ISSN 1066-369X, Russian Mathematics, 2023, Vol. 67, No. 8, pp. 7–17. © Allerton Press, Inc., 2023.
Russian Text © The Author(s), 2023, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2023, No. 8, pp. 10–22.

Inverse Source Problem for the Equation of Forced Vibrations of a Beam

U. D. Durdiev[a,b,*,**]

[a]*Bukhara State University, Bukhara, 200118 Republic of Uzbekistan*

[b]*Romanovskii Institute of Mathematics, Bukhara Branch, Tashkent, 100174 Republic of Uzbekistan*

**e-mail: umidjan93@mail.ru*

***e-mail: u.d.durdiev@buxdu.uz*

Received December 28, 2022; revised May 16, 2023; accepted May 29, 2023

Abstract—Direct and inverse problems for the equation of forced vibrations of a finite length beam with a variable stiffness coefficient at the lowest term are investigated. The direct problem is the initial–boundary value problem for this equation with boundary conditions in the form of a beam fixed at one end and free at the other. The unknown variable in the inverse problem is a multiplier in the right-hand side, which depends on the space variable x. This unknown is determined with respect to the solution of the direct problem by specifying an integral redefinition condition. The uniqueness of the solution of the direct problem is proved by the method of energy estimates. The eigenvalues and eigenfunctions of the corresponding elliptic operator are used to reduce the problems to integral equations. The method of successive approximations is used to prove existence and uniqueness theorems for solutions of these equations.

Keywords: integral equation, eigenvalue, eigenfunction, existence, uniqueness, redefinition condition

DOI: 10.3103/S1066369X23080017

INTRODUCTION

Many problems on vibrations of rods, beams, and plates have important applications in many fields, such as the design of structures, the theory of stability of rotating shafts, and the theory of vibrations of ships and pipelines, and they lead to at least second order differential equations [1, 2]. Recent years have witnessed increased interest in the study of linear and nonlinear initial–boundary value problems for the equation of beam vibrations [3–9]. The study [10] obtained an analytical solution to the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain for any type of boundary conditions.

Inverse problems for beam vibration equations are not as well studied as similar problems for classical equations of mathematical physics. It should be noted that inverse problems of determining variable coefficients and right-hand sides of second-order linear parabolic equations were considered in [11–14] (see also the literature cited in monographs [13, 14]). Various inverse problems for second-order equations of hyperbolic type can be found in the monographs [15–17] (see also the extensive bibliography therein). The studies [18–22] considered a new direction in the theory of inverse problems—reconstructing the convolution kernel in hyperbolic equations describing delay phenomena. Numerical methods for solving these problems were proposed in [23–25].

The problem of determining the time-dependent stiffness coefficient in the equation of transverse vibrations of a beam was considered in [26]. This study considers the initial–boundary value problem for the equation of transverse vibrations of a finite-length beam of and the inverse problem of determining the multiplier of the right-hand side, which depends on the spatial variable x.

Let us consider the equation of vibrations of a nonhomogeneous beam:

$$Lu = u_{tt} + a^2 u_{xxxx} + p(x)u = f(x,t) \tag{1}$$

in the domain

$$D = \{(x,t) \colon 0 < x < l,\ 0 < t < T\},$$

where l is the length of the beam and T is the end time, with initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in [0,l] \tag{2}$$

and boundary conditions

$$\begin{aligned} &u(0,t) = u_x(0,t) = 0 \quad \text{(fixed end)},\\ &u_{xx}(l,t) = u_{xxx}(l,t) = 0 \quad \text{(free end)}, \quad 0 \le t \le T. \end{aligned} \tag{3}$$

In the direct problem, it is required to find a function

$$u(x,t) \in C^{4,2}_{x,t}(D), \tag{4}$$

satisfying equalities (1)–(4) for given numbers a, l, and T and sufficiently smooth functions $p(x)$, $f(x,t)$, $\varphi(x)$, and $\psi(x)$.

Inverse problem. Let $f(x,t) = g_0(x)g_1(t)$, $g_0(x)$ be the unknown variable and $g_1(t)$ be a known function. It is required to find $g_0(x)$ given that the solution of direct problem (1)–(4) satisfies the integral redefinition condition

$$H(x) = \int_0^T u(x,t)h(t)dt. \tag{5}$$

Let the following conditions be satisfied with respect to the given functions:

$$(B_1) \quad \varphi(x) \in C^6[0,l], \quad \varphi(0) = \varphi'(0) = \varphi''(l) = \varphi'''(l) = \varphi^{IV}(0) = \varphi^{V}(0) = 0,$$

$$(B_2) \quad \psi(x) \in C^4[0,l], \quad \psi(0) = \psi'(0) = \psi''(l) = \psi'''(l) = 0,$$

$$(B_3) \quad f(x,t) \in C(\overline{D}) \cap C^4_x(D), \quad f(0,t) = f'(0,t) = f''(l,t) = f'''(l,t) = 0,$$

$$(B_4) \quad h(t) \in C^2(0,T), \quad h(0) = h(T) = h'(0) = h'(T) = 0,$$

$$(B_5) \quad H(x) \in C^4(0,l), \quad H(0) = H'(0) = H''(l) = H'''(l) = 0, \quad \beta = \int_0^T g_1(t)h(t)dt, \quad \beta \ne 0.$$

