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PARTIAL DIFFERENTIAL EQUATIONS

A Time-Nonlocal Inverse Problem for the Beam Vibration Equation with an Integral Condition

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Abstract—We study the direct problem for transverse vibrations of a homogeneous beam of finite length with time-nonlocal conditions and obtain necessary and sufficient conditions for the existence of its solution. For the direct problem, the inverse problem of determining the time-dependent coefficients of the lower derivative and the right-hand side in the equation is studied. The existence and uniqueness of the solution of the inverse problem are proved. The solution is based on separation of variables, which is used to reduce the problems to an integral equation and a system of integral equations.

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INTRODUCTION

Most beam vibration problems are widely used in structural mechanics and in the stability theory of running shafts, with their solutions leading to higher-order differential equations [1, pp. 143–147]. The paper [2] provides a detailed analysis of foreign publications and results in the field of dynamic behavior of inhomogeneous beams. The papers [3–6] study the solvability of initial–boundary value problems for the beam vibration equation with various boundary conditions. The paper [7] deals with the analytical solution of the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain for any kind of boundary conditions.

The problems of determining the coefficients or the right-hand side of a differential equation based on some known data of its solution are called *inverse problems* of mathematical physics. Inverse problems of determining the kernels of integro-differential equations from the theory of viscoelasticity were studied in [8–10]. The papers [11–13] applied a method for solving inverse problems that is based on the representation of the solution of two-dimensional inverse problems in the form of a trigonometric polynomial in one of the spatial variables. The paper [14] considers the inverse problem of determining the time-dependent coefficient in the equation of transverse vibrations of a beam, which, from a physical point of view, represents its stiffness. Numerical methods for solving inverse problems can be found in the papers [15–20]. The paper [21] considers an inverse problem related to reconstructing the second moment of the cross-section area for a beam using the scheme of a shearing uniformly distributed load. The efficiency of the variational method for solving the inverse problem of finding the coefficients in the Euler–Bernoulli equation was demonstrated in the paper [22].

In the present paper, we study the direct problem with time-nonlocal conditions and the inverse problem with integral overdetermination conditions for the equation of transverse vibrations of a homogeneous beam.

1. STATEMENT OF THE PROBLEM

Consider the following equation of transverse vibrations of a beam of length l resting on the ends:

$$w_{tt} + a^2 w_{xxxx} + q(t)w = f(x,t), \quad (x,t) \in \Sigma, \tag{1}$$

where $a^2 = EJ/(\rho S)$, $f(x,t)$ is the external force, ρ is the beam density, S is the beam cross-section area, E is the modulus of elasticity of the material, J is the moment of inertia of the beam cross section about the horizontal axis, $\Sigma = \{(x,t) : 0 < x < l,\ 0 < t \le T\}$ is a rectangular domain, and T is the time interval. The entire length of the beam is supported by an elastic base with stiffness coefficient $q(t)$.

Direct problem. In the domain Σ, find a solution of Eq. (1) with the following initial and boundary conditions:

$$w(x,0)+\delta_1 w(x,T)=\varphi(x), \quad w_t(x,0)+\delta_2 w_t(x,T)=\psi(x), \quad x\in[0,l], \tag{2}$$

$$\begin{gathered}\varphi(0)=\psi(0)=0, \quad \varphi(l)=\psi(l)=0,\\ w(0,t)=w_{xx}(0,t)=w(l,t)=w_{xx}(l,t)=0, \quad 0\le t\le T.\end{gathered} \tag{3}$$

Definition. A solution of problem (1)–(3) is a function $w(x,t)$ in the class $C^{4,2}_{x,t}(\Sigma)$ satisfying conditions (2) and (3) and making Eq. (1) an identity with positive numbers δ_1, δ_2 and sufficiently smooth functions $q(t)$, $f(x,t)$, and $\varphi(x)$, $\psi(x)$.

Inverse problem. Let $f(x,t)=g(t)f_0(x)$. Find functions $q(t)$ and $g(t)$, $t\in[0,T]$, based on the known equalities

$$\int\limits_0^l y_i(x)w(x,t)\,dx=Y_i(t), \quad i=1,2, \tag{4}$$

where $y_i(x)$ and $Y_i(t)$ are given sufficiently smooth functions satisfying the matching conditions

$$\int\limits_0^l y_i(x)\varphi(x)\,dx=Y_i(0)+\delta_1 Y_i(T), \quad \int\limits_0^l y_i(x)\psi(x)\,dx=Y_i'(0)+\delta_2 Y_i'(T), \quad Y(t)\ne 0. \tag{5}$$

2. STUDYING THE DIRECT PROBLEM

Using the notation $F(x,t):=f(x,t)-q(t)w(x,t)$, Eq. (1) becomes $w_{tt}+a^2 w_{xxxx}=F(x,t)$. A solution of problem (1)–(3) is sought as

$$w(x,t)=\sum_{k=1}^{\infty} w_k(t)X_k(x), \tag{6}$$

where $w_k(t)=\sqrt{(2/l)}\int\limits_0^l w(x,t)\sin(\mu_k x)\,dx$, $X_k(x)=\sqrt{(2/l)}\sin(\mu_k x)$, and $\mu_k=\pi k/l$, $k\in\mathbb{N}$.

