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Global Solvability of the Kernel Identification Problem for the Integro-Differential Equation of Beam Vibrations

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Abstract—We investigate the inverse problem of identifying the kernel that characterizes the memory effects of the medium in the integro-differential equation governing the forced vibrations of a beam. The direct problem is formulated as a Cauchy problem for this equation, which is subsequently reduced to a system of second-kind integral equations of the Volterra type involving the solution of the direct problem and the unknown kernel of the inverse problem. To address this system, we apply the method of compressive mappings within the space of continuous functions equipped with an exponential weight norm. Our analysis establishes the global solvability of the proposed inverse problem.

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1. INTRODUCTION

Beams serve as critical structural components in construction, providing essential support for various loads and facilitating their transfer to the underlying supports or walls of a building. Their widespread application in architecture and civil engineering stems from their inherent strength and efficiency in load distribution. The primary function of beams is to bear loads, which include the dead weight of the structure, the weights of additional elements such as roofs, floors, and walls, as well as external forces such as wind and seismic activity.

In contemporary engineering and applied physics, the analysis of vibrations in beams, rods, and plates is of significant importance in the realms of stability theory and structural mechanics. The characterization of these vibrational phenomena frequently necessitates the formulation of higher-order differential equations, as detailed in the literature [1–3].

An inverse problem is a significant category of problem that frequently arises across various scientific disciplines, wherein the objective is to deduce values of model parameters based on observed data. These problems are commonly encountered in diverse fields, including geophysics, astronomy, medical imaging, computed tomography, remote sensing of the Earth, spectral analysis, and scattering theory, among others. The complexity and relevance of inverse problems highlight their critical role in the interpretation and analysis of empirical data within these domains.

In recent years, there has been a marked increase in interest regarding the study of both direct and inverse problems associated with the beam vibration equation [4–16]. The author's work [8] addresses the direct problem related to the vibration equation of an infinite beam, specifically the Cauchy problem, as well as the inverse problem concerning the determination of the time-dependent coefficient of the leading term in this equation. The study establishes theorems pertaining to the existence, uniqueness,

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and conditional stability of solutions for the inverse problem, thereby contributing to the theoretical framework of this area of research.

Inverse problems in mathematical physics have been extensively investigated for various classes of differential equations. The methodologies employed to demonstrate local existence and uniqueness theorems, as well as uniqueness and conditional stability theorems for inverse dynamical problems, are comprehensively addressed in the literature [17–25]. Furthermore, this body of work encompasses numerical approaches aimed at identifying solutions to these problems, as discussed in the referenced studies.

In the studies presented in [26–29], the authors investigate inverse problems related to the identification of space-dependent and time-dependent source terms in the time-fractional diffusion equation. This is accomplished through the eigenfunction expansion of the non-self-adjoint spectral problem utilizing the generalized Fourier method. The principal findings of these studies include theorems concerning the existence and uniqueness of solutions, as well as stability estimates for the problem of determining the coefficients in both the time-fractional diffusion and wave equations.

In this paper, we examine the direct problem associated with the vibration equation of an infinite beam, characterized by an integral on the right-hand side, thereby framing it as a Cauchy problem. Additionally, we address the inverse problem related to the determination of the kernel of the convolution integral.

2. PROBLEM STATEMENT

Let us consider the integro-differential equation of beam vibration

$$u_{tt} + a^2 u_{xxxx} = \int\limits_0^t k(\tau) u(x, t-\tau) d\tau, \tag{1}$$

in half-plane $D = \{(x,t):\ -\infty < x < +\infty,\ t > 0\}$, with initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R}. \tag{2}$$

In the **direct problem**, we search for a function $u(x,t)$ satisfying equations (1), (2), and the condition

$$u(x,t) \in C^{4,2}_{x,t}(D) \cap C^1_t(D \cup \{t=0\}) \cap C(\overline{D}). \tag{3}$$

The functions $k(t)$, $\varphi(x)$, and $\psi(x)$ are assumed to be given and smooth enough.

Inverse problem: find the kernel $k(t)$ if the solution of the Cauchy problem (1)–(3) satisfies the condition

$$u(0,t) = g(t), \tag{4}$$

where $g(t)$ is a given sufficiently smooth function.

By $D_T := \{(x,t) : x \in \mathbb{R},\ 0 < t < T\}$ denote a strip of thickness T, where $T > 0$ is an arbitrary fixed number.

Let $u(x,t)$ be the solution of the problem (1)–(3). By introducing the notation $v(x,t) := u_t(x,t)$, we transform the direct problem. Differentiating (1) by t, we have

$$v_{tt} + a^2 v_{xxxxx} = k(t)\varphi(x) + \int\limits_0^t k(\tau) v(x, t-\tau) d\tau. \tag{5}$$

The initial conditions are easily obtained from (2) and equation (1) at $t = 0$:

$$v\big|_{t=0} = \psi(x), \quad v_t\big|_{t=0} = -a^2 \varphi^{(4)}(x), \quad x \in \mathbb{R}. \tag{6}$$

The additional condition for the function v, as follows from (4), takes the form

$$v(0,t) = g'(t), \quad t \in [0,T]. \tag{7}$$

Thus, the inverse problem (1)–(3) is reduced to the problem of determining the same function $k(t)$ from equation (5) when the solution of this equation satisfies the equalities (6) and (7). It is easy to see that, if the matching conditions $\varphi(0) = g(0)$ and $\psi(0) = g'(0)$ are satisfied, it is easy to obtain the equations (5)–(7), (1), (2), and (4) from the relations (5)–(7) by inverse transformations (by integrating the corresponding equations over t).