1. DIRECT PROBLEM

Moving the term $p(x)$ to the right-hand side of Eq. (1) and introducing the notation $F(x,t) = f(x,t) - p(x)u$, we obtain

$$u_{tt} + a^2 u_{xxxx} = F(x,t). \tag{6}$$

Solving Eq. (6) with initial (2) and boundary (3) conditions by the method of separation of variables $u(x,t) = X(x)T(t)$, we obtain a spectral problem with respect to $X(x)$. This problem was considered in [9]. Following that study, we look for a solution to Eq. (6) with conditions (2), (3) in the form of a sum of series:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)Y_n(x) + \sum_{n=1}^{\infty} v_n(t)Y_n(x), \tag{7}$$

where

$$\begin{gathered} u_n(t) = \int_0^l \tilde{u}(x,t)Y_n(x)dx, \quad v_n(t) = \int_0^l \tilde{v}(x,t)Y_n(x)dx,\\ Y_n(x) = \frac{X(x)}{\|X(x)\|}, \end{gathered} \tag{8}$$

the function $\tilde{u}(x,t)$ is a solution to Eq. (6) with a homogeneous right-hand side ($F(x,t) \equiv 0$), inhomogeneous initial (2), and homogeneous boundary conditions (3), and $\tilde{v}(x,t)$ is a solution to Eq. (6) with a nonhomogeneous right-hand side ($F(x,t) \neq 0$), zero initial (2) ($\varphi \equiv \psi = 0$), and boundary conditions (3). Next,

$$X_n(x) = \begin{cases} a_n \cosh d_n\left(x - \frac{l}{2}\right) + b_n \sin d_n\left(x - \frac{l}{2}\right), & n = 2k-1; \\ c_n \sinh d_n\left(x - \frac{l}{2}\right) + z_n \cos d_n\left(x - \frac{l}{2}\right), & n = 2k, \end{cases}$$

is the system of eigenfunctions, where

$$a_n = \sinh^{-1}\left(\frac{d_n l}{2}\right), \quad b_n = \cos^{-1}\left(\frac{d_n l}{2}\right), \quad c_n = -\cosh^{-1}\left(\frac{d_n l}{2}\right), \quad z_n = \sin^{-1}\left(\frac{d_n l}{2}\right),$$

$$\|X(x)\| = \begin{cases} \sqrt{l}\cot\left(\frac{d_n l}{2}\right), & n = 2k-1; \\ \sqrt{l}\tanh\left(\frac{d_n l}{2}\right), & n = 2k, \end{cases}, \quad d_n = \frac{\pi}{l}\left(n - \frac{1}{2} + (-1)^n \Theta_n\right), \quad \Theta_n = O\left(\frac{1}{n^2}\right). \tag{9}$$

It should be noted that system (8) is orthonormal and complete in $L_2[0,l]$ and forms an orthonormal basis in it.

By direct calculations of the functions $u_n(t)$ and $v_n(t)$ in (7), one can easily obtain

$$u_n(t) = \varphi_n \cos a d_n^2 t + \frac{\psi_n}{a d_n^2}\sin a d_n^2 t, \quad v_n(t) = \frac{1}{a d_n^2}\int_0^l F_n(s)\sin a d_n^2(t-s)ds,$$

where

$$\varphi_n = \int_0^l \varphi(x)Y_n(x)dx, \quad \psi_n = \int_0^l \psi(x)Y_n(x)dx,$$

$$F_n(t) = \int_0^l F(x,t)Y_n(x)dx.$$

Theorem 1. *If there exists a solution to initial–boundary value problem* (1)–(3), *then the following estimate holds for any* $t \in [0,T]$:

$$\int_0^l (u_t^2 + a^2 u_{xx}^2 + p u^2)dx \le e^T\left[\int_0^l \left(\psi^2(x) + a^2\varphi''^2(x) + p(x)\varphi^2(x)\right)dx + \iint_D f^2(x,t)dxdt\right], \tag{10}$$

where $p(x) \ge 0$, $x \in [0,l]$.

Proof. We use the methodology of [6]. The following relation holds:

$$E(t) = \frac{1}{2}\int_0^l \left[u_t^2(x,t) + a^2 u_{xx}^2(x,t) + p(x)u^2(x,t)\right]dx. \tag{11}$$

Integral (11) is a mathematical expression of the law of conservation of energy of free oscillations of a string under homogeneous (zero) boundary conditions, i.e., in the absence of energy influx from outside or when there is energy dissipation during the oscillation [27].

