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The problem of determining the electric prehistory of the electrically conductive medium

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Abstract. In this paper, we study the problem of determining the electrical prehistory of a homogeneous anisotropic medium from the integro-differential Maxwell's equations. As an additional condition is given with respect to the Fourier image of the stress of the electric field at the value $\nu = 0$ (ν is the transformation parameter). It is shown that if the inverse problem data $g(t)$ satisfies certain conditions of consistency and smoothness, then there is a unique solution to the inverse problem, and this solution continuously depends on the vector function $g(t)$ and on the known functions.

Keywords: inverse problem, Fourier transform, dirac function, Gronwall's inequality.

MSC (2010): 35L20; 35R30; 35Q99

1 Problem statement and main result

In rapidly changing electromagnetic fields, the frequencies of which are not limited by the condition that they are small in comparison with the frequencies typical for the establishment of the electric and magnetic polarization of the substance, the unambiguous dependence of D and B (induction of the electric and magnetic fields, respectively) on the values of E and H (the strength of the corresponding fields) is violated. the same moment in time. It turns out that the values of D and B at a given time depends not only on E and H, but also on the entire history of the action of these fields (such an environment is called an environment with aftereffect)[7]. This circumstance is an expression of the fact that the establishment of the electric and magnetic polarization of the substance does not have time to follow the change in the electromagnetic field. The most general view of the linear relationship between $D(x,t)$, $B(x,t)$ and the corresponding meanings of the functions $E(x,t)$, $H(x,t)$ at all previous points in time can be written as integral relations [7]:

$$D(x,t) = \hat{\epsilon}E + \int_0^t \varphi(t-\tau)E(x,\ \tau)d\tau,$$

$$B(x,t) = \hat{\mu}H, \tag{1.1}$$

$$E = (E_1, E_2, E_3),\ H = (H_1, H_2, H_3),\ D = (D_1, D_2, D_3),$$

$$B = (B_1, B_1, B_3),\ x = (x_1, x_2, x_3),$$

where the nondegenerate matrices $\hat{\epsilon} = (\hat{\epsilon}_{ij})_{3\times 3}$ and $\hat{\mu} = (\hat{\mu}_{ij})_{3\times 3}$ are the permittivity and permeability matrices, respectively; $\varphi(t) = diag\,(\varphi_1,\ \varphi_2,\ \varphi_3)$, is diagonal matrix function representing memory. These functions are finite for all values of their argument and tend to zero at $t \to \infty$. The latter circumstance is an expression of the fact that the values of $D(x,t)$, $B(x,t)$ at a given time cannot be noticeably influenced by the values of $E(x,t)$, $H(x,t)$ at very old moments. The physical mechanism underlying the integral dependencies of the form (1.1) is in the process of establishing electromagnetic polarization. Therefore, the interval of values in which the function $\varphi(t)$, differs noticeably from zero, the magnitude of the relaxation time characterizing the rate of these processes.

Let, in accordance with equalities (1.1), the set of vectors E, H be the solution of the Cauchy problem for the system of Maxwell equations with zero initial data:

$$\nabla \times H = \frac{\partial D(x,t)}{\partial t} + j, \quad \nabla \times E = -\frac{\partial B(x,t)}{\partial t}, (x,t) \in \mathbb{R}^4,$$

$$(E,H)\Big|_{t\le 0} = 0. \tag{1.2}$$

Here $\nabla \times H$ is the cross product of vectors ∇ and H; $j = j(x,t)$ is a given function characterizing the external current density with components $j_i = j_i(x,t)$. Matrices $\hat{\epsilon}$ and $\hat{\mu}$ in equations (1.1) are assumed to be known and constant. Moreover, $\hat{\epsilon}$ is a symmetric positive definite matrix. We will consider problem (1.1), (1.2) for the case when the function $j(x,t)$ has the form

$$j(x,t) = \vec{e}\delta(x)f(t), \tag{1.3}$$

where $\vec{e} = (1,0,0)$ is a unit vector; $\delta(x) = \delta(x_1)\delta(x_2)\delta(x_3)$ is the Dirac function depending on spatial variables and concentrated at points $x_1 = 0, x_2 = 0, x_3 = 0$; $f(t)$ is an enough smooth function.

