(t)\right| \le \widetilde{C}_2 n^4\left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right). \tag{17} $$

By formal term-by-term differentiation of (10), we derive the series

$$ {}^{C}D_t^{\alpha}u(x,t) = \sum_{n=1}^{\infty}{}^{C}D_t^{\alpha}T_n(t)Y_n(x), \tag{18} $$

$$ u_{xxxx}(x,t) = \sum_{n=1}^{\infty}T_n(t)X_n^{(4)}(x) = \sum_{n=1}^{\infty}d_n^4 T_n(t)Y_n(x). \tag{19} $$

Because $|Y_n(x)| \le 1$, from series (10) and from Lemma 2 we obtain the estimate

$$|u(x,t)| \le \widetilde{C}_1 \sum_{n=1}^{\infty} \left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right).$$

Identically, for series (18) and (19), by Lemma 2 we have

$$\left|{}^C D_t^{\alpha} u(x,t)\right| \le \widetilde{C}_3 \sum_{n=1}^{\infty} \lambda_n \left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right),$$

$$|u_{xxxx}(x,t)| \le \sum_{n=1}^{\infty} d_n^4 |T_n(t)| \le \widetilde{C}_2 \sum_{n=1}^{\infty} n^4 \left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right).$$

By Lemma 2, series (10), (18), and (19) for any $(x,t) \in \overline{\Sigma}$ are majorized by

$$\widetilde{C}_3 \sum_{n=1}^{\infty} n^4 \left(|\varphi_n| + |\psi_n| + |F_n(t_m)|\right).$$

For the last series to converge we need

Lemma 3. *If the functions* $\varphi(x)$, $\psi(x)$, *and* $F(x,t)$ *satisfy the conditions*

$$\varphi(x) \in C^6[0,l],\ \ \varphi(0) = \varphi(l) = \varphi'(0) = \varphi'(l) = \varphi'''(0) = \varphi'''(l) = \varphi^{(4)}(0) = \varphi^{(4)}(l)$$

$$= \varphi^{(5)}(0) = \varphi^{(5)}(l) = 0,$$

$$\psi(x) \in C^6[0,l],\ \ \psi(0) = \psi(l) = \psi'(0) = \psi'(l) = \psi'''(0) = \psi'''(l) = \psi^{(4)}(0) = \psi^{(4)}(l)$$

$$= \psi^{(5)}(0) = \psi^{(5)}(l) = 0,$$

$$F(\cdot,t) \in C^6[0,l],\ \ F(0,t) = F(l,t) = F_x(0,t) = F_x(l,t) = F_{xxx}(0,t) = F_{xxx}(l,t)$$

$$= F_{xxxx}(0,t) = F_{xxxx}(l,t) = F_{xxxxx}(0,t) = F_{xxxxx}(l,t) = 0$$

for any $0 \le t \le T$, *then the following relations hold*:

$$\varphi_n = \frac{1}{d_n^6}\varphi_n^{(6)},\ \ \psi_n = \frac{1}{d_n^6}\psi_n^{(6)},\ \ F_n(t) = \frac{1}{d_n^6}F_n^{(6)}(t),$$

where

$$\varphi_n^{(6)} = \int_0^l \varphi^{(6)}(x) Y_n(x) dx,\ \ \psi_n^{(6)} = \int_0^l \psi^{(6)}(x) Y_n(x) dx,\ \ F_n^{(6)}(t) = \int_0^l \frac{\partial^6 F}{\partial x^6}(x,t) Y_n(x) dx.$$

By Lemma 3, series (10), (18), and (19) are majorized by a converging numerical series

$$\widetilde{C}_4 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\left|\varphi_n^{(6)}\right| + \left|\psi_n^{(6)}\right| + \left|F_n^{(6)}(t_m)\right|\right),$$

that is, they converge absolutely and uniformly in $\overline{\Sigma}$. Therefore, the sum of the series satisfies all the conditions of the problem (1)–(4). Thus, the following theorem is proved.

Theorem 1. *Let the function* $F(x,\cdot) \in AC[0,T]$ *for any* $x \in [0,l]$. *If the functions* $\varphi(x)$, $\psi(x)$, *and* $F(x,t)$ *satisfy the conditions of Lemma* 3, *then there exists a unique solution to the problem* (1)–(4), *which is defined as the sum of series* (10), *where the coefficients are found using formula* (12).