Let us consider the identity

$$u_t L u = \frac{1}{2}\left(u_t^2 + a^2 u_{xx}^2 + p(x)u^2\right)'_t + a^2\left(u_t u_{xxx} - u_{tx}u_{xx}\right)'_x.$$

Integrating this identity over the domain

$$D_\tau = D \cap \{t < \tau\}, \quad 0 < \tau \le T$$

and taking Green's formula into account, we obtain the relation

$$E(\tau) - E(0) + a^2\int_0^\tau (u_t u_{xxx} - u_{xt}u_{xx})\big|_{x=l}dt - a^2\int_0^\tau (u_t u_{xxx} - u_{xt}u_{xx})\big|_{x=0}dt = \iint_{D_\tau} u_t f(x,t)dxdt.$$

In combination with boundary conditions (3), this relation means that

$$E(\tau) = E(0) + \iint\limits_{D_\tau} u_t f(x,t)dxdt. \tag{12}$$

Due to the well-known inequality $2ab \le a^2 + b^2$, relation (12) can be rewritten as

$$E(\tau) \le E(0) + \frac{1}{2}\iint\limits_{D_\tau} f^2 dxdt + \frac{1}{2}\iint\limits_{D_\tau} u_t^2 dxdt = B + \frac{1}{2}\int\limits_0^\tau dt \int\limits_0^l u_t^2 dx$$

$$\le B + \frac{1}{2}\int\limits_0^\tau dt \int\limits_0^l \left(u_t^2 + a^2 u_{xx}^2 + pu^2 dx\right) dx = B + \int\limits_0^\tau E(t)dt,$$

where

$$B = E(0) + \frac{1}{2}\iint\limits_{D_\tau} f^2 dxdt.$$

Thus, for any $\tau \in [0, T]$, we have

$$E(\tau) \le B + \int\limits_0^\tau E(t)dt. \tag{13}$$

Multiplying inequality (13) by $e^{-\tau}$, we obtain

$$\frac{d}{d\tau}\left[e^{-\tau}\int\limits_0^\tau E(t)dt\right] \le Be^{-\tau}. \tag{14}$$

Integrating inequality (13) with respect to τ from 0 to T, we obtain

$$B + \int\limits_0^T E(t)dt \le Be^T.$$

In view of inequality (13), it follows from here that estimate (10) is true.

□

Corollary 1. If there exists a solution to initial–boundary value problem (1)–(4) and $f(x,t) \equiv 0$, then the relation

$$E(t) = E(0) = \frac{1}{2}\int\limits_0^l \left[\psi^2(x) + a^2\varphi''^2(x) + p(x)\varphi^2(x)\right] dx \tag{15}$$

holds for any $t \in [0, T]$.

In fact, we obtain from relation (15) that the total energy of free vibrations of a homogeneous beam remains constant and equal to its initial energy throughout the entire vibration process.

Corollary 2 (uniqueness)**.** If there exists a function $u(x,t)$ satisfying relations (1)–(4), then it is unique.

Proof. Assume that there exist two solutions to direct problem (1)–(4). Then, their difference

$$u_1(x,t) - u_2(x,t) = u(x,t)$$

belongs to class (4) and satisfies the homogeneous equation $Lu = 0$ in D with zero initial and boundary conditions (3). For this solution, we obtain from (15)

$$E(t) = \frac{1}{2}\int\limits_0^l \left(u_t^2 + a^2 u_{xx}^2 + p(x)u^2\right) dx \equiv 0.$$

If $p(x) > 0$, this identity is possible if and only if $u_t(x,0) \equiv 0$, $u_{xx} \equiv 0$ and $u = 0$ in D. If $p(x) = 0$, then $u(x,t) = c_1 x + c_2$, where c_1 and c_2 are positive constants. The function $u(x,t)$ satisfies boundary conditions (3); then, $c_1 = c_2 = 0$. Therefore, $u(x,t) \equiv 0$ in D.

□

Let us investigate the existence of solution.

