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Mubina Sharipova
Bukhara State University, Bukhara, Uzbekistan, m.sh.sharipova@buxdu.uz

USUAL, QUADRATIC AND CUBIC NUMERICAL RANGES CORRESPONDING TO A 3×3 OPERATOR MATRICES

Sharipova Mubina
Bukhara State University, Bukhara, Uzbekistan
e-mail: m.sh.sharipova@buxdu.uz

Abstract

In the present paper we consider bounded self-adjoint 3×3 operator matrices $\mathcal{A}$. An alternative formula for the calculating the cubic numerical range of the operator matrices $\mathcal{A}$ is derived. The components of the quadratic numerical range with respect to the expansion of the Hilbert space are found.

Keywords: *block operator matrix, quadratic numerical range, cubic numerical range.*

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Introduction

A matrix with linear operators as its entries is called a block operator matrix [1]. Every bounded linear operator can be expressed as a block operator matrix if the space in which it acts is decomposed in two or more components. They arise in the various areas of mathematics and its applications. One of the category of such block operator matrices are Hamiltonians associated with systems of non conserved number of quasi-particles on a lattice. Their number can be unbounded as in the case of spin-boson models (in this case we obtain an infinite-dimensional block operator matrix) [2, 3] or bounded as in the case of "truncated" spin-boson models (in this case we obtain finite-dimensional block operator matrix) [4, 5, 6, 7].

Spectral theory provides us with one of the effective methods for determining the position of the spectrum of a linear operator $\mathcal{A}$ in the Banach or Hilbert space $\mathcal{H}$ is the numerical range:

$$W(\mathcal{A}) := \{(\mathcal{A}f, f) : \, f \in \mathcal{H}, \, \|f\| = 1\}.$$

This idea was initially investigated by O.Toeplitz in 1918 (see [8]); he proved that the numerical range $W(\mathcal{A})$ of a matrix $\mathcal{A}$ contains all its eigenvalues, that is,

$$\sigma_{\mathrm{p}}(\mathcal{A}) = \sigma(\mathcal{A}) \subset W(\mathcal{A})$$

and that its boundary is a convex curve. Here by $\sigma_{\mathrm{p}}(\mathcal{A})$ and $\sigma(\mathcal{A})$ we denote the point spectrum and spectrum of $\mathcal{A}$. In 1919 F.Hausdorff showed that indeed the set $W(\mathcal{A})$ is convex [9]. It turned out that this still holds true for general bounded linear operators and that the spectrum is contained in the closure $W(\mathcal{A})$ (see [10]).

In [11] the structure of the closure of the numerical range $W(\mathcal{A}_1)$ of the generalized Friedrichs model $\mathcal{A}_1$ is studied in detail by terms of its matrix entries for all dimensions of the torus $\mathbb{T}^{\mathrm{d}}$. The cases when the set $W(\mathcal{A}_1)$ is closed is given and necessary and sufficient conditions under which the spectrum of $\mathcal{A}_1$ coincides with its numerical range is found.

At first sight, the convexity of the numerical range seems to be a useful property, e.g. to show that the spectrum of an operator lies in a half plane. However, the numerical range often gives a poor localization of the spectrum and it cannot capture finer structures such as the separation of the spectrum in two parts. In view of these shortcomings, the new concept of quadratic numerical range was introduced in 1998 in [12].

In [13] the alternative formula for the quadratic numerical range $W^2(\mathcal{A}_1)$ of the generalized Friedrichs model $\mathcal{A}_1$ is given. It is shown that the set $W^2(\mathcal{A}_1)$ consists no more that two components and estimates for the bound of these components are obtained. These outcomes made it possible to get estimates for eigenvalues of $\mathcal{A}_1$.

In this paper we consider self adjoint 3×3 operator matrix in the direct sum of three Hilbert spaces. First, cubic numerical range of this operator matrix found by formulas. Then, given formulas of finding quadratic numerical range of 3×3 operator matrix respect to different decompositions of Hilbert spaces.

1 Preliminaries

If the Hilbert space $\mathcal{H}$ is the product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$, then every operator $\mathcal{A} \in L(\mathcal{H})$ has a block operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

with linear operators $A \in L(\mathcal{H}_1), B \in L(\mathcal{H}_2, \mathcal{H}_1), C \in L(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in L(\mathcal{H}_2)$. The following generalization of the numerical range of $\mathcal{A}$ takes into account the block structure (1) of $\mathcal{A}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$.