Substituting the function (6) into Eq. (1) and conditions (2), after separating the variables we obtain the problem

$$w_k''(t)+\lambda_k^2 w_k(t)=F_k(t;q,w), \quad \lambda_k=a\mu_k^2, \quad k\in\mathbb{N}, \quad 0<t\le T, \tag{7}$$

$$w_k(0)+\delta_1 w_k(T)=\varphi_k, \quad w_k'(0)+\delta_2 w_k'(T)=\psi_k, \quad k\in\mathbb{N}, \tag{8}$$

where

$$F_k(t;q,w)=f_k(t)-q(t)w_k(t), \tag{9}$$

$$f_k(t)=\sqrt{\frac{2}{l}}\int\limits_0^l f(x,t)\sin(\mu_k x)\,dx, \tag{10}$$

$$\begin{gathered}\varphi_k=\sqrt{\frac{2}{l}}\int\limits_0^l \varphi(x)\sin(\mu_k x)\,dx,\\ \psi_k=\sqrt{\frac{2}{l}}\int\limits_0^l \psi(x)\sin(\mu_k x)\,dx, \quad k\in\mathbb{N}.\end{gathered} \tag{11}$$

We write the solution of problem (7), (8) in the form [23]

$$w_k(t) = \frac{1}{\rho_k(T)} E_k(t) + \int\limits_0^T G_k(t,s) F_k(s;q,w)\,ds, \tag{12}$$

where

$$\rho_k(T) = 1 + (\delta_1 + \delta_2)\cos(\lambda_k T) + \delta_1\delta_2,$$

$$E_k(t) = \varphi_k\Big(\cos\lambda_k t + \delta_2 \cos\big(\lambda_k(T-t)\big)\Big) + \frac{\psi_k}{\lambda_k}\Big(\sin(\lambda_k t) - \delta_1 \sin\big(\lambda_k(T-t)\big)\Big),$$

$$G_k(t,s) = \begin{cases} -\dfrac{1}{\lambda_k\rho_k(T)}\Big[\delta_1 \sin\big(\lambda_k(T-t)\big)\cos(\lambda_k s) + \delta_2\cos\big(\lambda_k(T-t)\big)\sin(\lambda_k s) \\ \qquad + \delta_1\delta_2 \sin\big(\lambda_k(s-t)\big)\Big], \qquad s\in[0,t], \\ -\dfrac{1}{\lambda_k\rho_k(T)}\Big[\delta_1 \sin\big(\lambda_k(T-t)\big)\cos(\lambda_k s) + \delta_2\cos\big(\lambda_k(T-t)\big)\sin(\lambda_k s) \\ \qquad + \delta_1\delta_2 \sin\big(\lambda_k(s-t)\big)\Big] + \dfrac{1}{\lambda_k}\sin\big(\lambda_k(s-t)\big), \quad s\in[t,T]. \end{cases}$$

By substituting the expression (12) into (6), we obtain

$$w(x,t) = \sum_{k=1}^{\infty}\left\{\frac{1}{\rho_k(T)}E_k(t) + \int\limits_0^T G_k(t,s)F_k(s;q,w)\,ds\right\}\sin(\mu_k x). \tag{13}$$

It can readily be seen that for $\delta_1 > 0$ and $\delta_2 > 0$ and under the condition $1+\delta_1\delta_2 > \delta_1+\delta_2$ we have the inequality

$$\frac{1}{\rho_k(T)} \le \frac{1}{1-(\delta_1+\delta_2)+\delta_1\delta_2} \equiv \beta > 0. \tag{14}$$

Theorem 1 [24]. *Let $\alpha, \lambda, \mu > 0$, and let $g(t)$ be a continuously differentiable nonnegative function on the interval $[a,b]$ with $a < b \le +\infty$. Furthermore, assume that $\omega(t)$ is integrable and nonnegative on $[a,b]$ and satisfies the inequality*

$$\omega(t) \le \frac{\lambda}{\Gamma(\alpha)}\int\limits_0^t (t-\zeta)^{\alpha-1}\omega(\zeta)\,d\zeta + \mu\int\limits_a^b \omega(\zeta)\,d\zeta + g(t)$$

for every $t\in[a,b)$. If $0 \le \mu(b-a)E_{\alpha,2}(\lambda(b-a)^\alpha) < 1$, then

$$\omega(t) \le E_\alpha\big(\lambda(t-a)^\alpha\big)\omega_0 + g(t) - E_\alpha\big(\lambda(t-a)^\alpha\big)g(a) + \lambda\int\limits_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\big(\lambda(t-\zeta)^\alpha\big)g(\zeta)\,d\zeta,$$

where

$$\omega_0 \le \frac{1}{1-\mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)} \times \left(\mu\int\limits_a^b g(\zeta)\,d\zeta - \mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)g(a) + \mu\lambda\int\limits_a^b (b-\zeta)^\alpha E_{\alpha,\alpha+1}\big(\lambda(b-\zeta)^\alpha\big)g(\zeta)\,d\zeta\right).$$

Theorem 1 uses the two-parameter Mittag-Leffler function

$$E_{\alpha,\gamma} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma+\alpha k)}, \quad \alpha,\gamma>0, \quad z\in\mathbb{R}; \tag{15}$$

in particular, for $\gamma = 1$ it becomes the one-parameter Mittag-Leffler function, $E_{\alpha,1}(z) = E_\alpha(z)$ (see [25, pp. 40–49]).

