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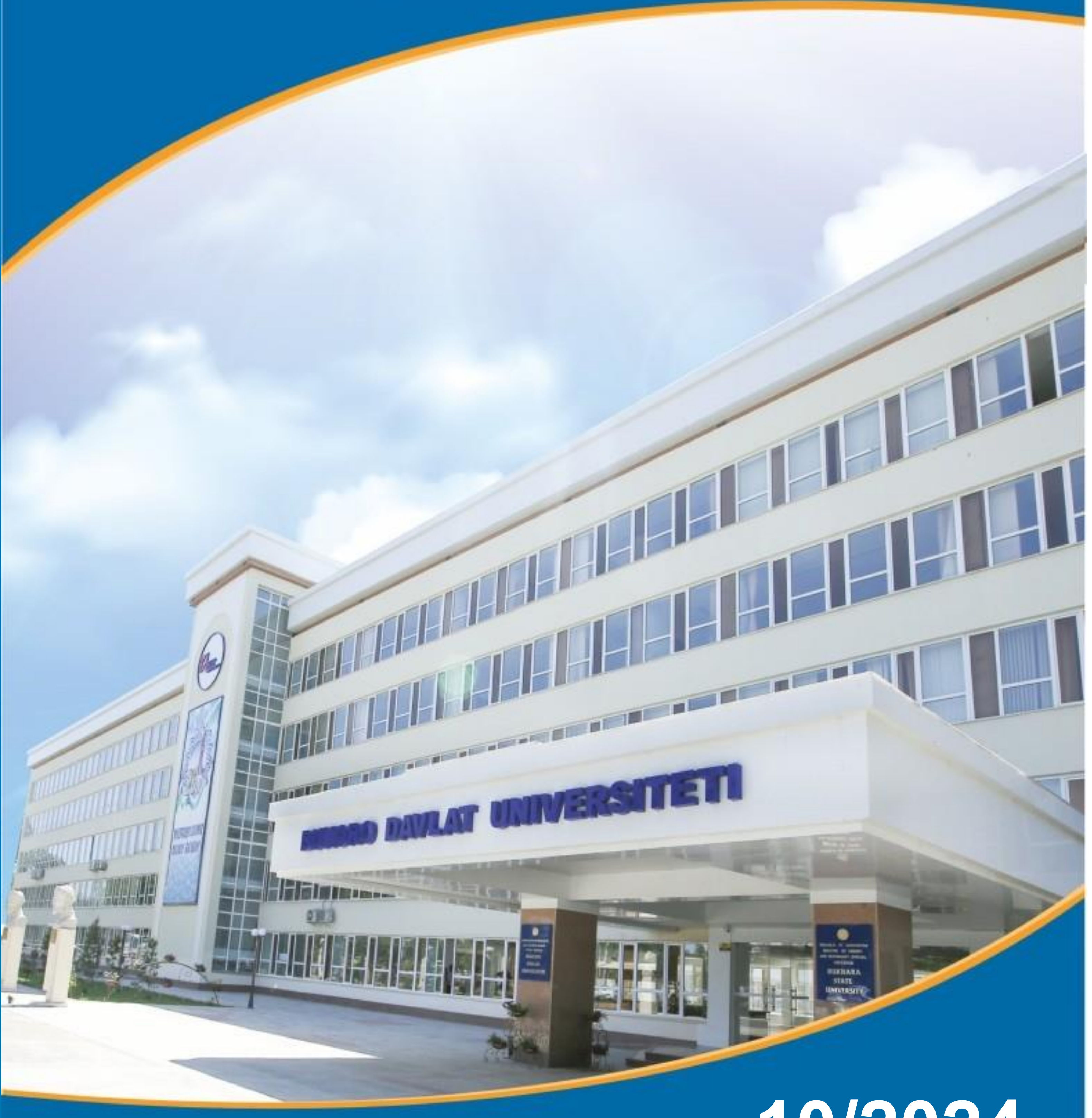

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Elektron manzil: nashriyot_buxdu@buxdu.uz

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UDC 51

AN INVESTIGATION INTO THE INHOMOGENEOUS INTEGRO-DIFFERENTIAL EQUATION FOR THE TRANSVERSE VIBRATION OF A BEAM

Durdiev Umidjon Durdimuratovich
Head of the department Differential equations, Bukhara State university,
Senior Research Fellow at the Institute of Mathematics named after V.I.Romanovskiy
at the Academy of sciences of the Republic of Uzbekistan
umidjan93@list.ru, u.d.durdiev@buxdu.uz