Substituting $f(x,t) - p(x)u$ for $F(x,t)$ in (7), we obtain

$$u(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos ad_n^2 t + \frac{\psi_n}{ad_n^2}\sin ad_n^2 t\right)Y_n(x) + \sum_{n=1}^{\infty}\frac{Y_n(x)}{ad_n^2}\int_0^l \sin ad_n^2(t-s)\int_0^l f(\xi,s)d\xi ds$$
$$- \sum_{n=1}^{\infty}\frac{Y_n(x)}{ad_n^2}\int_0^l \sin ad_n^2(t-s)\int_0^l p(\xi)u(\xi,s)d\xi ds. \tag{16}$$

For convenience, we introduce the notation

$$\Phi(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos ad_n^2 t + \frac{\psi_n}{ad_n^2}\sin ad_n^2 t\right)Y_n(x) + \sum_{n=1}^{\infty}\frac{Y_n(x)}{ad_n^2}\int_0^l \sin ad_n^2(t-s)\int_0^l f(\xi,s)d\xi ds,$$
$$K(x,\xi,t-s) = \sum_{n=1}^{\infty}\frac{Y_n(x)}{ad_n^2}\sin ad_n^2(t-s)p(\xi). \tag{17}$$

Then, (16) can be rewritten as

$$u(x,t) = \Phi(x,t) - \int_0^l\int_0^l K(x,\xi,t-s)u(\xi,s)d\xi ds.$$

Thus, we have obtained a Fredholm integral equation of the second kind. To solve this equation, we use the method of successive approximations, representing the solution in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{18}$$

where

$$u_0(x,t) = \Phi(x,t),$$
$$u_k(x,t) = -\int_0^l\int_0^l K(x,\xi,t-s)u_{k-1}(\xi,s)d\xi ds.$$

Estimating u_n in the domain D, we have

$$|u_0| \le f_0,$$
$$|u_1| \le \left|\int_0^l\int_0^l K(x,\xi,t-s)u_0(\xi,s)d\xi ds\right| \le K_0 f_0 l^2,$$
$$|u_2| \le \left|\int_0^l\int_0^l K(x,\xi,t-s)u_1(\xi,s)d\xi ds\right| \le K_0^2 f_0 l^4,$$
$$\ldots$$
$$|u_k| \le \left|\int_0^l\int_0^l K(x,\xi,t-s)u_{k-1}(\xi,s)d\xi ds\right| \le f_0(K_0 l^2)^k,$$

where

$$K_0 = \max_{\xi,x\in[0,l],\ s\in[0,T]}|K(x,\xi,t-s)| \le \frac{C_0}{\sqrt{l}}\cdot\frac{l^2 p_0}{a\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \le \frac{2C_0 l^{\frac{3}{2}} p_0}{a\pi^2},$$
$$C_0 = \max\{C_1, 6\}, \quad C_1 = \frac{4}{\left(1-e^{-d_1 l}\right)^2},$$
$$p_0 = \max_{\xi\in[0,l]}|p(\xi)|, \quad f_0 = \max_{x\in[0,l],\ t\in[0,T])}|f(x,t)|.$$

Here, C_i, $i = 0,1$ are positive constants depending on l and T.

A necessary condition for the convergence of series (18) is the inequality $K_0 l^2 < 1$; hence, we obtain the condition for l:

$$l < \left(\frac{a\pi^2}{2C_0 p_0}\right)^{\frac{2}{7}}. \tag{19}$$

Then, series (18) satisfies the estimate

$$|u(x,t)| \le f_0 \sum_{k=1}^{\infty} \left(K_0 l^2\right)^k = f_0 \frac{K_0 l^2}{1 - K_0 l^2}. \tag{20}$$

Thus, the following lemma holds.

Lemma 1. *For any* $(x,t) \in D$ *and for all* l *satisfying inequality* (19), *estimate* (20) *is true*.

Formal term-by-term differentiation of integral equation (16) gives

$$\begin{aligned} u_{tt}(x,t) = &-\sum_{n=1}^{\infty}\left(a^2 d_n^4 \varphi_n \cos a d_n^2 t + a d_n^2 \sin a d_n^2 t\right) Y_n(x) \\ &- \sum_{n=1}^{\infty} Y_n(x) a d_n^2 \int_0^l \sin a d_n^2 (t-s) \int_0^l f(\xi,s) d\xi ds \\ &+ \sum_{n=1}^{\infty} Y_n(x) a d_n^2 \int_0^l \sin a d_n^2 (t-s) \int_0^l p(\xi) u(\xi,s) d\xi ds; \end{aligned} \tag{21}$$

$$\begin{aligned} u_{xxxx}(x,t) = &\sum_{n=1}^{\infty} d_n^4 \left(\varphi_n \cos a d_n^2 t + \frac{\psi_n}{a d_n^2} \sin a d_n^2 t\right) Y_n(x) \\ &+ \sum_{n=1}^{\infty} \frac{Y_n(x) d_n^2}{a} \int_0^l \sin a d_n^2 (t-s) \int_0^l f(\xi,s) d\xi ds \\ &- \sum_{n=1}^{\infty} \frac{Y_n(x) d_n^2}{a} \int_0^l \sin a d_n^2 (t-s) \int_0^l p(\xi) u(\xi,s) d\xi ds. \end{aligned} \tag{22}$$