Problem (1.1)-(1.3) of finding vectors $E(x,t)$, $H(x,t)$ for given matrix functions $\hat{\epsilon}$, $\hat{\mu}$, $f(t)$, $\varphi(t)$ is called the *direct problem* for Maxwell's integro-differential equations in a homogeneous anisotropic medium.

Let $\tilde{E} = \left(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\right)(\nu,t)$, $\tilde{H} = \left(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3\right)(\nu,t)$ be the Fourier transform of functions $(E,H)(x,t)$ with respect to variables $(x_1,x_2,x_3) \in \mathbb{R}^3$, i.e.

$$\tilde{E}_j(\nu,t) = \int\limits_{\mathbb{R}^3} E_j(x,t)e^{i(x,\nu)}dx, \quad \tilde{H}_j(\nu,t) = \int\limits_{\mathbb{R}^3} H_j(x,t)e^{i(x,\nu)}dx,$$

$$\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{R}^3, \quad (x,\nu) = \sum_{\lambda=1}^{3} x_\lambda \nu_\lambda,\ j = 1,2,3,$$

where ν is the transformation parameter.

Let us pose the following inverse problem: determine the matrix functions $\varphi(t) = diag(\varphi_1, \varphi_2, \varphi_3)$ included in the integrals of equations (1.1), if the information

$$\left(\tilde{E}\right)(0,t) = g(t), \quad g(t) = (g_1, g_2, g_3), \tag{1.4}$$

is given with respect to the Fourier image of the solution of direct problem (1.1),(1.2) for any time $t \geq 0$ and values $\nu = 0$.

Everywhere below, the numbers $\varphi_i(0), i = 1, 2, 3$ are considered known and we will be interested in the question of finding $\varphi_i(t), i = 1, 2, 3$ at $t > 0$.

Definition 1.1. The solution of the inverse problem is called matrix functions $\varphi(t)$ such that the corresponding solution of the direct problem (1.1)-(1.3) satisfies condition (1.4).

Many technically important materials and crystals that are becoming popular in new technologies are anisotropic. The physical properties of homogeneous isotropic crystals do not depend on the direction and position inside the medium. At the same time, the physical properties of anisotropic crystals strongly depend on the orientation and position. An anisotropic crystal is called homogeneous if its physical properties depend on orientation and do not depend on position.

Among works devoted to the determination of the integrand in a hyperbolic equation with sources concentrated at points and at the boundaries of considered domain, we note the papers [2], [3], [5]. Similar problems with distributed sources are investigated in papers [6], [8]. The paper [4] presents a numerical method for determining the dependence of the dielectric constant on frequency, when the unchanged part of the dielectric constant was considered known. In article [1], the problem of restoring the memory of the electric field from the integro-differential equations of electrodynamics is investigated. In this article, we study the problem of determining the electrical prehistory of a homogeneous anisotropic medium. It is shown that if the vector function $g(t)$ satisfies certain conditions of consistency and smoothness, then there is a unique solution to the inverse problem, and this solution continuously depends on the vector function $g(t)$.

The mains result of this article are the following theorems.

Theorem 1.2. *Suppose that $g(t) \in C^3[0,T]$, $f(t) \in C^2[0,T]$, $f(0) \neq 0$ and the matching conditions are satisfied*

$$g(0) = 0, \quad g'(0) = A_0^{-1}\vec{e}_0 f(0), \quad i\Phi_0 g'(0) - g''(0) = A_0^{-1}\vec{e}_0 f'(0).$$

Then inverse problem (1.1)-(1.4) has a unique solution $\varphi(t) \in C^1[0,T]$ for fixed $T > 0$.