Substituting the function $F_k(t;q,w)$ (9) into (12), we have

$$w_k(t) = \frac{1}{\rho_k(T)} E_k(t) + \int_0^T f_k(s) G_k(t,s)\,ds - \int_0^T q(s) w_k(s) G_k(t,s)\,ds.$$

Let us estimate this function considering the form of the function $E_k(t)$ for $t\in[0,T]$,

$$\big|w_k(t)\big| \le \beta(1+\delta_2)|\varphi_k| + \frac{\beta(1+\delta_1)}{\lambda_k}|\psi_k| + \frac{\beta\delta}{\lambda_k}\int_0^T \big|f_k(s)\big|\,ds + \frac{\beta}{\lambda_k^2}\int_t^T \big|f_k(s)\big|\,ds$$
$$+ \frac{\bar q\beta\delta}{\lambda_k}\int_0^t \big|w_k(s)\big|\,ds + \frac{\bar q\beta}{\lambda_k}\left(\delta + \frac{1}{\lambda_k}\right)\int_0^T \big|w_k(s)\big|\,ds,$$

where $\delta = \delta_1+\delta_2+\delta_1\delta_2$ and $\bar q = \max\limits_{s\in[0,T]} |q(s)|$. Applying Theorem 1 with $\alpha = 1$ to the last inequality, in view of (15), we obtain the following assertion.

Lemma 1. *Let* $0 < C_{2k}(e^{C_{1k}T}-1)/C_{1k} < 1$; *then one has the estimate*

$$\big|w_k(t)\big| \le \lambda_k \widetilde{C} g_k, \quad k\in\mathbb{N}, \tag{16}$$

where

$$C_{1k} = \frac{\bar q\beta\delta}{\lambda_k}, \quad C_{2k} = \frac{\bar q\beta}{\lambda_k}\left(\delta+\frac{1}{\lambda_k}\right), \quad \widetilde{C} = \frac{\delta}{\lambda_1\delta(2-e^{C_{1k}T}) + e^{C_{1k}T}-1}, \quad k\in\mathbb{N},$$
$$g_k = \beta(1+\delta_2)|\varphi_k| + \frac{\beta(1+\delta_1)}{\lambda_k}|\psi_k| + \frac{\beta}{\lambda_k}\left(\delta+\frac{1}{\lambda_1}\right)\int_0^T \big|f_k(s)\big|\,ds, \quad k\in\mathbb{N}. \tag{17}$$

Further, taking into account (17), from the estimate (16) we have

$$\big|w_k(t)\big| \le \overline{C}_1\Big(\lambda_k|\varphi_k| + |\psi_k| + \big\|f_k(t)\big\|\Big),$$

where $\|f_k\| = \max\limits_{0\le t\le T}|f_k(t)|$. Using Eq. (7) and inequality (12), we obtain the estimate

$$\big|w_k''(t)\big| \le \overline{C}_2\Big(\lambda_k^3|\varphi_k| + \lambda_k^2|\psi_k| + \lambda_k^2\big\|f_k(t)\big\| + \bar q|w_k|\Big) \le \overline{C}_2(\bar q+\lambda_k^2)\Big(\lambda_k|\varphi_k| + |\psi_k| + \big\|f_k(t)\big\|\Big).$$

Thus, we have proved the following assertion.

Lemma 2. *For each* $t\in[0,T]$ *and for sufficiently large* k, *one has the estimates*

$$\big|w_k(t)\big| \le \overline{C}_1\Big(k^2|\varphi_k| + |\psi_k| + \big\|f_k(t)\big\|_C\Big),$$
$$\big|w_k''(t)\big| \le \overline{C}_2\Big(k^6|\varphi_k| + k^4|\psi_k| + k^4\big\|f_k(t)\big\|_C\Big).$$

Here and in the following, the $\overline{C}_i$ *are positive constants.*

Formally, from (6) we compose the series

$$w_{tt} = \sum_{k=1}^{\infty} w_k''(t)\sin(\mu_k x), \tag{18}$$

$$w_{xxxx} = \sum_{k=1}^{\infty} \mu_k^4 w_k(t)\sin(\mu_k x). \tag{19}$$

For any $(x,t)\in\overline{\Sigma}$, based on Lemma 1, the series (6), (18), and (19) are majorized by the series

$$\overline{C}_3\sum_{k=1}^{\infty}\Big(k^6|\varphi_k| + k^4|\psi_k| + k^4\big\|f_k(t)\big\|\Big).$$

The following auxiliary assertion holds true.