3. RESEARCHING DIRECT PROBLEM

Let us use the results of [3, 6], in which the solution of the problem

$$u_{tt} + a^2 u_{xxxxxx} = F(x,t), \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \mathbb{R},$$

at $F(x,t) = 0$, is constructed using the fundamental solutions in the form

$$u(x,t) = \int\limits_{-\infty}^{\infty} \varphi(\xi) G_1(x,t,\xi) d\xi + \int\limits_{-\infty}^{\infty} \psi(\xi) G_2(x,t,\xi) d\xi, \tag{8}$$

where

$$G_1(x,t,\xi) = \frac{1}{2\sqrt{\pi a t}} \sin\left[\frac{(\xi - x)^2}{4at} + \frac{\pi}{4}\right], \quad G_2(x,t,\xi) = \frac{1}{\pi a} \int\limits_{-\infty}^{\infty} \frac{1}{\lambda^2} \sin(a\lambda^2 t) \cos(\lambda(\xi - x)) d\lambda.$$

It is proved in [6] that the functions $G_i(x,t,\xi)$, $i = 1, 2$, satisfy the following conditions

$$(A_1) \quad G_1(x,t,\xi) \in C^\infty(D),$$

$$(A_2) \quad G_2(x,t,\xi) \in C^\infty(D),$$

$$(B_1) \quad \lim_{t \to 0+0} G_1(x,t,\xi) = \infty,$$

$$(B_2) \quad \lim_{t \to 0+0} G_2(x,t,\xi) = 0,$$

$$(C_1) \quad \int\limits_{-\infty}^{\infty} G_1(x,t,\xi) d\xi = 1,$$

$$(C_2) \quad \int\limits_{-\infty}^{\infty} \frac{\partial G_2(x,t,\xi)}{\partial t} d\xi = 1.$$

In fact, $G_1(x,t,\xi)$ is the partial derivative of $G_2(x,t,\xi)$ on the variable t, i.e., $\partial/\partial t(G_2(x,t,\xi)) = G_1(x,t,\xi)$.

When $F(x,t) \neq 0$, let us use Duhamel's principle and taking into account (8) to solve the direct problem (5) and (6), we obtain the integral equation

$$v(x,t) = \int\limits_{-\infty}^{\infty} \psi(\xi) G_1(x,t,\xi) d\xi - a^2 \int\limits_{-\infty}^{\infty} \varphi^{(4)}(\xi) G_2(x,t,\xi) d\xi + \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau) \varphi(\xi) G_2(x,t-\tau,\xi) d\xi d\tau$$

$$+ \int\limits_{-\infty}^{\infty} G_2(x,t-\tau,\xi) \int\limits_0^\tau k(s) v(\xi,\tau - s) ds d\xi d\tau. \tag{9}$$

Let $C_0^m(\mathbb{R})$ be the class of m times continuously differentiable and finite in $\mathbb{R}$ functions, and let $C_{x,t}^{m,k}(D_T)$ be the class of m times x, k times t continuously differentiable and bounded at each fixed $t \in [0,T]$ in D_T functions.

The following statement is true.

Lemma 1. *If $k(t) \in C[0,T]$, $\varphi(x) \in C_0^6(\mathbb{R})$, and $\psi(x) \in C_0^4(\mathbb{R})$, furthermore the functions $\varphi''(x)$, $\varphi^{(4)}(x)$, $\varphi^{(6)}(x)$, $\psi''(x)$, $\psi^{(4)}(x)$, $\varphi^{(4)}(x)$, $\varphi(x)$, $x^4\varphi(x)$, $x^4\varphi^{(4)}(x)$, and $x^4\psi(x)$ are absolutely integrable on $(-\infty, \infty)$, then there exists a unique classical solution of the integral equation (9) such that $v(x,t) \in C_{x,t}^{4,2}(D_T)$.*

To prove Lemma 1, let us use the method of successive approximations and consider a sequence of functions, defining them by formulas

$$v_n(x,t) = \int_0^t \int_{-\infty}^{\infty} G_2(x,t-\tau,\xi) \int_0^{\tau} k(s) v_{n-1}(\xi,\tau-s) ds d\xi d\tau, \quad n=1,2,\ldots, \tag{10}$$

where

$$v_0(x,t) = \int_{-\infty}^{\infty} \psi(\xi) G_1(x,t,\xi) d\xi - a^2 \int_{-\infty}^{\infty} \varphi^{(4)}(\xi) G_2(x,t,\xi) d\xi + \int_0^t \int_{-\infty}^{\infty} k(\tau)\varphi(\xi) G_2(x,t-\tau,\xi) d\xi d\tau. \tag{11}$$

Note that, from the construction of the functions $G_1(x,t,\xi)$ and $G_2(x,t,\xi)$, the following equality [6] follows:

$$\int_{-\infty}^{\infty} G_1(x,t,\xi) d\xi = 1, \quad \int_{-\infty}^{\infty} G_2(x,t,\xi) d\xi = 2t. \tag{12}$$

Let's put $\varphi_0 = \sup_{x\in\mathbb{R}} |\varphi(x)|$, $\varphi_0^{(4)} = \sup_{x\in\mathbb{R}} |\varphi^{(4)}(x)|$, $\psi_0 = \sup_{x\in\mathbb{R}} |\psi(x)|$, and $k_0 = \max_{0<t\leq T} |k(t)|$. Using (10)–(12), we estimate $v_n(x,t)$ in the domain D_T as follows

$$|v_0(x,t)| \leq \psi_0 + 2a^2 \varphi_0^{(4)} T + k_0 \varphi_0 T^2 =: \lambda_0,$$

$$|v_1(x,t)| \leq \lambda_0 k_0 \int_0^t 2(t-\tau)\tau d\tau = \lambda_0 k_0 \frac{2t^3}{(3\cdot 1)!},$$