Definition 1. *For $f \in \mathbb{S}_{\mathcal{H}_1}, g \in \mathbb{S}_{\mathcal{H}_2}$, we define the 2×2 matrix*

$$\mathcal{A}_{f,g} := \begin{pmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{pmatrix} \in M_2(\mathbb{C}).$$

Then the set

$$W^2(\mathcal{A}) := \bigcup_{f \in \mathbb{S}_{\mathcal{H}_1}, g \in \mathbb{S}_{\mathcal{H}_2}} \sigma_{\mathrm{p}}(\mathcal{A}_{f,g})$$

is called the quadratic numerical range of $\mathcal{A}$ *with respect to the block operator matrix representation* (1).

If the operator $\mathcal{A}$ is in the form of a 3×3 block operator matrix, then the concept of quadratic numerical range for 2×2 block operator matrices has an obvious generalization to 3×3 block operator matrices.

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be complex Hilbert spaces and consider $\mathcal{H} := \mathcal{H}_1 \bigoplus \mathcal{H}_2 \bigoplus \mathcal{H}_3$. With respect to this decomposition, every bounded self-adjoint linear operator $\mathcal{A} \in L(\mathcal{H})$ has a 3×3 block operator matrix representation

$$\mathcal{A} := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix} \tag{2}$$

with bounded linear entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i), i, j = 1, 2, 3$ such that $A_{ii} = A_{ii}^*$. In the following we denote by

$$\mathbb{S}_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3} := \mathbb{S}_{\mathcal{H}_1} \times \mathbb{S}_{\mathcal{H}_2} \times \mathbb{S}_{\mathcal{H}_3} = \{f_1, f_2, f_3)^t \in \mathcal{H} : \|f_i\| = 1,\ i = 1, 2, 3\}$$

the product of the unit spheres $\mathbb{S}_{\mathcal{H}_i}$ in $\mathcal{H}_i$; we also write $\mathbb{S}^3$ or $\mathbb{S}_{\mathcal{H}}$ instead of $\mathbb{S}_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}$ if the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is clear. In this case

$$\mathbb{S}^3 := \{f = (f_1, f_2, f_3)^t \in \mathcal{H} :\ \|f_i\| = 1,\ i = 1, 2, 3\}.$$

For $f = (f_1, f_2, f_3)^t \in \mathbb{S}_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}$, we introduce the 3×3 matrix

$$\mathcal{A}_f := \begin{pmatrix} (A_{11}f_1, f_1) & (A_{12}f_2, f_1) & (A_{13}f_3, f_1) \\ (A_{12}^*f_1, f_2) & (A_{22}f_2, f_2) & (A_{23}f_3, f_2) \\ (A_{13}^*f_1, f_3) & (A_{23}^*f_2, f_3) & (A_{33}f_3, f_3) \end{pmatrix} \in M_3(\mathbb{C}),$$

that is, $(\mathcal{A}_f)_{i,j} := (A_{ij}f_j, f_i)$, $i, j = 1, 2, 3$. Then the set

$$W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A}) := \bigcup_{f \in \mathbb{S}^3} \sigma_{\mathrm{p}}(\mathcal{A}_f)$$

is called *block numerical range* of $\mathcal{A}$ with respect to the block operator matrix representation (2). For a fixed decomposition of $\mathcal{H}$, we also write

$$W^3(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A}).$$

Clearly, since

$$\sigma_{\mathrm{p}}(\mathcal{A}_f) = \{\lambda \in \mathbb{C} :\ \det(\mathcal{A}_f - \lambda) = 0\}$$

for all $f \in \mathbb{S}^3$, the set $W^3(\mathcal{A})$ has the equivalent representation

$$W^3(\mathcal{A}) = \{\lambda \in \mathbb{C} :\ \exists f \in \mathbb{S}^3,\ \det(\mathcal{A}_f - \lambda) = 0\}.$$

It is preferable to use the above definition in deriving an alternative formula for the cubic numerical range.

2 Formula for the cubic numerical range

In this subsection first we derive an alternative formula for the cubic numerical range of the 3×3 block operator matrix $\mathcal{A}$. In some spacial cases a Hilbert space $\mathcal{H}$ has to be written as a direct sum of two Hilbert spaces. In such cases, the operator is described in the form of a 2×2 block operator matrix, and there is a need to study its quadratic numerical range [14]. At the end we give some information about the numerical range.