Lemma 3. *If the conditions*

$$\begin{gathered}\varphi(x)\in C^6[0,l],\quad \varphi^{VII}(x)\in L_2[0,l],\\ \varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=\varphi^{IV}(0)=\varphi^{IV}(l)=\varphi^{VI}(0)=\varphi^{VI}(l)=0,\\ \psi(x)\in C^4[0,l],\quad \psi^V(x)\in L_2[0,l],\quad \psi(0)=\psi(l)=\psi''(0)=\psi''(l)=\psi^{IV}(0)=\psi^{IV}(l)=0,\\ f(x,t)\in C(\overline{\Sigma})\cap C_x^4(\Sigma),\quad f^V_{xxxxx}(x,t)\in L_2(\Sigma),\\ f(0,t)=f(l,t)=f''_{xx}(0,t)=f''_{xx}(l,t)=f^{IV}_{xxxx}(0,t)=f^{IV}_{xxxx}(l,t)=0\end{gathered}$$

are satisfied, then one has the equalities

$$\varphi_k = \frac{1}{\mu_k^7}\varphi_k^{VII},\quad \psi_k = \frac{1}{\mu_k^5}\psi_k^V,\quad f_k(t) = \frac{1}{\mu_k^5}f_k^V(t), \tag{20}$$

where

$$\varphi_k^{VII} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{VII}(x)\cos(\mu_k x)\,dx,$$

$$\psi_k^V = \sqrt{\frac{2}{l}}\int_0^l \psi^V(x)\cos(\mu_k x)\,dx,$$

$$f_k^V(t) = \sqrt{\frac{2}{l}}\int_0^l f^V_{xxxxx}(x,t)\cos(\mu_k x)\,dx,$$

and the following estimates hold true:

$$\begin{gathered}\sum_{n=1}^{\infty}|\varphi_k^{VII}|^2 \le \|\varphi^{VII}\|_{L_2[0,l]},\\ \sum_{n=1}^{\infty}|\psi_k^{V}|^2 \le \|\psi^{V}\|_{L_2[0,l]},\\ \sum_{n=1}^{\infty}|f_k^{V}(t)|^2 \le \|f^{V}(t)\|_{L_2[0,l]\times C[0,T]}.\end{gathered} \tag{21}$$

By applying integration by parts seven times in the integral for φ_k and five times in the integrals for ψ_k and $f_k(t)$ (see (10) and (11)) and by taking into account the conditions in Lemma 3, we obtain

Eqs. (20). Inequalities (21) are the Bessel inequalities for the coefficients of the Fourier expansions of the functions φ_k^{VII} and ψ_k^V in the cosine system $\{\sqrt{(2/l)}\cos(\mu_k x)\}$ on the interval $(0,l)$. If the functions $\varphi(x)$, $\psi(x)$, and $f(x,t)$ satisfy the conditions in Lemma 2, then, by virtue of the representations (20) and (21), the series (6), (18), and (19) converge uniformly in the rectangle $\overline{\Sigma}$, and so the function (13) is a solution of problem (1)–(3).

3. STUDYING THE INVERSE PROBLEM

Let $f(x,t)=f_0(x)g(t)$, where $f_0(x)$ is a known function. Then $F_k(t;g,q,w)=f_{0k}g(t)-q(t)w_k(t)$. Multiplying both sides of Eq. (1) by $y_i(x)$, $i=1,2$, integrating from 0 to l over the variable x, and taking into account conditions (4), we obtain the equations

$$g(t)\int\limits_0^l f_0(x)y_1(x)\,dx-q(t)Y_1(t)=Y_1''(t)+a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4 w_k(t)y_{1k}, \tag{22}$$

$$g(t)\int\limits_0^l f_0(x)y_2(x)\,dx-q(t)Y_2(t)=Y_2''(t)+a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4 w_k(t)y_{2k}, \tag{23}$$

where $y_{ik}=\sqrt{(2/l)}\int\limits_0^l y_i(x)\sin(\mu_k x)\,dx$, $k=1,2$.

Introduce the notation

$$\int\limits_0^l f_0(x)y_1(x)\,dx=\alpha_1,$$
$$\int\limits_0^l f_0(x)y_2(x)\,dx=\alpha_2$$

and assume that

$$\mathcal{Y}(t)=\alpha_2 Y_1(t)-\alpha_1 Y_2(t)\neq 0,\quad 0\le t\le T. \tag{24}$$

Then from (22) and (23) we find

$$g(t)=\big[\mathcal{Y}(t)\big]^{-1}\left\{Y_1(t)Y_2''(t)-Y_2(t)Y_1''(t)+a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4 w_k(t)\big(Y_1(t)y_{2k}-Y_2(t)y_{1k}\big)\right\}, \tag{25}$$

$$q(t)=\big[\mathcal{Y}(t)\big]^{-1}\left\{\alpha_1 Y_2''(t)-\alpha_2 Y_1''(t)+a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4 w_k(t)(\alpha_1 y_{2k}-\alpha_2 y_{1k})\right\}. \tag{26}$$

After substituting the function (12) into (25) and (26), we obtain the following integral equations for the functions $g(t)$ and $q(t)$:

$$\begin{aligned}g(t)=\big[\mathcal{Y}(t)\big]^{-1}\Bigg\{&Y_1(t)Y_2''(t)-Y_2(t)Y_1''(t)\\&+a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4\left\{\frac{1}{\rho_k(T)}E_k(t)+\int\limits_0^T G_k(t,s)F_k(s;g,q,w)\,ds\right\}\big(Y_1(t)y_{2k}-Y_2(t)y_{1k}\big)\Bigg\},\end{aligned} \tag{27}$$

$$q(t) = \big[\mathcal{Y}(t)\big]^{-1}\Bigg\{\alpha_1 Y_2''(t) - \alpha_2 Y_1''(t)$$
$$+ a^2\sqrt{\frac{l}{2}}\sum_{k=1}^{\infty}\mu_k^4\left\{\frac{1}{\rho_k(T)}E_k(t) + \int_0^T G_k(t,s)F_k(s;g,q,w)\,ds\right\}(\alpha_1 y_{2k} - \alpha_2 y_{1k})\Bigg\}. \tag{28}$$