$$|v_2(x,t)| \leq \lambda_0 k_0^2 \int_0^t 2(t-\tau)\frac{\tau^4}{12} d\tau = \lambda_0 k_0^2 \frac{(2t^3)^2}{(3\cdot 2)!}.$$

$$\ldots$$

Thus, for all $n=0,1,2,\ldots$, we obtain the estimators

$$|v_n(x,t)| \leq \lambda_0 \frac{(2k_0 t^3)^n}{(3n)!}.$$

From the above estimates it follows that the series $v(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ converges absolutely and uniformly in D_T since it can be majorized in D_T by a convergent number series

$$\lambda_0 \sum_{n=0}^{\infty} \frac{(2k_0 T^3)^n}{(3n)!}.$$

This means that the following evaluation of the solution of the integral equation (9) takes place

$$|v(x,t)| \leq \lambda_0 \sum_{n=0}^{\infty} \frac{\left(2k_0 T^3\right)^n}{(3n)!} = \mu, \quad (x,t) \in D_T.$$

Now let us prove that the solution of the integral equation (9) is indeed a classical solution of the problem (5) and (6). Let us first consider the fulfillment of the initial conditions.

Since for $t \to 0+0$, if the conditions $(A_1)-(C_2)$ for the fundamental solutions are satisfied, the integral in the right-hand side (11) diverges. Let us introduce the notation $\frac{(\xi - x)}{2\sqrt{at}} = \eta$ and transform these integrals. If the function $\varphi(x)$ is absolutely integrable on $(-\infty, \infty)$, then the integrals on the right-hand side of (11) converge uniformly at $t \geq 0$, and $x \in \mathbb{R}$. Then, passing to the limit at $t \to 0+0$, by virtue of

$$\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \sin\left(\eta^2 + \frac{\pi}{4}\right) d\eta = 1,$$

we get $\lim\limits_{t \to 0+0} v(x,t) = \psi(x)$.

To prove the fulfillment of the second initial condition, let us find the partial derivative of t from (11). It has the form

$$\frac{\partial v_0(x,t)}{\partial t} = \sqrt{\frac{a}{\pi t}} \int\limits_{-\infty}^{\infty} \psi\left(x + 2\eta\sqrt{at}\right) \eta \sin\left(\eta^2 + \frac{\pi}{4}\right) d\eta - a^2 \int\limits_{-\infty}^{\infty} \varphi^{(4)}(\xi) G_1(x,t,\xi) d\xi$$

$$+ \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau)\varphi(\xi) G_1(x, t-\tau, \xi) d\xi d\tau.$$

Hence, based on the above considerations, we obtain

$$\frac{\partial v_0(x,t)}{\partial t} = \sqrt{\frac{a}{\pi t}} \left[-\frac{1}{2}\cos\left(\eta^2 + \frac{\pi}{4}\right) \psi'\left(x + 2\eta\sqrt{at}\right)\right]\Bigg|_{-\infty}^{\infty}$$

$$+ \frac{a}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \psi''\left(x + 2\eta\sqrt{at}\right) \cos\left(\eta^2 + \frac{\pi}{4}\right) d\eta$$

$$- a^2 \int\limits_{-\infty}^{\infty} \varphi^{(4)}(\xi) G_1(x,t,\xi) d\xi + \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau)\varphi(\xi) G_1(x, t-\tau, \xi) d\xi d\tau.$$

If $\psi'(\pm\infty) = 0$, then get

$$\frac{\partial v_0(x,t)}{\partial t} = \frac{a}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \psi''(x + 2\eta\sqrt{at}) \cos\left(\eta^2 + \frac{\pi}{4}\right) d\eta - a^2 \int\limits_{-\infty}^{\infty} \varphi^{(4)}(\xi) G_1(x,t,\xi) d\xi$$

$$+ \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau)\varphi(\xi) G_1(x, t-\tau, \xi) d\xi d\tau. \tag{13}$$

If the functions ψ'' and $\varphi^{(4)}$ are absolutely integrable on $(-\infty, \infty)$, then going to the limit at $t \to 0+0$, by virtue of

$$\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \cos\left(x^2 + \frac{\pi}{4}\right) dx = 0,$$

we find

$$\lim_{t \to 0+0} \frac{\partial v(x,t)}{\partial t} = -a^2 \varphi(x).$$

If we prove that the function $u_0(x,t)$ defined by (11) will be from the class (3), then by virtue of (10) it is easy to see that all $u_n(x,t)$ will be from the same class of functions. Then, it follows from the general theory of integral equations that $u(x,t) \in C^{4,2}_{x,t}(D) \cap C^1_t(D \cup \{t=0\}) \cap C(\overline{D})$; i.e., the function $u(x,t)$ is a classical solution of the Cauchy problem (5) and (6). To that end, we have

$$\frac{\partial^2 v_0(x,t)}{\partial t^2} = \sqrt{\frac{a^3}{\pi t}} \int\limits_{-\infty}^{\infty} \psi'''(x+2\eta\sqrt{at})\eta \cos\left(\eta^2 + \frac{\pi}{4}\right) d\eta$$

$$+ \sqrt{\frac{a}{\pi t}} \int\limits_{-\infty}^{\infty} \varphi^{(5)}(x+2\eta\sqrt{at})\eta \sin\left(\eta^2 + \frac{\pi}{4}\right) d\eta$$

$$+ k(t)\varphi(x) + \int\limits_0^t \int\limits_{-\infty}^{\infty} \sqrt{\frac{a}{\pi(t-\tau)}} \left[(k(\tau)\varphi'\left(x+2\zeta\sqrt{a(t-\tau)}\right)\zeta \sin\left(\zeta^2 + \frac{pi}{4}\right)\right] d\zeta d\tau$$