Abstract. *In this paper, we investigate the initial-boundary value problem associated with the inhomogeneous integro-differential equation governing the transverse vibrations of a beam. We subsequently examine the vibration operator of the homogeneous beam utilizing the method of separation of variables, focusing on the eigenvalues and eigenfunctions. This approach effectively transforms the problem into a Fredholm integral equation of the second kind. Leveraging the completeness of the eigenfunction system, we establish a unique theorem, which is critical for ensuring the solvability of the problem. Furthermore, we employ the method of successive approximations to derive solutions for these equations. This methodological framework not only facilitates the resolution of the integro-differential equation but also allows us to prove an existence theorem, thereby affirming the robustness of the solutions obtained.*

Keywords: *intego-differential, initial-boundary, inhomogeneous, eigenfunctions, separation of variables, Fredholm integral equation, unique, existence.*

ИССЛЕДОВАНИЕ НЕОДНОРОДНОГО ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ДЛЯ ПОПЕРЕЧНЫХ КОЛЕБАНИЙ БАЛКИ

Аннотация. *В данной работе исследуется начально-краевая задача, связанная с неоднородным интегро-дифференциальным уравнением, управляющим поперечными колебаниями балки. Затем мы исследуем оператор колебаний однородной балки с помощью метода разделения переменных, сосредоточив внимание на собственных значениях и собственных функциях. Такой подход эффективно преобразует задачу в интегральное уравнение Фредгольма второго рода. Опираясь на полноту системы собственных функций, мы устанавливаем теорему о единственности, которая является критической для обеспечения разрешимости задачи. Кроме того, мы используем метод последовательных приближений для получения решений этих уравнений. Эта методологическая основа не только облегчает решение интегро-дифференциального уравнения, но и позволяет доказать теорему существования, тем самым подтверждая устойчивость полученных решений.*

Ключевые слова: *интегро-дифференциальное, начально-граничное, неоднородное, собственные функции, разделение переменных, интегральное уравнение Фредгольма, единственность, существование.*

BALKANING KO'NDALANG TEBRANISHLARI UCHUN BIR JINSLI BO‘LMAGAN INTEGRAL-DIFFERENSIAL TENGLAMANI O'RGANISH

Annotatsiya. *Ushbu ish balkaning ko'ndalang tebranishlarini boshqaradigan bir jinsli bo‘lmagan integral-differensial tenglama bilan bog'liq bo'lgan boshlang'ich-chegaraviy masalani o'rganadi. Keyin biz o'zgaruvchilarni ajratish usuli yordamida bir jinsli balkaning tebranish operatorini o'rganamiz, xos qiymatlari va xos funktsiyalariga e'tibor qaratamiz. Ushbu yondashuv masalani ikkinchi turdagi Fredgolm integral tenglamasiga samarali o'zgartiradi. Xos funksiyalar sistamasining to'liqligiga asoslanib, biz masalaning yechimini ta'minlash uchun juda muhim bo'lgan yagonalik teoremasini o'rnatamiz. Bundan tashqari, biz ushbu tenglamalarning yechimlarini olish uchun ketma-ket yaqinlashtirish usulidan foydalanamiz. Ushbu uslubiy asos nafaqat integral-differensial tenglamani yechishni osonlashtiradi, balki mavjudlik teoremasini isbotlashga imkon beradi.*

Kalit so'zlar: *integral-differensial, boshlang'ich-chegaraviy, bir jinsli bo‘lmagan, xos funksiyalar, o'zgaruvchilarni ajratish, Fredgolm integral tenglamasi, yagonalik, mavjudlik.*

Introduction and statement of the problem. The investigation of transverse vibrations in beams is a critical aspect of structural engineering, where the dynamic behavior of beams under various loading conditions plays a pivotal role in ensuring safety and performance. Among the various mathematical models employed to describe these vibrations, the integro-differential equation stands out for its ability to capture both immediate and historical effects of forces acting on the beam.

The integro-differential equation for the transverse vibration of a beam incorporates not only the classical elements of beam theory but also the complexities introduced by inhomogeneous loading and the time-dependent nature of vibrations. This equation provides a comprehensive framework for analyzing the behavior of beams subjected to distributed loads, accounting for the interactions between various points along the beam over time.