Consider the function space $B_{2,T}^7$ [23], i.e., the set of all functions of the form (6) considered in $\overline{\Sigma}$ with the norm $\|w(x,t)\|_{B_{2,T}^7} = J_T(u)$, where $w_k(t) \in C[0,T]$ and

$$J_T(w) \equiv \left\{\sum_{k=1}^{\infty}\left(\mu_k^7\big\|w_k(t)\big\|_{C[0,T]}\right)^2\right\}^{1/2} < +\infty.$$

In what follows, we denote by $E_{2,T}^7$ the topological product $B_{2,T}^7 \times C[0,T] \times C[0,T]$, where the norm of an element $z = \{w, g, q\}$ is defined by the formula

$$\|z\|_{E_{2,T}^7} = \big\|w(x,t)\big\|_{B_{2,T}^7} + \big\|g(t)\big\|_{C[0,T]} + \big\|q(t)\big\|_{C[0,T]}.$$

It is well known that the spaces $B_{2,T}^7$ and $E_{2,T}^7$ are Banach spaces [26].

Now consider the operator

$$\Lambda(w,g,q) = \big\{\Lambda_1(w,g,q), \Lambda_2(w,g,q), \Lambda_3(w,g,q)\big\}$$

in the space $E_{2,T}^7$, where

$$\Lambda_1(w,g,q) = \tilde{w}(x,t) \equiv \sum_{k=1}^{\infty}\tilde{w}_k(t)\sin(\mu_k x),$$
$$\Lambda_2(w,g,q) = \tilde{g}(t),$$
$$\Lambda_3(w,g,q) = \tilde{q}(t),$$

and the functions $\tilde{w}_k(t)$, $k \in \mathbb{N}$, $\tilde{g}(t)$, and $\tilde{q}(t)$ are equal to the right-hand sides of (12), (27), and (28), respectively.

Taking into account inequality (14), we have

$$\left\{\sum_{k=1}^{\infty}\left(\mu_k^7\big\|\tilde{w}_k(t)\big\|_{C[0,T]}\right)^2\right\}^{1/2} \le \sqrt{\frac{2}{l}}\beta(1+\delta_2)\left(\sum_{k=1}^{\infty}(\mu_k^7|\varphi_k|)^2\right)^{1/2}$$
$$+ \sqrt{\frac{2}{l}}\beta(1+\delta_2)\left(\sum_{k=1}^{\infty}\left(\mu_k^5|\psi_k|\right)^2\right)^{1/2}$$
$$+ \sqrt{\frac{2}{l}}\kappa T\big\|g(t)\big\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\mu_k^5|f_k|\right)^2\right)^{1/2}$$
$$+ \sqrt{\frac{2}{l}}\kappa T\big\|q(t)\big\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\mu_k^7\big\|w_k(t)\big\|_{C[0,T]}\right)^2\right)^{1/2}, \tag{29}$$

where $\kappa = 1 + 2\beta\delta$,

$$\big\|\tilde{g}(t)\big\|_{C[0,T]} \le \Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]} \Bigg\{ \big\|Y_1(t)Y_2''(t) - Y_2(t)Y_1''(t)\big\|_{C[0,T]}$$
$$+ a^2\sqrt{\frac{l}{2}} \left[\big\|Y_1(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{2k}^2\right)^{1/2} + \big\|Y_2(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{1k}^2\right)^{1/2} \right]$$
$$\times \left[\beta(1+\delta_2) \left(\sum_{k=1}^{\infty} \left(\mu_k^7|\varphi_k|\right)^2\right)^{1/2} + \beta(1+\delta_1) \left(\sum_{k=1}^{\infty} \left(\mu_k^5|\psi_k|\right)^2\right)^{1/2} \right. \tag{30}$$
$$+ \kappa T\big\|g(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\mu_k^5|f_k|\right)^2\right)^{1/2}$$
$$\left. + \ \kappa T\big\|q(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\mu_k^7\big\|w_k(t)\big\|_{C[0,T]}\right)^2\right)^{1/2} \right] \Bigg\},$$

$$\big\|\tilde{q}(t)\big\|_{C[0,T]} \le \Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]} \Bigg\{ \big\|\alpha_1 Y_2''(t) - \alpha_2 Y_1''(t)\big\|_{C[0,T]}$$
$$+ a^2\sqrt{\frac{l}{2}} \left[\alpha_1 \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{2k}^2\right)^{1/2} + \alpha_2 \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{1k}^2\right)^{1/2} \right]$$
$$\times \left[\beta(1+\delta_2) \left(\sum_{k=1}^{\infty} \left(\mu_k^7|\varphi_k|\right)^2\right)^{1/2} + \beta(1+\delta_1) \left(\sum_{k=1}^{\infty} (\mu_k^5|\psi_k|)^2\right)^{1/2} \right. \tag{31}$$
$$+ \kappa T\big\|g(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\mu_k^5|f_k|\right)^2\right)^{1/2}$$
$$\left. + \ \kappa T\big\|q(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \left(\mu_k^7\big\|w_k(t)\big\|_{C[0,T]}\right)^2\right)^{1/2} \right] \Bigg\}.$$