3. MAIN RESULT

Let us study the inverse problem 1. Suppose that $F(x,t) = f(x)g(t)$. Then series (10) can be rewritten as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)Y_n(x)$$
$$= \sum_{n=1}^{\infty}\left[\varphi_n E_\alpha\left[a^2\lambda_n t^\alpha\right] + \psi_n t E_{\alpha,2}\left[a^2\lambda_n t^\alpha\right] + f_n\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[a^2\lambda_n(t-s)^\alpha\right]g(s)ds\right]Y_n(x), \tag{20}$$

where

$$f_n = \int_0^l f(x)Y_n(x)dx.$$

Using condition (6), we get

$$u(x,t_0) = \sum_{n=1}^{\infty} T_n(t_0)Y_n(x) = h(x) = \sum_{n=1}^{\infty} h_n Y_n(x), \tag{21}$$

where

$$T_n(t_0) = \varphi_n E_\alpha\left[a^2\lambda_n t_0^\alpha\right] + \psi_n t_0 E_{\alpha,2}\left[a^2\lambda_n t_0^\alpha\right] + f_n\int_0^{t_0}(t_0-s)^{\alpha-1}E_{\alpha,\alpha}\left[a^2\lambda_n(t_0-s)^\alpha\right]g(s)ds.$$

Let $g(t) \equiv 1$. Then equality (21) becomes

$$\left(\int_0^{t_0}(t_0-s)^{\alpha-1}E_{\alpha,\alpha}\left[a^2\lambda_n(t_0-s)^\alpha\right]g(s)ds\right)f_n = h_n - \varphi_n E_\alpha\left[a^2\lambda_n t_0^\alpha\right] - \psi_n t_0 E_{\alpha,2}\left[a^2\lambda_n t_0^\alpha\right]. \tag{22}$$

From here we find the unknown coefficients

$$f_n = \frac{1}{\delta_n(t_0)}\left(h_n - \varphi_n E_\alpha\left[a^2\lambda_n t_0^\alpha\right] - \psi_n t_0 E_{\alpha,2}\left[a^2\lambda_n t_0^\alpha\right]\right) \tag{23}$$

provided that for all $n \in \mathbb{N}$

$$\delta_n(t_0) = \frac{1}{a^2\lambda_n^2}\left(-E_\alpha\left[a^2\lambda_n t_0^\alpha\right] + 1\right) \neq 0,$$

where we used $\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(cs^\alpha)ds = \frac{1}{c}\left(-E_\alpha\left[ct^\alpha\right]+1\right)$ for $c \neq 0$ and $0 \le E_\alpha(a^2\lambda_n t_0^\alpha) < 1$. By virtue of the theorem of uniqueness of the solution to problem 1, the expression $\delta_n(t_0)$ is not equal to zero for all $n \in \mathbb{N}$. Indeed, let $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$, and $h(x) \equiv 0$. Then $\varphi_n = \psi_n = h_n \equiv 0$. If for some $n = p \in \mathbb{N}$ the expression $\delta_p(t_0) = 0$, then from Eq. (22) it follows that f_p is an arbitrary constant, generally speaking, not equal to zero. Then the inverse problem 1 with zero boundary conditions at $g(t) = 1$ has a nonzero solution

$$u(x,t) = \frac{f_p}{a^2\lambda_p}\left(E_\alpha\left[a^2\lambda_n t_0^\alpha\right] - 1\right),\quad f(x) = f_p X_p(x).$$

Having substituted Eq. (23) into Eq. (20), we find the function $u(x,t)$.

Now, suppose that $g(t) \neq 1$, $g(t) \in C[0,T]$, and assume that $g(t) \ge g_0 = \text{const} > 0$. Then

$$g_n(t) = g(\xi)\int_0^{t_0}(t-s)^{\alpha-1}E_{\alpha,\alpha}[a^2\lambda_n(t-s)^\alpha]ds = g(\xi)\frac{E_\alpha[a^2\lambda_n t_0^\alpha]-1}{a^2\lambda_n},$$

where $0 < \xi < t_0$.

Theorem 2. *Let* $h(x) \in C^1[0,l]$, $\varphi(x) \in C^1[0,l]$, $\psi(x) \in C^1[0,l]$, *and* $g(t) = 1$ *or* $g(t) \in C[0,T]$, $|g(x)| \ge g_0 > 0$, *then the inverse problem* (1)–(6) *in the space of continuous functions has a unique solution.*

Let us study the inverse problem 2.