Lemma 2. *If* $\varphi(x)$, $\psi(x)$, $f(x,t)$ *satisfy the conditions* B_1, B_2, B_3, *then*

$$\varphi_n = \frac{\varphi_n^{(6)}}{d_n^6}, \quad \psi_n = \frac{\psi^{(4)}}{d_n^4}, \quad f_n(t) = \frac{f_n^{(4)}(t)}{d_n^4}, \tag{23}$$

where

$$\varphi_n^{(6)} = \begin{cases} \dfrac{1}{\|X_n\|} \displaystyle\int_0^l \varphi^{(6)}(x)\left(a_n d_n \left(x - \frac{l}{2}\right) + b_n \sin d_n \left(x - \frac{l}{2}\right)\right) dx, & n = 2k-1; \\ \dfrac{1}{\|X_n\|} \displaystyle\int_0^l \varphi^{(6)}(x)\left(c_n d_n \left(x - \frac{l}{2}\right) - z_n \cos d_n \left(x - \frac{l}{2}\right)\right) dx, & n = 2k, \end{cases}$$

$$\psi_n^{(4)} = \int_0^l \psi^{(4)} Y_n(x) dx,$$

$$f_n^{(4)}(t) = \int_0^l f^{(4)}(x,t) Y_n(x) dx;$$

where the coefficients a_n, b_n, c_n, *and* z_n *are determined from formula* (9).

Integrating by parts the integrals for φ_n six times and for ψ_n and $f_n(t)$ four times, and taking into account the conditions of Lemma 2, we obtain (23).

Based on Lemma 2, we can majorize series (16), (21), and (22) by the convergent numerical series

$$C^*\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left|\varphi_n^{(6)}\right|+\left|\psi_n^{(4)}\right|+\left|f_n^{(4)}(t)\right|\right);$$

therefore, they converge uniformly in $\overline{D}$.

This completes the proof of the following theorem.

Theorem 2. *If estimate* (19) *is true and the functions* $\varphi(x)$, $\psi(x)$, $f(x,t)$ *satisfy the conditions* B_1, B_2, B_3, *then there exists a unique solution to problem* (1)–(4), *represented as* (16).

To prove the stability of the solution of problem (1)–(3), we consider the space of square-summable functions $L_2[0,l]$.

Theorem 3. *Solution* (7) *of initial–boundary value problem* (1)–(4) *satisfies the following estimates*:

$$\|u(x,t)\|_{L_2[0,l]}\le C_2\left(\|\varphi(x)\|_{L_2[0,l]}+\|\psi(x)\|_{L_2[0,l]}+\|f(x,t)\|_{L_2[D]}+\|p(x)\|_{L_2[0,l]}\right), \tag{24}$$

$$\|u(x,t)\|_{C(\overline{D})}\le C_3\left(\left\|\varphi^{(4)}(x)\right\|_{C[0,l]}+\|\psi(x)\|_{C[0,l]}+\|f(x,t)\|_{C(\overline{D})}+\|p(x)\|_{C[0,l]}\right), \tag{25}$$

where C_i, $i=2,3$ *are positive constants*.

Proof. Since the system of functions $Y_n(x)$ is orthonormal, we obtain from representation (7) that

$$\|u_n(x,t)\|^2_{L_2[0,l]}\le 2\sum_{n=1}^{\infty}u_n^2(t)+2\sum_{n=1}^{\infty}v_n^2(t). \tag{26}$$

Each series in the right-hand side of this inequality is estimated separately:

$$2\sum_{n=1}^{\infty}u_n^2(t)\le C_4\sum_{n=1}^{\infty}\left(\varphi_n^2+\psi_n^2\right)=C_5\left(\|\varphi(x)\|_{L_2[0,l]}+\|\psi(x)\|_{L_2[0,l]}\right); \tag{27}$$

$$\begin{aligned}
2\sum_{n=1}^{\infty}v_n^2(t)&\le 2\sum_{n=1}^{\infty}\left[\frac{1}{ad_n^2}\int_0^t\int_0^l F(x,s)Y_n(x)\sin ad_n^2(t-s)dsdx\right]^2\\
&=2\sum_{n=1}^{\infty}\left[\frac{1}{ad_n^2}\int_0^t\int_0^l f(x,s)Y_n(x)\sin ad_n^2(t-s)dsdx\right.\\
&\left.\quad-\frac{1}{ad_n^2}\int_0^t\int_0^l p(x)u(x,s)Y_n(x)\sin ad_n^2(t-s)dsdx\right]^2\\
&\le 4\sum_{n=1}^{\infty}\left[\frac{1}{ad_n^2}\int_0^t\int_0^l f(x,s)Y_n(x)\sin ad_n^2(t-s)dsdx\right]^2\\
&\quad+4\sum_{n=1}^{\infty}\left[\frac{1}{ad_n^2}\int_0^t\int_0^l p(x)u(x,s)Y_n(x)\sin ad_n^2(t-s)dsdx\right]^2
\end{aligned} \tag{28}$$