By $G(\gamma)$ we denote the sets of functions $g(t)$, $f(t)$ satisfying the conditions of Theorem 1.2 and condition

$$\max\left\{\max_{1\leq i\leq 3} \|g_i\|_{C^3[0,T]}, \|f\|_{C^2[0,T]}\right\} \leq \gamma < \infty$$

here γ is a given number

Theorem 1.3. *Let $\varphi^{(m)}(t) \in C^1[0,T]$ be the solution to the inverse problem* (1.1)-(1.4) *with $(g^m(t), f^m(t)) \in G(\gamma)$, respectively. Then there is a positive constant depending on $T, \gamma, \hat{\epsilon}_{ij}, i,j = 1,2,3$ such that the stability estimate*

$$\sum_{j=1}^{3} \|\varphi_j^1 - \varphi_j^2\|_{C^1[0,T]} \leq C \sum_{j=1}^{3} \left[\|g_j^1 - g_j^2\|_{C^3[0,T]} + \|f^1 - f^2\|_{C^2[0,T]} \right], \tag{1.5}$$

is valid.

2 Transformation of the Maxwell system of equations to a symmetric hyperbolic system of the first order

It is not difficult to show that the system of equations (1.1)-(1.3) can be written in the form of the following symmetric hyperbolic system of the first order:

$$A_0 \frac{\partial V}{\partial t} + \sum_{i=0}^{3} A_i \frac{\partial V}{\partial x_i} + \Phi_0 V + \int_0^t \Phi'(t-\tau) V(x,\tau) d\tau = F(x,t), \tag{2.1}$$

with the initial condition

$$V \Big|_{t \leq 0} = 0, \tag{2.2}$$

where

$$A_0 := \begin{pmatrix} \hat{\epsilon} & 0 \\ 0 & \hat{\mu} \end{pmatrix}_{6\times 6}, \quad A_j := \begin{pmatrix} 0 & A_j^1 \\ (A_j^1)^* & 0 \end{pmatrix}_{6\times 6}, \quad A_1^1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$A_2^1 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3^1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Phi_0 := \begin{pmatrix} \varphi(0) & 0 \\ 0 & 0 \end{pmatrix}_{6\times 6},$$

$$\Phi'(t) := \begin{pmatrix} \varphi'(t) & 0 \\ 0 & 0 \end{pmatrix}_{6\times 6}, \quad V := (E,H)^\top, \quad F := (-j, 0_{1\times 3})^\top;$$

$\top$ is the transposition symbol; $0_{1\times 3}$ means a vector row with elements $0,0,0$; $\Phi'(t) := (\partial/\partial t)\Phi(t)$.

We apply the Fourier transform (1.4) to both sides of the equations (2.1)-(2.2). The Fourier transformation of the vector of function $V(x,t)$ for any finite t>0 exists, since function $V(x,t)$ as a solution to problem (2.1), (2.2) is the sum of a singular generalized vector of a function of finite order and a regular vector of a function whose supports are bounded [9]. For a fixed $\nu \in \mathbb{R}$, the vector of function $\tilde{V}(\nu,t)$ ($\tilde{V}(\nu,t)$ is the Fourier transform of the vector of function $V(\nu,t)$ with respect to variable x) satisfies the initial differential equations

$$A_0\frac{\partial \tilde{V}}{\partial t} - iB\tilde{V} + \int\limits_0^t \Phi'(t-\tau)\tilde{V}(\nu,\tau)d\tau = \tilde{F}(t), \tag{2.3}$$

initial and, as follows from (1.4) additional conditions, respectively

$$\tilde{V}\Big|_{t\leq 0} = 0, \tag{2.4}$$

$$\tilde{V}\Big|_{\nu=0} = g(t), \quad g(t) = (g_1, g_2, g_3), \quad t \geq 0. \tag{2.5}$$

In equation (2.3), we introduced the notation $B := \sum\limits_{j=1}^{3} \nu_j A_j + \Phi_0$, $\tilde{F}(t) = \vec{e}f(t)$, $\vec{e}_0 := (1,0,0,0,0,0)^\top$.