Let $a_{ij}(f) = (A_{ij} f_j, f_i)$ for $i, j = 1, 2, 3$ and

$$E_k(f) := \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)),$$

if $a_{11}(f) = a_{22}(f) = a_{33}(f)$, $a_{12}(f) = a_{23}(f) = a_{13}(f) = 0$;

$$E_k(f) := \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)) + 2\sqrt{-\frac{P(f)}{3}} \cos \frac{\Phi(f) + 2\pi k}{3}$$

otherwise, where

$$\begin{aligned} P(f) := &-\frac{1}{6}(a_{11}(f) - a_{22}(f))^2 + (a_{11}(f) - a_{33}(f))^2 + (a_{22}(f) - a_{33}(f))^2 \\ &- |a_{12}(f)|^2 - |a_{23}(f)|^2 - |a_{13}(f)|^2; \end{aligned} \tag{3}$$

$$\begin{aligned} Q(f) := &-\frac{2}{27}(a_{11}(f) + a_{22}(f) + a_{33}(f))^3 + \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)) \times \\ &\times (a_{11}(f)a_{22}(f) + a_{22}(f)a_{33}(f) + a_{11}(f)a_{33}(f) - |a_{12}(f)|^2 - |a_{23}(f)|^2 - |a_{13}(f)|^2) + \\ &+ a_{11}(f)a_{22}(f)a_{33}(f) + 2\mathrm{Re}(a_{12}(f)a_{23}(f)\overline{a_{13}(f)}) + \\ &+ |a_{12}(f)|^2 a_{33}(f) + |a_{23}(f)|^2 a_{11}(f) + |a_{13}(f)|^2 a_{22}(f); \end{aligned} \tag{4}$$

$$\Phi(f) = \arccos\left(-\frac{3Q(f)}{2P(f)}\sqrt{-\frac{3}{P(f)}}\right).$$

The main result of this paper is the following theorem.

Theorem 1. *For the cubic numerical range of $\mathcal{A}$ we have*

$$W^3(\mathcal{A}) = \bigcup_{k=1}^{3} \bigcup_{f \in \mathbb{S}^3} \{E_k(f)\}.$$

Proof. According to the definition of a cubic numerical range, first we construct the numerical matrix $\mathcal{A}_f$ corresponding to the operator matrix $\mathcal{A}$ in (2):

$$\mathcal{A}_f := \begin{pmatrix} (A_{11}f_1, f_1) & (A_{12}f_2, f_1) & (A_{13}f_3, f_1) \\ (A_{12}^* f_1, f_2) & (A_{22}f_2, f_2) & (A_{23}f_3, f_2) \\ (A_{13}^* f_1, f_3) & (A_{23}^* f_2, f_3) & (A_{33}f_3, f_3) \end{pmatrix}.$$

Then, the eigenvalue equation for $\mathcal{A}_f$ has form:

$$\begin{aligned}\lambda^3 &- (a_{11}(f)+a_{22}(f)+a_{33}(f))\lambda^2 + (a_{11}(f)a_{22}(f)+a_{11}(f)a_{33}(f)+a_{22}(f)a_{33}(f))\lambda - \\ &- a_{11}(f)a_{22}(f)a_{33}(f) = (|a_{12}(f)|^2+|a_{13}(f)|^2+|a_{23}(f)|^2)\lambda - |a_{12}(f)|^2 a_{33}(f) - \\ &- |a_{23}(f)|^2 a_{11}(f) - |a_{13}(f)|^2 a_{22}(f) - 2\mathrm{Re}(\overline{a_{13}(f)}a_{12}(f)a_{23}(f)). \end{aligned} \quad (5)$$

By the change of variables

$$x = \lambda - \frac{1}{3}(a_{11}(f)+a_{22}(f)+a_{33}(f)), \quad (6)$$

(5) is reduced to the depressed cubic equation $x^3+P(f)x+Q(f)=0$ with $P(f)$ and $Q(f)$ given by (3) and (4) respectively; note that $P(f)\le 0$. Since the matrix $\mathcal{A}_f$ is Hermitian, it has only real eigenvalues $E_1(f)\le E_2(f)\le E_3(f)$. This is the case of non-positive discriminant, $27(Q(f))^2+4(Q(f))^3\le 0$, and hence, the roots of the depressed equation are given by

$$2x_k(f)=0, \quad k=1,2,3, \quad \text{if} \quad P(f)=0;$$

$$x_k(f) = 2\sqrt{-\frac{P(f)}{3}}\cos\frac{\Phi(f)+2k\pi}{3}, \quad k=1,2,3, \quad \text{if} \quad P(f)<0;$$

note that we changed orders of the roots to achieve that they are enumerated increasingly, i.e., $x_1(f)\le x_2(f)\le x_3(f)$. This and (6) yield the formulae for $E_k(f)$, $k=1,2,3$. □