From (29)–(31), respectively, we obtain the estimates

$$\left\{ \sum_{k=1}^{\infty} \left(\mu_k^7\big\|\tilde{w}_k(t)\big\|_{C[0,T]}\right)^2 \right\}^{1/2} \le \frac{2\beta}{l}(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]} + \frac{2\beta}{l}(1+\delta_1)\big\|\psi^{V}(x)\big\|_{L_2[0,l]}$$
$$+ \frac{2\beta}{l}\kappa T\big\|g(t)\big\|_{C[0,T]}\big\|f_0^V(x)\big\|_{L_2[0,l]}$$
$$+ \sqrt{\frac{2}{l}}\kappa T\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)},$$

$$\big\|\tilde{g}(t)\big\|_{C[0,T]} \le \Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]} \Bigg\{ \big\|Y_1(t)Y_2''(t) - Y_2(t)Y_1''(t)\big\|_{C[0,T]}$$
$$+ a^2 \left[\big\|Y_1(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{2k}^2\right)^{1/2} + \big\|Y_2(t)\big\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \mu_k^{-6} y_{1k}^2\right)^{1/2} \right]$$

$$\times\Bigg[\beta(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]}+\beta(1+\delta_1)\big\|\psi^{V}(x)\big\|_{L_2[0,l]}$$

$$+\ \kappa T\big\|g(t)\big\|_{C[0,T]}\big\|f_0^V(x)\big\|_{L_2[0,l]}+\sqrt{\frac{l}{2}}\kappa T\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)}\Bigg]\Bigg\},$$

$$\big\|\tilde q(t)\big\|_{C[0,T]}\le\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}\Bigg\{\big\|\alpha_1Y_2''(t)-\alpha_2Y_1''(t)\big\|_{C[0,T]}$$

$$+a^2\Bigg[\alpha_1\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\bigg)^{1/2}+\alpha_2\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\bigg)^{1/2}\Bigg]$$

$$\times\Bigg[\beta(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]}+\beta(1+\delta_1)\|\psi^{V}(x)\|_{L_2[0,l]}$$

$$+\ \kappa T\big\|g(t)\big\|_{C[0,T]}\big\|f_0^V(x)\big\|_{L_2[0,l]}+\sqrt{\frac{l}{2}}\kappa T\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)}\Bigg]\Bigg\},$$

or

$$\left\{\sum_{k=1}^{\infty}\Big(\mu_k^7\big\|\tilde w_k(t)\big\|_{C[0,T]}\Big)^2\right\}^{1/2}\le L_1(T)+M_1(T)\big\|g(t)\big\|_{C[0,T]}$$
$$+N_1(T)\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)}, \tag{32}$$

$$\big\|\tilde g(t)\big\|_{C[0,T]}\le L_2(T)+M_2(T)\big\|g(t)\big\|_{C[0,T]}+N_2(T)\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)}, \tag{33}$$

$$\big\|\tilde q(t)\big\|_{C[0,T]}\le L_3(T)+M_3(T)\big\|g(t)\big\|_{C[0,T]}+N_3(T)\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B^7_{2,T}(x,t)}, \tag{34}$$

where

$$L_1(T)=\frac{2\beta}{l}(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]}+\frac{2\beta}{l}(1+\delta_1)\big\|\psi^{V}(x)\big\|_{L_2[0,l]},$$
$$M_1(T)=\frac{2\beta}{l}\kappa T\big\|f_0^V(x)\big\|_{L_2[0,l]},$$
$$N_1(T)=\sqrt{\frac{2}{l}}\kappa T,$$
$$L_2(T)=\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}\Bigg\{\big\|Y_1(t)Y_2''(t)-Y_2(t)Y_1''(t)\big\|_{C[0,T]}$$
$$+a^2\Bigg[\big\|Y_1(t)\big\|_{C[0,T]}\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\bigg)^{1/2}+\big\|Y_2(t)\big\|_{C[0,T]}\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\bigg)^{1/2}\Bigg]$$
$$\times\Big[\beta(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]}+\beta(1+\delta_1)\big\|\psi^{V}(x)\big\|_{L_2[0,l]}\Big]\Bigg\},$$
$$M_2(T)=a^2\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}$$
$$\times\Bigg[\big\|Y_1(t)\big\|_{C[0,T]}\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\bigg)^{1/2}+\big\|Y_2(t)\big\|_{C[0,T]}\bigg(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\bigg)^{1/2}\Bigg]\kappa T\big\|f_0^V(x)\big\|_{L_2[0,l]},$$

$$N_2(T) = a^2\sqrt{\frac{l}{2}}\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}$$
$$\times\left[\big\|Y_1(t)\big\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\right)^{1/2}+\big\|Y_2(t)\big\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\right)^{1/2}\right]\kappa T,$$