We will calculate the matrix A_0^{-1}, which is the inverse of A_0. If we denote by $\epsilon = (\epsilon_{ij})$, $\mu = (\mu_{ij})$ the inverse matrices of $\hat{\epsilon}, \hat{\mu}$, respectively, then

$$A_0^{-1} := \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix}_{6\times 6},$$

Multiplying equation (2.3) from the left by A_0^{-1}, we get

$$I\frac{\partial \tilde{V}}{\partial t} - iC\tilde{V} + \int\limits_0^t \Psi(t-\tau)\tilde{V}(\nu,\tau)d\tau = F_0, \tag{2.6}$$

where I is the sixth order identity matrix,

$$C := \begin{pmatrix} \epsilon\varphi(0) & \epsilon\sum\limits_{i=1}^{3}\nu_i A_i^1 \\ \mu\sum\limits_{i=1}^{3}\nu_i (A_i^1)^\top & 0 \end{pmatrix}_{6\times 6},$$

$\Psi(t) := A_0^{-1}\Phi'(t)$, $F_0 := A_0^{-1}\tilde{F} = A_0^{-1}\vec{e}_0 f(t)$.

Thus, inverse problem (1.1)-(1.4) has been reduced to the problem of determining the kernel $\Psi(t)$ of the integral term of equation (2.6) under the given conditions (2.4),(2.5).

3 Proofs of the main results

First, we prove Theorem 1.2. To do this, we integrate the differential equation (2.6). Then, using the initial condition (2.4), we find

$$\tilde{V}(\nu,t) = A_0^{-1}\vec{e}_0 \int_0^t f(\tau)d\tau + iC\int_0^t \tilde{V}(\nu,\tau)d\tau - \int_0^t\int_0^\tau \Psi(\alpha)\tilde{V}(\nu,\tau-\alpha)d\alpha d\tau,\ t\geq 0. \quad (3.1)$$

Considering (2.5), from equality (3.1) we get

$$g(t) = A_0^{-1}\vec{e}_0 \int_0^t f(\tau)d\tau + iC_0\int_0^t g(\tau)d\tau + \int_0^t \Psi(\alpha)\int_0^\tau g(\tau-\alpha)d\tau d\alpha, \quad t\geq 0, \quad (3.2)$$

where $C_0 := A_0^{-1}\Phi_0$. One of the conditions for agreement follows from (3.2): $g(0) = 0$. Differentiating equation (3.2), we arrive at the equality

$$\int_0^t \Psi(\tau)g(t-\tau)d\tau = g'(t) - iC_0 g(t) - A_0^{-1}\vec{e}_0 f(t). \quad (3.3)$$

Assuming $t = 0$ in this equation and considering $g(0) = 0$, we have

$$g'(0) = A_0^{-1}\vec{e}_0 f(0)$$

Differentiating equation (3.3) again, we get

$$\int_0^t \Psi(\tau)g'(t-\tau)d\tau = g''(t) - iC_0 g'(t) - A_0^{-1}\vec{e}_0 f'(t).$$

Multiplying this equality on the left by A_0, we find

$$\int_0^t \Phi'(\tau)g'(t-\tau)d\tau = A_0 g''(t) - i\Phi_0 g'(t) - \vec{e}_0 f'(t).$$

Taking the derivative again, we obtain a linear integral equation for the matrix function $\Phi'(t)$:

$$\Phi'(t)g'(0) + \int_0^t \Phi'(\tau)g''(t-\tau)d\tau = A_0 g'''(t) - i\Phi_0 g''(t) - \vec{e}_0 f''(t).$$

It can be written with respect to the components $\varphi_j'(t),\ j = 1,2,3$ of matrix $\Phi'(t)$ in the form

$$\varphi_j'(t) + \int_0^t \varphi_j'(\tau)\frac{g''(t-\tau)}{\epsilon_{j1} f_j(0)}d\tau =$$

$$= \frac{1}{\epsilon_{j1} f_j(0)} \sum_{j=1}^{3} \left[\hat{\epsilon}_{jk} g_k'''(t) - i\varphi_j(0) g_k''(t)\right] + \frac{f_j''(t)}{\epsilon_{j1} f_j(0)}, \quad j = 1, 2, 3. \tag{3.4}$$