$$= \sqrt{\frac{a^3}{\pi t}} \left[\frac{1}{2}\sin\left(\eta^2+\frac{\pi}{4}\right)\psi'''(x+2\eta\sqrt{at})\right]\Bigg|_{-\infty}^{\infty} + \frac{a^2}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \psi^{(4)}(x+2\eta\sqrt{at})\sin\left(\eta^2+\frac{\pi}{4}\right) d\eta$$

$$+ \sqrt{\frac{a}{\pi t}} \left[-\frac{1}{2}\cos\left(\eta^2+\frac{\pi}{4}\right)\varphi^{(5)}(x+2\eta\sqrt{at})\right]\Bigg|_{-\infty}^{\infty} - \frac{a}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \varphi^{(6)}(x+2\eta\sqrt{at})\cos\left(\eta^2+\frac{\pi}{4}\right) d\eta$$

$$+ k(t)\varphi(x) - \frac{1}{2}\int\limits_0^t \left[\sqrt{\frac{a}{\pi(t-\tau)}} k(\tau)\varphi'(x+2\zeta\sqrt{a(t-\tau)})\zeta \cos\left(\zeta^2+\frac{\pi}{4}\right)\right]\Bigg|_{-\infty}^{\infty} d\tau$$

$$+ \frac{a}{\sqrt{\pi}} \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau)\varphi''\left(x+2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right) d\zeta d\tau,$$

where $\zeta = \dfrac{\xi - x}{2\sqrt{a(t-\tau)}}$.

If $\varphi''(\pm\infty) = 0$, $\varphi^{(5)}(\pm\infty) = 0$, and $\psi''''(\pm\infty) = 0$, then we get the following expression

$$\frac{\partial^2 v_0(x,t)}{\partial t^2} = \frac{a^2}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \psi^{(4)}(x+2\eta\sqrt{at})\sin\left(\eta^2+\frac{\pi}{4}\right) d\eta$$

$$- \frac{a}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \varphi^{(6)}(x+2\eta\sqrt{at})\cos\left(\eta^2+\frac{\pi}{4}\right) d\eta$$

$$+ k(t)\varphi(x) + \frac{a}{\sqrt{\pi}} \int\limits_0^t \int\limits_{-\infty}^{\infty} k(\tau)\varphi''\left(x+2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right) d\zeta d\tau. \tag{14}$$

Then, in a similar manner, $\partial^4 v_0(x,t)/\partial x^4$ is calculated. It is not difficult to see that if the functions $\varphi''(x)$, $\varphi^{(6)}(x)$, and $\psi^{(4)}(x)$ are absolutely integrable on $(-\infty,\infty)$, then the non-singular integrals in (14) and if the functions $\varphi(x)$, $\psi(x)$, $x^4\varphi(x)$, $x^4\varphi^{(4)}(x)$, $x^4\psi(x)$, and $x^4\psi(x)$ are absolutely integrable on $(-\infty,\infty)$, then the non-singular integrals in $\dfrac{\partial^4 u_0(x,t)}{\partial x^4}$ converge uniformly for all $x \in \mathbb{R}$ and $t \geq 0$. Thus, we have proved Lemma 1.

Now let us estimate the norm of the difference between the solution of the original integral equation

(9) and the solution of the same equation by the perturbed functions $\tilde{k}$, $\tilde{\varphi}$, $\tilde{\varphi}^{(4)}$ and $\tilde{\psi}$ defining $\tilde{v}(x,t)$:

$$\tilde{v}(x,t)=\int\limits_{-\infty}^{\infty}\tilde{\psi}(\xi)G_1(x,t,\xi)d\xi-a^2\int\limits_{-\infty}^{\infty}\tilde{\varphi}^{(4)}(\xi)G_2(x,t,\xi)d\xi+\int\limits_0^t\int\limits_{-\infty}^{\infty}\tilde{k}(\tau)\tilde{\varphi}(\xi)G_2(x,t-\tau,\xi)d\xi d\tau$$
$$+\int\limits_{-\infty}^{\infty}G_2(x,t-\tau,\xi)\int\limits_0^{\tau}\tilde{k}(s)\tilde{v}(\xi,\tau-s)dsd\xi d\tau. \qquad (15)$$

Then, for $v-\tilde{v}$, using (9) and (15), we obtain a linear integral equation

$$v(x,t)-\tilde{v}(x,t)=\int\limits_{-\infty}^{\infty}(\psi(\xi)-\tilde{\psi}(\xi))G_1(x,t,\xi)d\xi$$
$$-a^2\int\limits_{-\infty}^{\infty}(\varphi^{(4)}(\xi)-\tilde{\varphi}^{(4)}(\xi))G_2(x,t,\xi)d\xi+\int\limits_0^t\int\limits_{-\infty}^{\infty}(k(\tau)\varphi(\xi)-\tilde{k}(\tau)\tilde{\varphi}(\xi))G_2(x,t-\tau,\xi)d\xi d\tau$$
$$+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_2(x,t-\tau,\xi)\int\limits_0^{\tau}(k(s)v(\xi,\tau-s)-\tilde{k}(s)\tilde{v}(\xi,\tau-s))dsd\xi d\tau$$

from which the following linear integral inequality is derived

$$|v(x,t)-\tilde{v}(x,t)|\leq||\psi-\tilde{\psi}||+2a^2T\left|\left|\varphi^{(4)}-\tilde{\varphi}^{(4)}\right|\right|+\tilde{k}_0T^2\,||\varphi-\tilde{\varphi}||+(\varphi_0T^2+\mu)||k-\tilde{k}||$$
$$+\tilde{k}_0\int\limits_0^t\int\limits_{-\infty}^{\infty}G_2(x,t-\tau,\xi)\int\limits_0^{\tau}(v(\xi,\tau-s)-\tilde{v}(\xi,\tau-s))dsd\xi d\tau, \qquad (16)$$

where $\tilde{k}_0=\max\limits_{t\in[0,T]}|\tilde{k}(t)|$.