This study focuses on the integro-differential equation governing the transverse vibrations of a beam. By examining its theoretical foundations and employing advanced solution techniques, such as separation of variables and successive approximations, we aim to uncover the intricate relationships between the beam's response and the external forces applied to it. The findings of this investigation will enhance our understanding of beam dynamics, contributing valuable insights for the design and analysis of structural elements in practical engineering applications.

We investigate an inverse problem aimed at identifying the unknown kernel $k(t)$ as well as the transverse bending vibrations $u(x,t)$ of a homogeneous beam of length l. These variables are governed by a fourth-order partial differential equation.

$$u_{tt} + a^2\, u_{xxxx} = \int_0^t k(\tau)u(x,t-\tau)d\tau \; + f(x,t), \tag{1}$$
$$(x,t) \in \; (0,l) \times \; (0,T] =: G,$$

with the initial conditions

$$u|_{t=0} = \; \varphi(x), \qquad u_t|_{t=0} = \psi(x), \qquad x \in [0,l], \tag{2}$$

and boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \qquad t \in [0,T]. \tag{3}$$

Problem. The problem requires to define the function

$$u(x,t) \in \; C_{x,t}^{4,2}\;(G) \cap \; C_{x,t}^{2,1}(\bar{G}), \tag{4}$$

satisfying relations (1) – (3) with the known positive numbers a, l, T and sufficiently smooth functions $\varphi(x)$, $\psi(x)$, $k(t)$, $f(x,t)$.

Study of the problem. The solution of the direct problem (1) – (4) will be found in the following form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x), \tag{5}$$

where $X(x)$ is the solution to the following problem:

$$X^4(x) + \lambda\, X(x) = 0, \qquad 0 < x < l,$$
$$X(0) = X''(0) = X(l) = X''(l) = 0.$$

Finding these solutions we get

$$X_n(x) = \sqrt{\frac{2}{l}} \sin \mu_n\, x, \qquad \lambda_n = -\mu_n^4 = -\left(\frac{\pi n}{l}\right)^4, \tag{6}$$

$$u_n(t) = \sqrt{\frac{2}{l}} \int_0^l u(x,t) \sin \mu_n\, x dx. \tag{7}$$

Formally, from (5) by term-by-term differentiation we compose the series

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} u_n''(t)X_n(x), \tag{8}$$

$$u_{xxxx}(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n^{(4)}(x) = \sum_{n=1}^{\infty} \mu_n^4\, u_n(t)X_n(x), \tag{9}$$

By applying the formal scheme of the Fourier method and utilizing equations (1) and (2), we derive the result

$$u_n''(t) + a^2\mu_n^4\, u_n(t) = \int_0^t k(\tau)u_n(t-\tau)d\tau \ + f_n(t), \qquad n = 1,2,\dots , \qquad 0 < t \le T, \tag{10}$$

$$u_n(0) = \varphi_n, \quad u_n'(0) = \psi_n, \quad n = 1,2,\dots, \tag{11}$$

where

$$\varphi_n = \int_0^l \varphi(x)X_n(x)\,dx, \qquad \psi_n = \int_0^l \psi(x)X_n(x)\,dx, \qquad f_n(t) = \int_0^l f(x,t)X_n(x)dx. \tag{12}$$

Using the method of [7], we present the solution of the problem (10), (11) as an integral equation

$$u_n(t) = q(t) + \frac{1}{a\mu_n^2}\int_0^t \sin[a\mu_n^2(t-\tau)]\int_0^\tau k(s)u_n(\tau-s)dsd\tau, \tag{13}$$

where

$$q(t) = \ \varphi_n \cos a\,\mu_n^2\,t \ + \frac{\psi_n}{a\mu_n^2}\sin a\mu_n^2\,t \ + \frac{1}{a\mu_n^2}\int_0^t f_n(s)\sin[a\mu_n^2(t-\tau)]ds.$$

Theorem 1. *If there exists a function $u(x,t)$ satisfying relations (1)-(4), then it is unique.*

Proof. Based on the completeness of the system $X_n(x)$ in the space $L_2(0,l)$ we can establish the uniqueness of the solution to the problem defined by equations (1) – (4). Assume, for contradiction, that there exist two distinct functions $u_1(x,t)$ and $u_2(x,t)$ that serve as solutions to this problem. The difference $u(x,t) = u_1(x,t) - u_2(x,t)$ then becomes a solution to the homogeneous problem (1) – (4), where $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$, $f(x,t) \equiv 0$. Consequently, we have $\varphi_n \equiv 0$, $\psi_n \equiv 0$, $f_n(t) \equiv 0$.