Equation (3.4) is a linear Volterra integral equation of the second kind for the unknown function φ_i', $i = 1, 2, 3$. Under the conditions of Theorem 1.2, it has continuous free terms and kernels. To solve equation (3.4), we use the method of successive approximations. For this purpose, we present solution (3.4) in the form of series

$$\varphi_j'(t) = \sum_{n=0}^{\infty} \left(\varphi_j'\right)_n (t), \tag{3.5}$$

where

$$\left(\varphi_j'\right)_0(t) = \frac{1}{\epsilon_{j1} f_j(0)} \sum_{j=1}^{3} \left[\hat{\epsilon}_{jk} g_k'''(t) - i\varphi_j(0) g_k''(t)\right] + \frac{f_j''(t)}{\epsilon_{j1} f_j(0)}$$

$$\left(\varphi_j'\right)_n(t) = \int_0^t \left(\varphi_j'\right)_{n-1}(\tau) \frac{g''(t-\tau)}{\epsilon_{j1} f_j(0)} d\tau, \quad n = 1, 2, \ldots. \tag{3.6}$$

Let

$$C_1 = \max_{t\in[0,T]} \left| \frac{1}{\epsilon_{j1} f_j(0)} \sum_{j=1}^{3} \left[\hat{\epsilon}_{jk} g_k'''(t) - i\varphi_j(0) g_k''(t)\right] + \frac{f_j''(t)}{\epsilon_{j1} f_j(0)} \right|,$$

$$C_2 = \max_{t\in[0,T]} \left| \frac{g''(t)}{\epsilon_{j1} f_j(0)} \right|$$

The constants C_1, C_2 depend on the set of given numbers T, γ, $\hat{\epsilon}_{ij}$, i, $j = 1, 2, 3$. From relations (3.6) we obtain the estimate

$$\left| \left(\varphi_j'\right)_n (t) \right| \le C_1 \sum_{i=0}^{n} C_2^i \frac{t^i}{i!}. \tag{3.7}$$

Estimate (3.7) shows that the series (3.5) converges evenly on the segment $[0, T]$, since it is mogged by the converging numerical series $C_1 \sum\limits_{n=0}^{\infty} C_2^i \frac{T^i}{i!} = C_1 exp(C_2 T)$, and therefore continuously determines the solution (3.4) on the segment $[0, T]$. As a rule, this solution is the only one.

Using the functions already found φ_i', $i = 1, 2, 3$, functions φ_i, $i = 1, 2, 3$ is determined by the formulas:

$$\varphi_j(t) = \varphi_j(0) + \int_0^t \varphi_j'(\tau) d\tau, \quad j = 1, 2, 3. \tag{3.8}$$

Thus Theorem 1.2 is proved.

To prove Theorem 1.3, consider, for example, the integral equation (3.4) for $\left(g_j^m(t), f^m(t)\right) \in G(\gamma)$. The solutions of the equation corresponding to these functions will be denoted by $(\varphi_j')^m(t),\ m = 1,2$. Composing the difference $(\varphi_j')^1 - (\varphi_j')^2$, we make estimates. As a result, we get

$$\left|(\varphi_j')^1(t) - (\varphi_j')^2(t)\right|_{C[0,T]} \leq C_3 \left\|g_j^1 - g_j^2\right\|_{C^3[0,T]} + C_4 \left\|f^1 - f^2\right\|_{C^2[0,T]} +$$

$$+C_5 \int\limits_0^t \left|(\varphi_j')^1(\tau) - (\varphi_j')^2(\tau)\right| d\tau.$$

Hence, using Gronwall's inequality, we have

$$\left\|(\varphi_j')^1(t) - (\varphi_j')^2(t)\right\|_{C[0,T]} \leq C_6 \left[\left\|g_j^1 - g_j^2\right\|_{C^3[0,T]} + \left\|f^1 - f^2\right\|_{C^2[0,T]}\right].$$

Constants C_i, $i = 3,4,5,6$ depends on the set of given numbers T, γ, $\hat{\epsilon}_{ij}$, $i,j = 1,2,3$. And so Theorem 1.3 is proved.

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