Let $\sigma=\max\left\{1,2a^2T,\tilde{k}_0T^2,\varphi_0T^2+\mu\right\}$. Applying the method of successive approximations to the inequality (16) and using the scheme

$$|v(x,t)-\tilde{v}(x,t)|_0\leq\sigma\left(||\psi-\tilde{\psi}||+\left|\left|\varphi^{(4)}-\tilde{\varphi}^{(4)}\right|\right|+||\varphi-\tilde{\varphi}||+||k-\tilde{k}||\right),$$

$$|v(x,t)-\tilde{v}(x,t)|_n\leq\tilde{k}_0\int\limits_0^t\int\limits_{-\infty}^{\infty}G_2(x,t-\tau,\xi)\int\limits_0^{\tau}(v(\xi,\tau-s)-\tilde{v}(\xi,\tau-s))_{n-1}dsd\xi d\tau,\quad n=1,2,\ldots,$$

we come to an assessment

$$|v(x,t)-\tilde{v}(x,t)|\leq\sigma\mu\left(||\psi-\tilde{\psi}||+\left|\left|\varphi^{(4)}-\tilde{\varphi}^{(4)}\right|\right|+||\varphi-\tilde{\varphi}||+||k-\tilde{k}||\right), \qquad (17)$$

which will be used in the next section. In fact, the expression (17) is an estimate of the stability of the solution of the Cauchy problem (5) and (6). The uniqueness of this solution follows from (17).

4. INVERSE PROBLEM RESEARCH (5)–(7)

Theorem 1. *Suppose that $g(t)\in C^3[0,T]$ and the conditions of Lemma 1 and the agreement conditions are satisfied $\psi(0)=g'(0)$ and $-a^2\varphi^{(4)}(0)=g''(0)$. Then, the inverse problem (5)−(7) has a unique solution $k(t)\in C[0,T]$, for any $T>0$.*

To investigate the inverse problem, differentiate (9) by t twice and find

$$\frac{\partial^2 v(x,t)}{\partial t^2}=\frac{\partial^2 v_0(x,t)}{\partial t^2}+\frac{\partial}{\partial t}\left[\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int\limits_0^{\tau}k(s)v(\xi,\tau-s)dsd\xi d\nu\right],$$

we introduce the notation $t-\tau=\nu$:

$$\frac{\partial^2 v(x,t)}{\partial t^2}=\frac{\partial^2 v_0(x,t)}{\partial t^2}+\frac{\partial}{\partial t}\left[\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,\nu,\xi)\int\limits_0^{t-\nu}k(s)v(\xi,t-\nu-s)dsd\xi d\nu\right],$$

further we have

$$\frac{\partial^2 v(x,t)}{\partial t^2}=\frac{\partial^2 v_0(x,t)}{\partial t^2}+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,\nu,\xi)\left[k(t-\nu)v(\xi,0)+\int\limits_0^{t-\nu}k(s)v_t(\xi,t-\nu-s)ds\right]d\xi d\nu.$$

Passing to the old variable τ, we obtain

$$\frac{\partial^2 v(x,t)}{\partial t^2}=\frac{\partial^2 v_0(x,t)}{\partial t^2}+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)k(\tau)\psi(\xi)d\xi d\tau$$
$$+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int\limits_0^{\tau}k(s)v_t(\xi,\tau-s)dsd\xi d\tau.$$

Substituting (14) into the last equality, we obtain

$$v_{tt}(x,t)=\frac{a^2}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\psi^{(4)}\left(x+2\eta\sqrt{at}\right)\sin\left(\eta^2+\frac{\pi}{4}\right)d\eta-\frac{a}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\varphi^{(6)}\left(x+2\eta\sqrt{at}\right)\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta$$
$$+k(t)\varphi(x)+\frac{a}{\sqrt{\pi}}\int\limits_0^t\int\limits_{-\infty}^{\infty}k(\tau)\varphi''\left(x+2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right)d\zeta d\tau$$
$$+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)k(\tau)\psi(\xi)d\xi d\tau+\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int\limits_0^{\tau}k(s)v_t(\xi,\tau-s)dsd\xi d\tau.$$

In the last equality by putting $x=0$ and using the additional condition (7), we obtain

$$k(t)=\frac{1}{\varphi(0)}\left[g'''(t)-\frac{a^2}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\psi^{(4)}\left(2\eta\sqrt{at}\right)\sin\left(\eta^2+\frac{\pi}{4}\right)d\eta\right.$$
$$+\frac{a}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\varphi^{(6)}\left(2\eta\sqrt{at}\right)\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta-\frac{a}{\sqrt{\pi}}\int\limits_0^t\int\limits_{-\infty}^{\infty}k(\tau)\varphi''\left(2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right)d\zeta d\tau$$
$$\left.-\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)k(\tau)\psi(\xi)d\xi d\tau-\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int\limits_0^{\tau}k(s)v_t(\xi,\tau-s)dsd\xi d\tau\right].\qquad(18)$$