From equation (13), we obtain $u_n \equiv 0$, as u_n satisfies the homogeneous equation:

$$u_n(t) = \frac{1}{a\mu_n^2}\int_0^t \sin[a\mu_n^2(t-\tau)]\int_0^\tau k(s)u_n(\tau-s)dsd\tau$$

This, in conjunction with equation (7), is equivalent to the condition:

$$\int_0^l u(x,t)X_n(x)dx = 0$$

Given the completeness of the system $Y_n(x)$ in the space $L_2(0,l)$, we conclude that the function $u(x,t)$ must be zero almost everywhere in the interval $[0,l]$ and at any $t \in [0,T]$. Furthermore, due to the continuity of u on $\bar{G}$, as stipulated by condition (4), we deduce that $u(x,t) \equiv 0$ on $\bar{G}$.

Thus, we have proven the uniqueness of the solution to the problem defined by equations (1) – (4).

For each fixed n, equation (13) represents a Volterra integral equation of the second kind concerning v_n . According to the established theory of integral equations, provided that the functions $\varphi(x)$, $\psi(x)$, $k(t)$, and $f(x,t)$ satisfy the necessary conditions, this equation has a unique solution.

The solution to this integral equation can be effectively obtained using the method of successive approximations. This iterative method constructs a sequence of approximations that converges to the unique solution of the equation, thereby providing a practical approach for solving v_n in this context.

Furthermore, from (13) we can obtain an estimate for $u_n(t)$:

$$|u_n(t)| \le |q(t)| + \left|\frac{1}{a\mu_n^2}\int_0^t \sin[a\mu_n^2(t-\tau)]\int_0^\tau k(s)u_n(\tau-s)dsd\tau\right|,$$

since

$$|q(t)| \le |\varphi_n \cos a\,\mu_n^2\,t| + \left|\frac{\psi_n}{a\mu_n^2}\sin a\mu_n^2\,t\right| + \left|\frac{1}{a\mu_n^2}\int_0^t f_n(s)\sin[a\mu_n^2(t-\tau)]ds\right|$$
$$\le |\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{T}{a\mu_n^2}\|f_n\|.$$

Thus, we have

$$|u_n(t)| \le |\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{T}{a\mu_n^2}\|f_n\| + \frac{\|k\|}{a\mu_n^2}\left|\int_0^t u_n(s)\int_0^{t-s} \sin a\mu_n^2\tau d\tau ds\right|$$

$$\le |\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{T}{a\mu_n^2}\|f_n\| + \frac{2\|k\|}{a^2\mu_n^4}\int_0^t |u_n(s)|ds, \qquad t\in[0,T],$$

where $\|k\| = \max_{t\in[0,T]}|k(t)|$, $\|f_n\| = \max_{t\in[0,T]}|f_n(t)|$.

Hence, by virtue of Gronwall's inequality, we can conclude that the solution $u_n(t)$ satisfies:

$$|u_n(t)| \le \left(|\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{T}{a\mu_n^2}\|f_n\|\right) exp\left\{\frac{2T\|k\|}{a^2\mu_n^4}\right\}, \qquad t\in[0,T],\ \ n=1,2,\dots. \tag{14}$$

$$|u_n''(t)| \le \ \|f_n\| + \left(|\varphi_n| + \frac{1}{a\mu_n^2}|\psi_n| + \frac{T}{a\mu_n^2}\|f_n\|\right)(\|k\|T + a^2\mu_n^4) exp\left\{\frac{2T\|k\|}{a^2\mu_n^4}\right\}.$$

Thus, the following statement is true:

Lemma 1. *For any $t\in[0,T]$ the following estimates*

$$|u_n(t)| \le \ C_1\left(|\varphi_n| + \frac{1}{n^2}|\psi_n| + \frac{1}{n^2}\|f_n\|\right), \tag{15}$$

$$|u_n''(t)| \le \ C_2\,(n^4|\varphi_n| + n^2|\psi_n| + n^2\|f_n\|), \tag{16}$$

are valid, where C_i, $i=1,2$ are positive constants depending on T, l, and $\|k\|$.