To do this, we use condition (7) and get

$$u(x_0,t) = \sum_{n=1}^{\infty} T_n(t) Y_n(x_0) = \sum_{n=1}^{\infty} \left(\varphi_n E_\alpha \left[a^2 \lambda_n t^\alpha \right] + \psi_n t E_{\alpha,2} \left[a^2 \lambda_n t^\alpha \right] + f_n g_n(t) \right) Y_n(x_0) = q(t), \tag{24}$$

where

$$g_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left[a^2 \lambda_n (t-s)^\alpha \right] g(s) ds.$$

Resolving (24) for $g_n(t)$, we obtain the Volterra integral equation of the first kind

$$\int_0^t \sum_{n=1}^{\infty} f_n (t-s)^{\alpha-1} E_{\alpha,\alpha} \left[a^2 \lambda_n (t-s)^\alpha \right] g(s) Y_n(x_0) ds = \tilde{q}(t), \tag{25}$$

where

$$\tilde{q}(t) = q(t) - \sum_{n=1}^{\infty} \left(\varphi_n E_\alpha \left[a^2 \lambda_n t^\alpha \right] + \psi_n t E_{\alpha,2} \left[a^2 \lambda_n t^\alpha \right] \right) Y_n(x_0). \tag{26}$$

First, let us show that $\tilde{q}(t) \in C^2[0,T]$. To do this, we find the first derivative of (26) with respect to t:

$$\frac{d}{dt} \tilde{q}(t) = q'(t) + \sum_{n=1}^{\infty} \left(\varphi_n a^2 \lambda_n t^{\alpha-1} E_{\alpha,\alpha}[a^2 \lambda_n t^\alpha] + \psi_n E_\alpha [a^2 \lambda_n t^\alpha] \right) Y_n(x_0).$$

Let $q'(t) \in C[0,T]$, then the series in (26) is majorized by a number series

$$\sum_{n=1}^{\infty} \left(n^4 |\varphi_n| + |\psi_n| \right).$$

This series converges by Lemma 3; hence, we obtain $d/dt[\tilde{q}(t)] \in C[0,T]$. Next, let us look at the function $p(t) = t^\gamma \frac{d^2}{dt^2} \tilde{q}(t)$. Calculating the derivative of the function $\frac{d}{dt} \tilde{q}(t)$, we have

$$p(t) = t^\gamma q''(t) - \sum_{n=1}^{\infty} \left(\varphi_n a^2 \lambda_n^2 t^{\gamma+\alpha-2} E_{\alpha,\alpha-1} \left[a^2 \lambda_n t^\alpha \right] + \psi_n a^2 \lambda_n t^{\gamma+\alpha-1} E_{\alpha,\alpha} \left[a^2 \lambda_n t^\alpha \right] \right) Y_n(x_0),$$

where $2 - \alpha \le \gamma < 1$ and γ is a given number. The inclusion $d^2/dt^2[\tilde{q}(t)] \in C_\gamma[0,T]$ follows from the following easily proven statement.

Lemma 4. *Let* $\varphi(x) \in C^{10}[0,l]$, $\varphi(0) = \varphi(l) = \varphi'(0) = \varphi'(l) = \varphi'''(0) = \varphi'''(l) = \varphi^{(4)}(0) = \varphi^{(4)}(l) = \varphi^{(5)}(0) = \varphi^{(5)}(l) = \varphi^{(7)}(0) = \varphi^{(7)}(l) = \varphi^{(8)}(0) = \varphi^{(8)}(l) = \varphi^{(9)}(0) = \varphi^{(9)}(l) = 0$, *and suppose that the function* $\psi(x)$ *is subject to the conditions of Lemma* 3. *Then the formula holds*

$$\varphi_n = \frac{1}{d_n^{10}} \varphi_n^{(10)},$$

where

$$\varphi_n^{(10)} = \int_0^l \varphi^{(10)}(x) Y_n(x) dx.$$

Because it is an ill-posed problem to find the unknown function in the Volterra integral equation of the first kind, we take the fractional Caputo derivative of the integral equation (25) and use Proposition 3 to get

$$\sum_{n=1}^{\infty} f_n Y_n(x_0) \left(g(t) - a^2 \lambda_n \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left[a^2 \lambda_n (t-s)^\alpha \right] g(s) ds \right) = {}^C D_t^\alpha \tilde{q}(t). \tag{27}$$

Hence, $\sum_{n=1}^{\infty} f_n Y_n(x_0) = f(x_0)$. If $f(x_0) \ne 0$ and the kernel of the integral equation is a continuous function, then, defining the desired function $g(t)$ from (27) and, after that, substituting $g(t)$ into (24), we find the function $u(x,t)$.

Theorem 3. *Let the conditions of Lemma* 4 *be satisfied. Then if*

$$f(x_0) \neq 0, \quad q(t) \in C_\gamma^2[0,T], \quad \varphi(x_0) = q(0), \quad \psi(x_0) = q'(0),$$

then Eq. (27) *has a unique solution* $g(t)$ *in the class of functions* $AC[0,T]$.

The theorem follows from the theory of Volterra integral equations of the second kind.

FUNDING

The work was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement no. 075-02-2024-1447.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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