Thus, we obtain a closed system of integral equations for finding the functions $k(t)$, $v(x,t)$, and $v_t(x,t)$, where $v(x,t)$ and $v_t(x,t)$ as follows from (9) and (13) are defined by the following integral equations

$$v(x,t)=\int\limits_{-\infty}^{\infty}\psi(\xi)G_1(x,t,\xi)d\xi-a^2\int\limits_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_2(x,t,\xi)d\xi+\int\limits_0^t\int\limits_{-\infty}^{\infty}k(\tau)\varphi(\xi)G_2(x,t-\tau,\xi)d\xi d\tau$$

$$+\int_{-\infty}^{\infty} G_2(x,t-\tau,\xi)\int_0^{\tau} k(s)v(\xi,\tau-s)dsd\xi d\tau, \tag{19}$$

$$v_t(x,t)=\frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\psi''\left(x+2\eta\sqrt{at}\right)\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta-a^2\int_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_1(x,t,\xi)d\xi$$

$$+\int_0^t\int_{-\infty}^{\infty} k(\tau)\varphi(\xi)G_1(x,t-\tau,\xi d\xi d\tau+\int_0^t\int_{-\infty}^{\infty} G_1(x,t-\tau,\xi)\int_0^{\tau} k(s)v(\xi,\tau-s)dsd\xi d\tau. \tag{20}$$

Let us write the system of equations (18)–(20) as an operator equation

$$\omega = A\omega, \tag{21}$$

where

$$\omega=[\omega_1,\omega_2,\omega_3]^T=[k(t),v(x,t),v_t(x,t)]^T$$

is a vector function, ^{T} is a transpose sign, and the operator A is defined on the set of functions $\omega\in C(D_T)$ and according to the equations (18)–(20) has the form $A=(A_1,A_2,A_3)$:

$$A_1\omega=\frac{1}{\varphi(0)}\Bigg[g'''(t)-\frac{a^2}{\sqrt{\pi}}\int_{-\infty}^{\infty}\psi^{(4)}\left(2\eta\sqrt{at}\right)\sin\left(\eta^2+\frac{\pi}{4}\right)d\eta$$

$$+\frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi^{(6)}(2\eta\sqrt{at})\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta$$

$$-\frac{a}{\sqrt{\pi}}\int_0^t\int_{-\infty}^{\infty}\omega_1(\tau)\varphi''\left(2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right)d\zeta d\tau-\int_0^t\int_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\omega_1(\tau)\psi(\xi)d\xi d\tau$$

$$-\int_0^t\int_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int_0^{\tau}\omega_1(s)\omega_3(\xi,\tau-s)dsd\xi d\tau\Bigg];$$

$$A_2\omega=\int_{-\infty}^{\infty}\psi(\xi)G_1(x,t,\xi)d\xi-a^2\int_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_2(x,t,\xi)d\xi+\int_0^t\int_{-\infty}^{\infty}\omega_1(\tau)\varphi(\xi)G_2(x,t-\tau,\xi)d\xi d\tau$$

$$+\int_0^t\int_{-\infty}^{\infty}G_2(x,t-\tau,\xi)\int_0^{\tau}\omega_1(s)\omega_2(\xi,\tau-s)dsd\xi d\tau,$$

$$A_3\omega=\frac{a}{\sqrt{\pi}}\int_{-\infty}^{\infty}\psi''(x+2\eta\sqrt{at})\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta-a^2\int_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_1(x,t,\xi)d\xi$$

$$+\int_0^t\int_{-\infty}^{\infty}\omega_1(\tau)\varphi(\xi)G_1(x,t-\tau,\xi)d\xi d\tau+\int_0^t\int_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int_0^{\tau}\omega_1(s)\omega_2(\xi,\tau-s)dsd\xi d\tau.$$

We denote by C_ρ the Banach space of continuous functions generated by the family of weight norms

$$||\omega||_\rho=\max\left\{\sup_{t\in[0,T]}\left|\omega_1(t)e^{-\rho t}\right|;\ \sup_{(x,t)\in D_T}\left|\omega_2(x,t)e^{-\rho t}\right|;\ \sup_{(x,t)\in D_T}\left|\omega_3(x,t)e^{-\rho t}\right|;\right\},\quad \rho\geq 0,$$

$\sigma \in (0,1)$ is some fixed number.

Obviously, at $\rho = 0$ this space is the space of continuous functions with the usual norm. This norm will be denoted hereafter by $|||\omega||$. By virtue of the inequality

$$e^{-\rho T}||\omega|| \leq ||\omega||_\rho \leq ||\omega||,$$

the norms $||\omega||_\rho$ and $||\omega||$ are equivalent for any fixed $T \in (0,\infty)$. We will choose the number ρ later. Let $B_\rho(\omega_0, r)$ be a ball of radius r centered at the point ω_0 of a certain weight space $C_\rho(\rho \geq 0)$, where

$$\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03})^T$$

$$= \left[\frac{1}{\varphi(0)}\left[g'''(t) - \frac{a^2}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\psi^{(4)}\left(2\eta\sqrt{at}\right)\sin\left(\eta^2+\frac{\pi}{4}\right)d\eta\right.\right.$$

$$\left.+\frac{a}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\varphi^{(6)}\left(2\eta\sqrt{at}\right)\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta\right];$$

$$\int\limits_{-\infty}^{\infty}\psi(\xi)G_1(x,t,\xi)d\xi - a^2\int\limits_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_2(x,t,\xi)d\xi;$$

$$\left.\frac{a}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\psi''(x+2\eta\sqrt{at})\cos\left(\eta^2+\frac{\pi}{4}\right)d\eta - a^2\int\limits_{-\infty}^{\infty}\varphi^{(4)}(\xi)G_1(x,t,\xi)d\xi\right].$$

It is easy to see that for $\omega \in B_\rho(\omega_0, r)$ there is an estimate $||\omega||_\rho \leq ||\omega_0||_\rho + r \leq r_0$, where $r_0 = ||\omega_0|| + r$ is a known number.