The series (5), (8) and (9) for any $(x,t)\in\ \bar{G}$ based on Lemma 1 are majorized by the series

$$C_3\sum_{n=1}^{\infty}(n^4|\varphi_n| + n^2|\psi_n| + n^2\|f_n\|), \tag{17}$$

where the constant C_3 depends on T, l, and $\|k\|$.

Lemma 2. *Under the conditions*

$$\varphi(x)\in C^5[0,l], \qquad \varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=\varphi^{(4)}(0)=\varphi^{(4)}(l)=0,$$

$$\psi(x)\in C^3[0,l], \qquad \psi(0)=\psi(l)=\psi''(0)=\psi''(l)=0,$$

$$f(x,t)\in C^{3,1}_{x,t}(\bar{G}), \quad f(0,t)=f(l,t)=f_{xx}(0,t)=f_{xx}(l,t)=0, \quad 0\le t\le T,$$

one has the relations

$$\varphi_n = \frac{1}{\mu_n^5}\varphi_n^{(5)}, \quad \psi_n = -\frac{1}{\mu_n^3}\psi_n^{(3)}, \quad f_n(t) = -\frac{1}{\mu_n^3}f_n^{(3)}(t), \tag{18}$$

where

$$\varphi_n^{(5)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(5)}(x)\,cos\,\mu_n x\,dx, \quad \psi_n^{(5)} = \sqrt{\frac{2}{l}}\int_0^l \psi^{(3)}(x)\,cos\,\mu_n x\,dx,$$

$$f_n^{(3)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\,cos\,\mu_n x\,dx,$$

with the estimates

$$\sum_{n=1}^{\infty}\left|\varphi_n^{(5)}\right|^2 \le \left\|\varphi^{(5)}(x)\right\|_{L_2[0,l]}, \quad \sum_{n=1}^{\infty}\left|\psi_n^{(3)}\right|^2 \le \left\|\psi^{(3)}(x)\right\|_{L_2[0,l]},$$

$$\sum_{n=1}^{\infty}\left|f_n^{(3)}(t)\right|^2 \le \left\|f_{xxx}^{(3)}(x,t)\right\|_{L_2[0,l]}. \tag{19}$$

Proof. We integrate equation (13) by parts multiple times: specifically, we perform integration by parts on the integral involving the functions $f(x,t)$ and $\psi(x)$ three times, and on the integral involving the function $\varphi(x)$ five times. Under the conditions specified in Lemma 2, we arrive at the equalities stated in (18). The inequality presented in (19) corresponds to the Bessel inequality for the coefficients of the Fourier expansions of the functions $\varphi_n^{(5)}$, $\psi_n^{(3)}$ and $F_n^{(3)}(t)$ in the cosine system $\left\{\sqrt{\frac{2}{l}}\cos\mu_n x\right\}$ on the interval $[0,l]$. This inequality reflects the relationship between the coefficients derived from the Fourier expansions and demonstrates the convergence properties of the series associated with these functions, thus reinforcing the theoretical foundation for our analysis.

It is important to note that if the conditions of Lemma 2 are satisfied, the series in equation (17) converge. To demonstrate this convergence for the first series, we can apply a similar approach to the other series as well. Specifically, using equation (18) and the Cauchy-Bunyakovsky inequality, we obtain:

$$\sum_{n=1}^{\infty} n^4|\varphi_n| = \sum_{n=1}^{\infty}\frac{1}{n}\left|\varphi_n^{(5)}\right| \le \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}\left|\varphi_n^{(5)}\right|^2}.$$

Thus, according to the first inequality in (19), we conclude that the series $\sum_{n=1}^{\infty} n^4|\varphi_n|$ converges, confirming the convergence of the numerical series as desired.

If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ meet the conditions outlined in Lemma 2.2, then, based on inequalities (18) and (19), the series (17) converge. Consequently, this implies that the series in (5), (8) and (9) converge absolutely and uniformly within the rectangle $\bar{G}$. As a result of this uniform convergence, we can assert that the sum of the series in (5) satisfies the relations given in (1)–(4).

Using the above results, we obtain the following statement.

Theorem 2. *Let $k(t) \in C[0,T]$. If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 2, then there exists a unique solution to the problem (1) – (4).*

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