Let $\widehat{\mathcal{H}}_1 := \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\widehat{\mathcal{H}}_2 := \mathcal{H}_3$. To define the quadratic numerical range we consider the operator matrix $\mathcal{A}$ with respect to the decomposition $\mathcal{H} = \widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2$:

$$\mathcal{A} = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{12}^* & \widehat{A}_{22} \end{pmatrix}$$

with the entries $\widehat{A}_{ij} : \widehat{\mathcal{H}}_j \to \widehat{\mathcal{H}}_i$, $i\le j$, $i,j=1,2$:

$$\widehat{A}_{11} := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \quad \widehat{A}_{12} := \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \quad \widehat{A}_{22} := A_{33}.$$

It is easy to see that the adjoint operator $\widehat{A}_{12}^*$ to $\widehat{A}_{12}$ is defined as

$$\widehat{A}_{12}^* : \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_2, \quad \widehat{A}_{12}^* = (A_{13}^* \, A_{23}^*).$$

For $\widehat{f} = (\widehat{f}_1, \widehat{f}_2) \in \mathbb{S}_{\widehat{\mathcal{H}}_1} \times \mathbb{S}_{\widehat{\mathcal{H}}_2}$ we define the 2×2 matrix

$$\mathcal{A}_{\widehat{f}} := \begin{pmatrix} (\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1) & (\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1) \\ (\widehat{A}_{12}^*\widehat{f}_1, \widehat{f}_2) & (\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2) \end{pmatrix}.$$

For $\widehat{f}_i \in \widehat{\mathcal{H}}_i$, $\widehat{f}_i \neq 0$, $i = 1, 2$, we define

$$\widehat{\lambda}_\pm \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} : \quad = \frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{2\|\widehat{f}_1\|^2} + \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{2\|\widehat{f}_2\|^2} \pm \frac{1}{2}\sqrt{\left(\frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{\|\widehat{f}_1\|^2} - \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{\|\widehat{f}_2\|^2}\right)^2 + 4\frac{\left|(\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1)\right|^2}{\|\widehat{f}_1\|\|\widehat{f}_2\|}}$$

and we let

$$\widehat{\Lambda}_\pm(\mathcal{A}) := \left\{\widehat{\lambda}_\pm \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} : \widehat{f}_i \in \widehat{\mathcal{H}}_i,\ \widehat{f}_i \neq 0,\ i = 1, 2\right\}.$$

Then for the quadratic numerical range $W^2(\mathcal{A})$ of $\mathcal{A}$ with respect to the decomposition $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2$ we have the equality

$$W^2(\mathcal{A}) = \widehat{\Lambda}_+(\mathcal{A}) \cup \widehat{\Lambda}_-(\mathcal{A}).$$

Let $\widetilde{\mathcal{H}}_1 := \mathcal{H}_1$ and $\widetilde{\mathcal{H}}_2 := \mathcal{H}_2 \oplus \mathcal{H}_3$. Similarly, to define the quadratic numerical range we consider the operator matrix $\mathcal{A}$ with respect to the another decomposition $\mathcal{H} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$:

$$\mathcal{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{12}^* & \widetilde{A}_{22} \end{pmatrix}$$

with the entries $\widetilde{A}_{ij} : \widetilde{\mathcal{H}}_j \to \widetilde{\mathcal{H}}_i$, $i \leq j$, $i, j = 1, 2$:

$$\widetilde{A}_{11} := A_{33}, \quad \widetilde{A}_{12} := (A_{12}\, A_{13}), \quad \widetilde{A}_{22} := \begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix}.$$

One can show that

$$\widetilde{A}_{12}^* : \widetilde{\mathcal{H}}_2 \to \widetilde{\mathcal{H}}_1, \quad \widetilde{A}_{12}^* = \begin{pmatrix} A_{12}^* \\ A_{13}^* \end{pmatrix}.$$