Let $\omega(x,t) \in B_\rho(\omega_0, r)$. We show that given a suitable choice $\rho > 0$, the operator A converts a ball into a ball, i.e., $A\omega \in B_\rho(\omega_0, r)$. In fact, making the norm of differences, for $(x,t) \in D_T$ we have

$$||A_1\omega - \omega_{01}||_\rho = \sup_{t\in[0,T]}\left|(A_1\omega - \omega_{01})e^{-\rho t}\right|$$

$$= \sup_{(x,t)\in D_T}\left|-\frac{a}{\sqrt{\pi}}\int\limits_0^t\int\limits_{-\infty}^{\infty}\omega_1(\tau)e^{-\rho\tau}\varphi''\left(2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right)e^{-\rho(t-\tau)}d\zeta d\tau\right.$$

$$-\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\omega_1(\tau)e^{-\rho\tau}\psi(\xi)e^{-\rho(t-\tau)}d\xi d\tau$$

$$\left.-\int\limits_0^t\int\limits_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int\limits_0^\tau\omega_1(s)e^{-\rho s}\omega_3(\xi,\tau-s)e^{-\rho(\tau-s)}e^{-\rho(t-\tau)}ds d\xi d\tau\right|$$

$$\leq \left(2||\omega_1||_\rho\left[\frac{a}{\sqrt{\pi}}||\varphi''||_{C_0(\mathbb{R})} + ||\psi||_{C_0(\mathbb{R})}\right] + ||\omega_1||_\rho||\omega_3||_\rho\left(\frac{2}{\rho}+T\right)\right)\frac{1}{\rho}$$

$$\leq r_0\left(2\left[\frac{a}{\sqrt{\pi}}||\varphi''||_{C_0(\mathbb{R})} + ||\psi||_{C_0(\mathbb{R})}\right] + r_0\left(\frac{2}{\rho}+T\right)\right)\frac{1}{\rho} := \alpha_1\frac{1}{\rho},$$

$$||A_2\omega - \omega_{02}||_\rho = \sup_{(x,t)\in D_T}\left|(A_2\omega - \omega_{02})e^{-\rho t}\right|$$

$$= \sup_{(x,t)\in D_T}\left|\int\limits_0^t\int\limits_{-\infty}^{\infty}\omega_1(\tau)e^{-\rho\tau}\varphi(\xi)G_2(x,t-\tau,\xi)e^{-\rho(t-\tau)}d\xi d\tau\right.$$

$$+\int_0^t\int_{-\infty}^{\infty} G_2(x,t-\tau,\xi)\int_0^{\tau}\omega_1(s)e^{-\rho s}\omega_2(\xi,\tau-s)e^{-\rho(\tau-s)}e^{-\rho(t-\tau)}dsd\xi d\tau$$

$$\leq\left|\frac{4}{\rho}T||\omega_1||_\rho||\varphi||_{C_0(\mathbb{R})}+\frac{2}{\rho}T||\omega_1||_\rho||\omega_2||_\rho\left(\frac{2}{\rho}+T\right)\right|\leq 2r_0T\left(2||\varphi||_{C_0(\mathbb{R})}+r_0\left(\frac{2}{\rho}+T\right)\right)\frac{1}{\rho}:=\alpha_2\frac{1}{\rho},$$

$$||A_3\omega-\omega_{03}||_\rho=\sup_{(x,t)\in D_T}\left|(A_3\omega-\omega_{03})e^{-\rho t}\right|$$

$$=\sup_{(x,t)\in D_T}\left|\int_0^t\int_{-\infty}^{\infty}\omega_1(\tau)e^{-\rho\tau}\varphi(\xi)G_1(x,t-\tau,\xi)e^{-\rho(t-\tau)}d\xi d\tau\right.$$

$$\left.+\int_0^t\int_{-\infty}^{\infty} G_1(x,t-\tau,\xi)\int_0^{\tau}\omega_1(s)e^{-\rho s}\omega_2(\xi,\tau-s)e^{-\rho(\tau-s)}e^{-\rho(t-\tau)}dsd\xi d\tau\right|$$

$$\leq\frac{2}{\rho}||\omega_1||_\rho||\varphi||_{C_0(\mathbb{R})}+\frac{1}{\rho}||\omega_1||_\rho||\omega_2||_\rho\left(\frac{2}{\rho}+T\right)\leq r_0\left(2||\varphi||_{C_0(\mathbb{R})}+r_0\left(\frac{2}{\rho}+T\right)\right)\frac{1}{\rho}:=\alpha_3\frac{1}{\rho},$$

where

$$\alpha_1=r_0\left(2\left[\frac{a}{\sqrt{\pi}}||\varphi''||_{C_0(\mathbb{R})}+||\psi||_{C_0(\mathbb{R})}\right]+r_0\left(\frac{2}{\rho}+T\right)\right),$$

$$\alpha_2=2r_0T\left(2||\varphi||_{C_0(\mathbb{R})}+r_0\left(\frac{2}{\rho}+T\right)\right),\quad \alpha_3=r_0\left(2||\varphi||_{C_0(\mathbb{R})}+r_0\left(\frac{2}{\rho}+T\right)\right).$$

Choosing $\rho\geq\alpha_0$, where $\alpha_0=\frac{1}{r}\max(\alpha_1,\alpha_2,\alpha_3)$, we obtain that the operator A translates the ball $B_\rho(\omega_0,r)$ into the ball $B_\rho(\omega_0,r)$. Now let ω_1 and ω_2 be any two elements of $B_\rho(\omega_0,r)$. Then, using auxiliary inequalities of the form

$$\left|\omega_i^1\omega_j^1-\omega_i^2\omega_j^2\right|e^{-\rho t}\leq\left|\omega_i^1\right|\left|\omega_j^1-\omega_j^2\right|e^{-\rho t}+\left|\omega_j^2\right|\left|\omega_i^1-\omega_i^1\right|e^{-\rho t}\leq 2r_0\left|\left|\omega^1-\omega^2\right|\right|_\rho,\quad (x,t)\in D_T,$$