$$\le 4\sum_{n=1}^{\infty}\frac{1}{a^2d_n^4}\int_0^t f_n^2(s)ds+4\sum_{n=1}^{\infty}\frac{1}{a^2d_n^4}p_n^2\int_0^t u(x,s)ds\le C_6\int_0^t\sum_{n=1}^{\infty}f_n^2(s)ds+C_7\sum_{n=1}^{\infty}p_n^2$$

$$=C_6\int_0^t\int_0^l f^2(x,s)dxds+C_7\int_0^l p^2(x)dx=C_6\int_0^t\|f(x,t)\|^2_{L_2[0,l]}ds+C_7\|p(x)\|^2_{L_2[0,l]}$$

$$\le C_6\|f(x,t)\|^2_{L_2(D)}+C_7\|p(x)\|^2_{L_2[0,l]}.$$

Substituting estimates (27) and (28) into (26), one can easily verify that estimate (24) is true.

It follows from (7) that

$$|u(x,t)| \le C_8 \sum_{n=1}^{\infty}\left(|\varphi_n| + \frac{|\psi_n|}{n^2}\right) + C_9 \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\int_0^t |f_n(s)|\,ds + \frac{1}{n^2}|p_n|\right)$$

$$\le C_8 \sum_{n=1}^{\infty}\left(\frac{|\varphi_n^{(4)}|}{n^4} + \frac{|\psi_n|}{n^2}\right) + C_9 \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\int_0^t |f_n(s)|\,ds + \frac{1}{n^2}|p_n|\right)$$

$$\le C_8 \sum_{n=1}^{\infty}\frac{1}{n}\left(|\varphi_n^{(4)}| + |\psi_n|\right) + C_9 \sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^t |f_n(s)|\,ds + |p_n|\right)$$

for any $(x,t) \in \overline{D}$.

Hence, using the Cauchy–Bunyakovsky inequality, we obtain

$$|u(x,t)| \le C_8 \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left[\left(\sum_{n=1}^{\infty}|\varphi_n^{(4)}|^2\right)^{1/2} + \left(\sum_{n=1}^{\infty}|\psi_n|^2\right)^{1/2}\right]$$

$$+ C_9 \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left[\left(\int_0^t \sum_{n=1}^{\infty}|f_n(s)|^2\,ds\right)^{1/2} + \left(\sum_{n=1}^{\infty}|p_n|^2\right)^{1/2}\right]$$

$$= C_3\left(\|\varphi^{(4)}(x)\|_{L_2[0,l]} + \|\psi(x)\|_{L_2[0,l]} + \|f(x,t)\|_{L_2(D)} + \|p(x)\|_{L_2[0,l]}\right).$$

Estimate (25) follows directly from this estimate, where C_i, $i = \overline{4,10}$, are positive constants.

□

2. INVERSE PROBLEM

Let $f(x,t) = g_0(x)g_1(t)$. Multiplying both sides of Eq. (1) by $h(t)$; integrating with respect to t from 0 to T; and taking into account conditions (5), B_4, and B_5; we obtain

$$\int_0^T u(x,t)h''(t)dt + a^2 H^{(4)}(x) + p(x)H(x) = \beta g_0(x).$$

Resolving this equation with respect to $g_0(x)$, we have

$$g_0(x) = \frac{1}{\beta}\left[\int_0^T u(x,t)h''(t)dt + a^2 H^{(4)}(x) + p(x)H(x)\right]. \tag{29}$$

Substituting the resulting expression into (16), we find an integral equation for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos a d_n^2 t + \frac{\psi_n}{a d_n^2}\sin a d_n^2 t\right)Y_n(x)$$

$$+ \sum_{n=1}^{\infty}\frac{Y_n(x)}{a\beta d_n^2}\int_0^l \sin a d_n^2(t-s)\int_0^l\int_0^T u(\xi,\eta)h''(\eta)d\eta d\xi ds + \sum_{n=1}^{\infty}\frac{Y_n(x)}{\beta d_n^2}\int_0^l \sin a d_n^2(t-s)\int_0^l H^{(4)}(\xi)d\xi ds \tag{30}$$

$$+ \sum_{n=1}^{\infty}\frac{Y_n(x)}{a\beta d_n^2}\int_0^l \sin a d_n^2(t-s)\int_0^l p(\xi)H(\xi)d\xi ds - \sum_{n=1}^{\infty}\frac{Y_n(x)}{a d_n^2}\int_0^l \sin a d_n^2(t-s)\int_0^l p(\xi)u(\xi,s)d\xi ds.$$