For $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) \in \mathbb{S}_{\widetilde{\mathcal{H}}_1} \times \mathbb{S}_{\widetilde{\mathcal{H}}_2}$ we define the 2×2 matrix

$$\mathcal{A}_{\widetilde{f}} := \begin{pmatrix} (\widetilde{A}_{11}\widetilde{f}_1, \widetilde{f}_1) & (\widetilde{A}_{12}\widetilde{f}_2, \widetilde{f}_1) \\ (\widetilde{A}_{12}^*\widetilde{f}_1, \widetilde{f}_2) & (\widetilde{A}_{22}\widetilde{f}_2, \widetilde{f}_2) \end{pmatrix}.$$

For $\widetilde{f}_i \in \widetilde{\mathcal{H}}_i$, $\widetilde{f}_i \neq 0$, $i = 1, 2$, we define

$$\widetilde{\lambda}_\pm \begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix} : \quad = \frac{(\widetilde{A}_{11}\widetilde{f}_1, \widetilde{f}_1)}{2\|\widetilde{f}_1\|^2} + \frac{(\widetilde{A}_{22}\widetilde{f}_2, \widetilde{f}_2)}{2\|\widetilde{f}_2\|^2} \pm \frac{1}{2}\sqrt{\left(\frac{(\widetilde{A}_{11}\widetilde{f}_1, \widetilde{f}_1)}{\|\widetilde{f}_1\|^2} - \frac{(\widetilde{A}_{22}\widetilde{f}_2, \widetilde{f}_2)}{\|\widetilde{f}_2\|^2}\right)^2 + 4\frac{\left|(\widetilde{A}_{12}\widetilde{f}_2, \widetilde{f}_1)\right|^2}{\|\widetilde{f}_1\|\|\widetilde{f}_2\|}}$$

and we let

$$\widetilde{\Lambda}_{\pm}(\mathcal{A}) := \left\{ \widetilde{\lambda}_{\pm} \left(\begin{array}{c} \widetilde{f}_1 \\ \widetilde{f}_2 \end{array} \right) : \widetilde{f}_i \in \widetilde{\mathcal{H}}_i,\ \widetilde{f}_i \neq 0,\ i = 1, 2 \right\}.$$

Then for the quadratic numerical range $W^2(\mathcal{A})$ of $\mathcal{A}$ with respect to the decomposition $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_1 \oplus \widetilde{\mathcal{H}}_2$ we have the equality

$$W^2(\mathcal{A}) = \widetilde{\Lambda}_{+}(\mathcal{A}) \cup \widetilde{\Lambda}_{-}(\mathcal{A}).$$

It is known that for $\mathcal{A}^* = \mathcal{A}$ we have $W(\mathcal{A}) \subset \mathbb{R}$. From the boundedness of $\mathcal{A}$ we obtain the fact $W(\mathcal{A}) \subset [-\|\mathcal{A}\|, \|\mathcal{A}\|]$. From the definition of the numerical range and continuity of the scalar product we obtain that $\overline{W(\mathcal{A})} = [\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})]$. If $\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A}) \in \sigma_{\rm p}(\mathcal{A})$, then $W(\mathcal{A}) = [\min \sigma(\mathcal{A}), \max \sigma(\mathcal{A})]$. It is easy to see that the cubic numerical range is contained in the in the numerical range, that is, $W^3(\mathcal{A}) \subset W(\mathcal{A})$. By the so-called spectral inclusion property [1] for $\mathcal{A}$ we have $\sigma_{\rm p}(\mathcal{A}) \subset W^3(\mathcal{A})$ and $\sigma(\mathcal{A}) \subset \overline{W^3(\mathcal{A})}$. Therefore, through the main result of this paper one can obtain a set, where located the spectrum of $\mathcal{A}$.

Conclusion. In the present paper the bounded self-adjoint 3×3 block operator matrices $\mathcal{A}$ in the direct sum of three Hilbert spaces is considered. For the reader's convenience and completeness, the brief information about the usual, quadratic and cubic ranges are given. The formula for the calculating of the cubic numerical range of 3×3 block operator matrices $\mathcal{A}$ is derived. Obtained formula allows to better determine the location of the spectrum of the block operator matrices $\mathcal{A}$. Then, describing the sum of three Hilbert spaces as a sum of two Hilbert spaces, an exact form of the components of the corresponding quadratic numerical range are found. At the end, an information about the numerical range of $\mathcal{A}$ are given.

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