we get

$$\left|\left|(A\omega^1-A\omega^2)_1\right|\right|_\rho=\sup_{t\in[0,T]}\left|(A\omega^1-A\omega^2)_1e^{-\rho t}\right|$$

$$=\sup_{(x,t)\in D_T}\left|-\frac{a}{\sqrt{\pi}}\int_0^t\int_{-\infty}^{\infty}(\omega_1^1-\omega_1^2)(\tau)e^{-\rho\tau}\varphi''\left(2\zeta\sqrt{a(t-\tau)}\right)\cos\left(\zeta^2+\frac{\pi}{4}\right)e^{-\rho(t-\tau)}d\zeta d\tau\right.$$

$$-\int_0^t\int_{-\infty}^{\infty}G_1(x,t-\tau,\xi)(\omega_1^1-\omega_1^2)(\tau)e^{-\rho\tau}\psi(\xi)e^{-\rho(t-\tau)}d\xi d\tau$$

$$-\int_0^t\int_{-\infty}^{\infty}G_1(x,t-\tau,\xi)\int_0^{\tau}\Big[\omega_1^1(s)e^{-\rho s}(\omega_3^1-\omega_3^2)(\xi,\tau-s)e^{-\rho(t-s)}e^{-\rho(t-\tau)}$$

$$\left.+\,\omega_3^2(\xi,\tau-s)e^{-\rho(\tau-s)}(\omega_1^1-\omega_1^2)(s)e^{-\rho s}e^{-\rho(t-\tau)}\Big]dsd\xi d\tau\right|$$

$$\leq\left[\frac{2a}{\sqrt{\pi}}||\varphi''||_{C_0(\mathbb{R})}+2||\psi||_{C_0(\mathbb{R})}+2r_0\left(\frac{2}{\rho}+T\right)\right]||\omega^1-\omega^2||_\rho\frac{1}{\rho}:=\beta_1\frac{1}{\rho}||\omega^1-\omega^2||_\rho,$$

$$\left|\left|(A\omega^1-A\omega^2)_2\right|\right|_\rho=\sup_{(x,t)\in D_T}\left|(A\omega^1-A\omega^2)_2e^{-\rho t}\right|$$

$$= \sup_{(x,t)\in D_T} \Bigg| \int_0^t \int_{-\infty}^{\infty} (\omega_1^1 - \omega_1^2)(\tau) e^{-\rho\tau} \varphi(\xi) G_2(x, t-\tau, \xi) e^{-\rho(t-\tau)} d\xi d\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} G_2(x, t-\tau, \xi) \int_0^\tau \Big[\omega_1^1(s) e^{-\rho s} (\omega_2^1 - \omega_2^2)(\xi, \tau - s) e^{-\rho(\tau - s)} e^{-\rho(t-\tau)}$$

$$+ \omega_2^2(\xi, \tau - s) e^{-\rho(\tau-s)} (\omega_1^1 - \omega_1^2)(s) e^{-\rho s} e^{-\rho(t-\tau)} ds d\xi \Big] d\tau$$

$$\leq 4T \left[||\varphi||_{C_0(\mathbb{R})} + r_0 \left(\frac{2}{\rho} + T \right) \right] ||\omega^1 - \omega^2||_\rho \frac{1}{\rho} := \beta_2 \frac{1}{\rho} ||\omega^1 - \omega^2||_\rho,$$

$$||(A\omega^1 - A\omega^2)_3||_\rho = \sup_{(x,t)\in D_T} \left| (A\omega^1 - A\omega^2)_3 e^{-\rho t} \right|$$

$$= \sup_{(x,t)\in D_T} \Bigg| \int_0^t \int_{-\infty}^{\infty} (\omega_1^1 - \omega_1^2)(\tau) e^{-\rho\tau} \varphi(\xi) G_1(x, t-\tau, \xi) e^{-\rho(t-\tau)} d\xi d\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} G_1(x, t-\tau, \xi) \int_0^\tau \Big[\omega_1^1(s) e^{-\rho s} (\omega_2^1 - \omega_2^2)(\xi, \tau - s) e^{-\rho(\tau - s)} e^{-\rho(t-\tau)}$$

$$+ \omega_2^2(\xi, \tau - s) e^{-\rho(\tau-s)} (\omega_1^1 - \omega_1^2)(s) e^{-\rho s} e^{-\rho(t-\tau)} ds d\xi \Big] d\tau$$

$$\leq \left[2||\varphi||_{C_0(\mathbb{R})} + 2r_0 \left(\frac{2}{\rho} + T \right) \right] ||\omega^1 - \omega^2||_\rho \frac{1}{\rho} := \beta_3 \frac{1}{\rho} ||\omega^1 - \omega^2||_\rho,$$

where

$$\beta_1 = \frac{2a}{\sqrt{\pi}} ||\varphi''||_{C_0(\mathbb{R})} + 2||\psi||_{C_0(\mathbb{R})} + 2r_0 \left(\frac{2}{\rho} + T \right),$$

$$\beta_2 = 4T \left[||\varphi||_{C_0(\mathbb{R})} + r_0 \left(\frac{2}{\rho} + T \right) \right], \quad \beta_3 = 2||\varphi||_{C_0(\mathbb{R})} + 2r_0 \left(\frac{2}{\rho} + T \right).$$

Let $\beta_0 = \max(\beta_1, \beta_2, \beta_3)$. As follows from the estimations made, if the number ρ is chosen from the condition $\rho > \max(\alpha_0, \beta_0)$, then the operator A is compressive on $B_\rho(\omega_0, r)$. Then, according to Banach's principle [30], the equation (21) has and yet a unique solution in $B_\rho(\omega_0, r)$ for any fixed $T > 0$. Thus, Theorem 1 is proved.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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