Using (17) and some notations of the form

$$\Psi(x,t) = \sum_{n=1}^{\infty}\left(\varphi_n \cos ad_n^2 t + \frac{\psi_n}{ad_n^2}\sin ad_n^2 t\right)Y_n(x)$$

$$+\sum_{n=1}^{\infty}\frac{Y_n(x)}{\beta d_n^2}\int_0^l \sin ad_n^2(t-s)\int_0^l H^{(4)}(\xi)d\xi ds + \sum_{n=1}^{\infty}\frac{Y_n(x)}{a\beta d_n^2}\int_0^l \sin ad_n^2(t-s)\int_0^l p(\xi)H(\xi)d\xi ds,$$

$$K_1(x,t,\xi,s,\eta) = \sum_{n=1}^{\infty}\frac{Y_n(x)}{ad_n^2}\sin ad_n^2(t-s)h''(\eta),$$

we write Eq. (30) in a more convenient form as

$$u(x,t) = \Psi(x,t) + \int_0^l\int_0^l\int_0^T K_1(x,t,\xi,s,\eta)u(\xi,\eta)d\eta d\xi ds - \int_0^l\int_0^l K(x,t,\xi,s)u(\xi,s)d\xi ds. \tag{31}$$

It should be noted that (31) is a Fredholm integral equation of the second kind with respect to the unknown function $u(x,t)$. We use the method of successive approximations to find a solution to Eq. (31), representing it in the form (18), where

$$u_0(x,t) = \Psi(x,t),$$

$$u_k(x,t) = \int_0^l\int_0^l\int_0^T K_1(x,t,\xi,s,\eta)u_{k-1}(\xi,\eta)d\eta d\xi ds - \int_0^l\int_0^l K(x,t,\xi,s)u_{k-1}(\xi,s)d\xi ds.$$

Let us estimate u_k in D:

$$|u_0| \le |M|,$$

$$|u_1| \le \left|\int_0^l\int_0^l\int_0^T K_1(x,t,\xi,s,\eta)u_0 d\eta d\xi ds - \int_0^l\int_0^l K(x,t,\xi,s)u_0 d\xi ds\right| \le M(K_{01}T + K_0)l^2,$$

$$|u_2| \le \left|\int_0^l\int_0^l\int_0^T K_1(x,t,\xi,s,\eta)u_1 d\eta d\xi ds - \int_0^l\int_0^l K(x,t,\xi,s)u_1 d\xi ds\right| \le M(K_{01}T + K_0)^2 l^4,$$

$$\ldots$$

$$|u_k| \le \left|\int_0^l\int_0^l\int_0^T K_1(x,t,\xi,s,\eta)u_{k-1} d\eta d\xi ds - \int_0^l\int_0^l K(x,t,\xi,s)u_{k-1} d\xi ds\right| \le M\left((K_{01}T + K_0)l^2\right)^k,$$

where

$$K_{01} = \max_{x,\xi\in[0,l],\ t,s,\eta\in[0,T]}|K_1(x,t,\xi,s,\eta)| \le \frac{2C_0 l^{\frac{3}{2}} h_0}{a\pi^2}, \quad h_0 = \max_{\eta\in[0,T]} h''(\eta).$$

A necessary condition for the convergence of the series is $(K_{01}T + K_0)l^2 < 1$. Hence, we obtain

$$l < \left[\frac{a\pi^2}{2C_0(Th_0 + p_0)}\right]^{\frac{2}{7}}. \tag{32}$$

Then, series (18) satisfies the estimate

$$|u(x,t)| \le M\sum_{k=1}^{\infty}\left((K_{01}T + K_0)l^2\right)^n = M\frac{K_{01}T + K_0 l^2}{1 - K_{01}T + K_0 l^2}. \tag{33}$$

This completes the proof of the following theorem.

Theorem 4. *If conditions* B_1, B_2, B_3, B_4, B_5, *and estimate* (32) *are satisfied, then there exists a unique solution* $u(x,t) \in C^{4,2}_{x,t}(D)$ *to integral equation* (31).

Using the function $u(x,t) \in C^{4,2}_{x,t}(D)$ and formula (29), one can find the unknown function $g_0(x)$—the solution to the inverse problem.