$$L_3(T) = \Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}\Bigg\{\big\|\alpha_1 Y_2''(t)-\alpha_2 Y_1''(t)\big\|_{C[0,T]}$$
$$+\,a^2\left[\alpha_1\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\right)^{1/2}+\alpha_2\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\right)^{1/2}\right]$$
$$\times\left[\beta(1+\delta_2)\big\|\varphi^{VII}(x)\big\|_{L_2[0,l]}+\beta(1+\delta_1)\big\|\psi^{V}(x)\big\|_{L_2[0,l]}\right]\Bigg\},$$

$$M_3(T) = a^2\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}\left[\alpha_1\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\right)^{1/2}+\alpha_2\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\right)^{1/2}\right]\kappa T\big\|f_0^{V}(x)\big\|_{L_2[0,l]},$$

$$N_3(T) = a^2\sqrt{\frac{l}{2}}\Big\|\big[\mathcal{Y}(t)\big]^{-1}\Big\|_{C[0,T]}\left[\alpha_1\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{2k}^2\right)^{1/2}+\alpha_2\left(\sum_{k=1}^{\infty}\mu_k^{-6}y_{1k}^2\right)^{1/2}\right]\kappa T.$$

Inequalities (32)–(34) imply the estimate

$$\begin{aligned}&\big\|\tilde w(x,t)\big\|_{B_{2,T}^7}+\big\|\tilde g(t)\big\|_{C[0,T]}+\big\|\tilde q(t)\big\|_{C[0,T]}\\&\qquad\le L(T)+M(T)\big\|g(t)\big\|_{C[0,T]}+N(T)\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B_{2,T}^7(x,t)},\end{aligned}\tag{35}$$

where $L(T)=L_1(T)+L_2(T)+L_3(T)$, $M(T)=M_1(T)+M_2(T)+M_3(T)$, and $N(T)=N_1(T)+N_2(T)+N_3(T)$.

Theorem 2. *Let the conditions in Theorem 1 and Lemma 2, (24), and the following condition be satisfied*:

$$\big(L(T)+2\big)\big(M(T)+N(T)(L(T)+2)\big)<2.\tag{36}$$

Then problem (1)–(4) has a unique solution in the ball $B_R=\{z:\|z\|_{E_{2,T}^7}\le R\}$.

Proof. Introduce the notation $z=(w(x,t),g(t),q(t))^*$ and write system (13), (27), (28) in operator form as

$$z=Az,\tag{37}$$

where $A=(A_1,A_2,A_3)^*$, $A_1(z)$, $A_2(z)$, and $A_3(z)$ are determined by the right-hand sides of (13), (27), and (28), respectively.

Likewise, from (35) we conclude that for any $z,z_1,z_2\in B_R$ one has the estimates

$$\|Az\|_{E_{2,T}^7}\le L(T)+M(T)\big\|g(t)\big\|_{C[0,T]}+N(T)\big\|q(t)\big\|_{C[0,T]}\big\|w(x,t)\big\|_{B_{2,T}^7},\tag{38}$$

$$\begin{aligned}\|Az_1-Az_2\|_{E_{2,T}^7}&\le M(T)\big\|g_1(t)-g_2(t)\big\|_{C[0,T]}\\&\quad+N(T)R\Big(\big\|q_1(t)-q_2(t)\big\|_{C[0,T]}+\big\|w_1(x,t)-w_2(x,t)\big\|_{B_{2,T}^7}\Big).\end{aligned}\tag{39}$$

Then, by virtue of (36), it follows from (38) and (39) that the operator A acts on the ball B_R and satisfies the contraction mapping principle. Therefore, by Banach's theorem, the operator A has a unique fixed point $\{w,g,q\}$ in the ball B_R, which is a solution of the operator equation (37).

Thus, the function $w(x,t)$ as an element of the space $B^7_{2,T}$ is continuous and has continuous derivatives $w_{tt}(x,t)$ and $w_{xxxx}(x,t)$ in the rectangle $\overline{\Sigma}$.

From (9) it is easy to see that the inequality

$$\left(\sum_{k=1}^{\infty}\left(\mu_k\left\|w_k''(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} \le \left(\sum_{k=1}^{\infty}\mu_k^{-6}\right)^{1/2}\left[\left(\sum_{k=1}^{\infty}\left(\mu_k^7\left\|w_k(t)\right\|_{C[0,T]}\right)^2\right)^{1/2} + \left\|f_0'(x)g(t)+q(t)w_x(x,t)\right\|_{L_2[0,l]}\right]$$

holds; this implies that $w_{tt}(x,t)$ is continuous in $\overline{\Sigma}$.

Remark. Inequality (36) is satisfied for sufficiently small T.

Theorem 3. *Let all conditions in Theorem 2 and conditions (5) and (24) be satisfied. Then problem (1)–(4) has a unique classical solution in the ball B_R of the space $E^7_{2,T}$.*

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REFERENCES

1. Tikhonov, A.N. and Samarskii, A.A., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Nauka, 1966.
2. Gusev, B.V. and Saurin, V.V., On vibrations of inhomogeneous beams, *Inzh. Vestn. Dona,* 2017. http://ivdon.ru/ru/magazine/archive/n3y2017/4312.
3. Baysal, O. and Hasanov, A., Solvability of the clamped Euler–Bernoulli beam equation, *Appl. Math. Lett.*, 2019, vol. 93, pp. 85–90.
4. Sabitov, K.B., Beam vibrations with pinned ends, *Vestn. Samarsk. Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauki*, 2015, vol. 19, no. 2, pp. 311–324.
5. Sabitov, K.B. and Akimov, A.A., Initial–boundary value problem for a nonlinear beam vibration equation, *Differ. Equations*, 2020, vol. 56, no. 5, pp. 621–634.
6. Kasimov, S.G. and Madrakhimov, U.S., Initial–boundary value problem for the beam vibration equation in the multidimensional case, *Differ. Equations*, 2019, vol. 55, no. 10, pp. 1336–1348.
7. Karchevsky, A.L., Analytical solutions to the differential equation of transverse vibrations of a piecewise homogeneous beam in the frequency domain for the boundary conditions of various types, *J. Appl. Ind. Math.*, 2020, vol. 14, no. 4, pp. 648–665.
8. Durdiev, D.K. and Totieva, Zh.D., The problem of determining the one-dimensional kernel of the electroviscoelasticity equation, *Sib. Math. J.*, 2017, vol. 58, no. 3, pp. 427–444.
9. Durdiev, D.K. and Rakhmonov, A.A., The problem of determining the 2D kernel in a system of integro-differential equations of a viscoelastic porous medium, *J. Appl. Ind. Math.*, 2020, vol. 14, no. 2, pp. 281–295.
10. Durdiev, U.D., An inverse problem for the system of viscoelasticity equations in homogeneous anisotropic media, *J. Appl. Ind. Math.*, 2019, vol. 13, no. 4, pp. 281–295.
11. Romanov, V.G., A problem of recovering a special two dimension potential in a hyperbolic equation, *Eur. J. Math. Comput. Appl.*, 2016, vol. 4, no. 1, pp. 32–46.
12. Durdiev, U.D., A problem of identification of a special 2D memory kernel in an integro-differential hyperbolic equation, *Eur. J. Math. Comput. Appl.*, 2019, vol. 7, no. 2, pp. 4–19.
13. Durdiev, U.D. and Totieva, Zh.D., A problem of determining a special spatial part of 3D memory kernel in an integro-differential hyperbolic equation, *Math. Methods Appl. Sci.*, 2019, vol. 42, no. 18, pp. 1–12.
14. Durdiev, U.D., Inverse problem of determining an unknown coefficient in the beam vibration equation, *Differ. Equations*, 2022, vol. 58, no. 1, pp. 36–43.
15. Karchevskii, A.L. and Fat'yanov, A.G., Numerical solution of the inverse problem for an elasticity system with aftereffect for a vertically inhomogeneous medium, *Sib. Zh. Vychisl. Mat.*, 2001, vol. 4, no. 3, pp. 259–268.

16. Karchevsky, A.L., Determination of the possibility of rock burst in a coal seam, *J. Appl. Ind. Math.*, 2017, vol. 11, no. 4, pp. 527–534.
17. Durdiev, U.D., Numerical determination of the dependence of the permittivity of a layered medium on the temporal frequency, *Sib. Elektron. Mat. Izv.*, 2020, vol. 17, pp. 179–189.
18. Maciag, A. and Pawinska, A., Solution of the direct and inverse problems for beam, *Comput. Appl. Math.*, 2016, vol. 35, pp. 187–201.
19. Maciag, A. and Pawinska, A., Solving direct and inverse problems of plate vibration by using the Trefftz functions, *J. Theor. Appl. Mech.*, 2013, vol. 51, no. 3, pp. 543–552.
20. Guojin Tan, Jinghui Shan, Chunli Wu, and Wensheng Wang, Direct and inverse problems on free vibration of cracked multiple I-section beam with different boundary conditions, *Adv. Mech. Eng.*, 2017, vol. 9, no. 11, pp. 1–17.
21. Moaveni, S. and Hyde, R., Reconstruction of the area-moment-of-inertia of a beam using a shifting load and the end-slope data, *Inverse Probl. Sci. Eng.*, 2016, vol. 24, no. 6, pp. 990–1010.
22. Marinov, T.T. and Vatsala, A.S., Inverse problem for coefficient identification in the Euler–Bernoulli equation, *Comput. Math. Appl.*, 2008, vol. 56, pp. 400–410.
23. Megraliev, Ya.T. and Azizbayov, E.I., A time-nonlocal inverse problem for a hyperbolic equation with an integral overdetermination condition, *Electron. J. Qual. Theory Diff. Equat.*, 2021, no. 28, pp. 1–12.
24. Xiao-Li Ding and Bashir Ahmad, A generalized Volterra–Fredholm integral inequality and its applications to fractional differential equations, *Adv. Difference Equat.*, 2018, vol. 2018, p. 91.
25. Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J., *Theory and Application of Fractional Differential Equations*, Amsterdam: Elsevier, 2006.
26. Tekin, I., Mehraliyev, Y.T., and Ismailov, M.I., Existence and uniqueness of an inverse problem for nonlinear Klein–Gordon equation, *Math. Methods Appl. Sci.*, 2019, vol. 42, no. 10, pp. 3739–3753.