FUNDING

This work was supported by the Ministry of Science and Higher Education of the Russian Federation, project no. 075-02-2023-939.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* (Nauka, Moscow, 1977; Dover, New York, 2013).
2. A. N. Krylov, *Vibrations of Vessels* (ONTI, Glav. Red. Sudostroit. Literatury, Moscow, 1936).
3. S. Li, E. Reynders, K. Maes, and G. De Roeck, "Vibration-based estimation of axial force for a beam member with uncertain boundary conditions," J. Sound Vib. **332**, 795–806 (2013). https://doi.org/10.1016/j.jsv.2012.10.019
4. Yi. Wang and Z. Fang, "Vibrations in an elastic beam with nonlinear supports at both ends," J. Appl. Mech. Tech. Phys. **56**, 337–346 (2015). https://doi.org/10.1134/S0021894415020200
5. K. B. Sabitov and A. A. Akimov, "Initial–boundary value problem for a nonlinear beam vibration equation," Differ. Equations **56**, 621–634 (2020). https://doi.org/10.1134/S0012266120050079
6. K. B. Sabitov, "A remark on the theory of initial-boundary value problems for the equation of rods and beams," Differ. Equations **53**, 86–98 (2017). https://doi.org/10.1134/S0012266117010086
7. S. G. Kasimov and U. S. Madrakhimov, "Initial-boundary value problem for the beam vibration equation in the multidimensional case," Differ. Equations **55**, 1336–1348 (2019). https://doi.org/10.1134/S0012266119100094
8. K. B. Sabitov, "Initial–boundary value problems for equation of oscillations of a rectangular plate," Russ. Math. **65**, 52–62 (2021). https://doi.org/10.3103/S1066369X21100054
9. K. B. Sabitov and O. V. Fadeeva, "Initial-boundary value problem for the equation of forced vibrations of a cantilever beam," Vestn. Samarsk. Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauki **25** (1), 51–66 https://doi.org/10.14498/vsgtu1845
10. A. L. Karchevsky, "Analytical solutions to the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain for the boundary conditions of various types," J. Appl. Ind. Math. **14**, 648–665 (2020). https://doi.org/10.1134/S1990478920040043
11. A. I. Prilepko, A. V. Kostin, and V. V. Solov'ev, "Inverse source and inverse coefficients problems for elliptic and parabolic equations in Hölder and Sobolev spaces," Sib. Zh. Chistoi Prikl. Mat. **17** (3), 67–85. https://doi.org/10.17377/PAM.2017.17.7
12. N. I. Ivanchov, "On the inverse problem of simultaneous determination of thermal conductivity and specific heat capacity," Sib. Math. J. **35**, 547–555 (1994). https://doi.org/10.1007/bf02104818
13. A. M. Denisov, *Introduction to the Theory of Inverse Problems* (Izd-vo Mosk. Univ., Moscow, 1994).
14. A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics New York Marcel Dekker* (Marcel Dekker, New York, 1999).
15. V. G. Romanov, *Inverse Problems for Hyperbolic Equations and Energy Inequalities* (Nauka, Moscow, 1984).
16. S. I. Kabanikhin, *Inverse and Ill-Posed Problems* (Sib. Nauchn. Izd-vo, Novosibirsk, 2009).
17. A. Hasanov Hasanoğlu and V. G. Romanov, *Introduction to Inverse Problems for Differential Equations* (Springer, Cham, 2017). https://doi.org/10.1007/978-3-319-62797-7

18. D. K. Durdiev and Zh. D. Totieva, "The problem of finding the kernels in the system of integro-differential Maxwell's equations," J. Appl. Ind. Math. **15**, 190–211 (2021). https://doi.org/10.1134/S1990478921020022
19. D. K. Durdiev and A. A. Rahmonov, "A 2D kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity," Math. Methods Appl. Sci. **43**, 8776–8796 (2020). https://doi.org/10.1002/mma.6544
20. U. Durdiev and Zh. Totieva, "A problem of determining a special spatial part of 3D memory kernel in an integro-differential hyperbolic equation," Math. Methods Appl. Sci. **42**, 7440–7451 (2019). https://doi.org/10.1002/mma.5863
21. U. D. Durdiev, "A problem of identification of a special *D* memory kernel in an integro-differential hyperbolic equation," Eurasian J. Math. Comput. Appl. **7** (2), 4–19
22. U. D. Durdiev, "An inverse problem for the system of viscoelasticity equations in homogeneous anisotropic media," J. Appl. Ind. Math. **13**, 623–628 (2019). https://doi.org/10.1134/S1990478919040057
23. A. L. Karchevsky and A. A. Fatianov, "Numerical solution of the inverse problem for a system of elasticity with the aftereffect for a vertically inhomogeneous medium," Sib. Zh. Vychislit. Mat. **4** (3) (2001).
24. A. L. Karchevsky, "Determination of the possibility of rock burst in a coal seam," J. Appl. Ind. Math. **11**, 527–534 (2017). https://doi.org/10.1134/S199047891704010X
25. U. D. Durdiev, "Numerical method for determining the dependence of the dielectric permittivity on the frequency in the equation of electrodynamics with memory," Sibirskie Elektronnye Mat. Izv. **17**, 179–189 (2020). https://doi.org/10.33048/semi.2020.17.013
26. U. D. Durdiev, "Inverse problem of determining an unknown coefficient in the beam vibration equation," Differ. Equations **58**, 36–43 (2022). https://doi.org/10.1134/s0012266122010050
27. K. B. Sabitov, *Equations of Mathematical Physics* (Fizmatlit, Moscow, 2013).
28. V. A. Trenogin, *Functional Analysis* (Nauka, Moscow, 1980).

Translated by V. Arutyunyan