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MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

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UCHINCHI TARTIBLI OPERATORLI MATRITSANING MUHIM SPEKTR TARMOQLARI: 1 O‘LCHAMLI HOL

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Annotatsiya: *Ushbu maqolada Fok fazosining uch zarrachali qism fazosida ta’sir qiluvchi* $\mathcal{A}_\mu, \mu > 0$ *uchinchi tartibli operatorli matritsa o‘rganilgan.* $\mathcal{A}_\mu$ *operatorli matritsa muhim spektrining ikki va uch zarrachali tarmoqlari mos umumlashgan Fridrixs modeli spektri yordamida tavsiflangan. Umumlashgan Fridrixs modeli xos qiymatlarining mavjudligi ko‘rsatilgan hamda ikki va uch zarrachali tarmoqlarning chegaralari uchun baholashlar olingan.*

Kalit so‘zlar: *operatorli matritsa, xos qiymat, muhim spektr, Fredgolm determinanti, umumlashgan Fridrixs modeli.*

КЛЮЧЕВЫЕ СПЕКТРАЛЬНЫЕ СЕТИ МАТРИЦЫ С ОПЕРАТОРОМ ТРЕТЬЕГО ПОРЯДКА: ОДНОМЕРНЫЙ СЛУЧАЙ

Аннотация: *В данной работе изучена операторная матрица третьего порядка* $\mathcal{A}_\mu, \mu > 0$, *действующая в трёхчастичной обрезанном подпространстве Фоковского пространства. С помощью спектра соответствующей обобщенной модели Фридрихса описаны двухчастичная и трёхчастичная ветви существенного спектра операторной матрицы* $\mathcal{A}_\mu$. *Показано существование собственных значений обобщенной модели Фридрихса, и получена оценка для границ двухчастичной и трёхчастичной ветвей.*

Ключевые слова: *операторная матрица, собственное значение, существенный спектр, определитель Фредгольма, обобщенная модель Фридрихса.*

KEY SPECTRAL NETWORKS OF A MATRIX WITH A THIRD-ORDER OPERATOR: 1-DIMENSIONAL CASE

Annotation: *In this paper the operator matrix of order three* $\mathcal{A}_\mu, \mu > 0$, *acting in the three-particle cut subspace of the Fock space is studied. Two-particle and three-particle branches of the essential spectrum of* $\mathcal{A}_\mu$ *is described by the spectrum of the corresponding generalized Friedrichs model. The existence of the eigenvalues of the generalized Friedrichs model is shown and the estimates for the bounds of the two-particle and three-particle branches are given.*

Key words: *operator matrix, eigenvalue, essential spectrum, Fredholm determinant, generalized Friedrichs model.*

Kirish. Bizga yaxshi ma’lumki, elementlari Banax yoki Gilbert fazosida ta’sir qiluvchi chiziqli operatorlardan iborat matritsaga operatorli matritsa deyiladi [1]. Operatorli matritsalarning muhim sinflaridan biri-bu $\mathbb{R}^d$ fazodagi yoki $\mathbb{Z}^d$ panjaradagi soni saqlanmaydigan zarrachalar sistemasiga mos Gamiltonianlardir. Zarrachalar soni “spin-bozon” modelidagi kabi cheksiz yoki qirqilgan “spin-bozon” modelidagi kabi chekli bo‘lishi mumkin [2], [3]. Birinchi holda cheksiz o‘lchamli matritsa va ikkinchi holda chekli o‘lchamli matritsa hosil bo‘ladi. Odatda, yuqorida qayd etib o‘tilgan sistemalar qattiq jismlar fizikasi, kvant maydonlar nazariyasi, statistik fizika, kvant mexanikasining ko‘plab masalalarida uchraydi [3-6].

Operatorli matritsalarning muhim spektri tushunchasi operatorlar nazariyasida chuqur o‘rganilayotgan muammolardan biri bo‘lib, bu turdagi operatorlarning muhim spektrining joylashuv o‘rnini tasvirlash spektral analizning muhim muammolari qatorida o‘rganiladi. Muhim spektrning joylashuv o‘rnini tadqiq etishda Veyl mezoni va Hunziker-van Winter-Zhislin (HWZ) teoremasi keng qo‘llaniladi. Ko‘plab tadqiqotlarda ushbu usullardan foydalanib, 3×3 va 4×4 operatorli matritsalar muhim spektri o‘rganilgan. Xususan, [7] maqolada 4×4 operatorli matritsa muhim spektri unga mos kanal operatori spektri orqali tasvirlangan va HWZ teoremasi isbotlangan.

Ushbu maqolada panjaradagi ko‘pi bilan uchta zarrachalar sistemasiga mos 3×3 operatorli matritsa $\mathcal{A}_\mu$ qaralgan bo‘lib, ushbu operatorli matritsa nol, bir va ikki zarrachali qirqilgan Fok fazosida ta’sir qiladi. $\mathcal{A}_\mu$ operatorli matritsaning muhim spektrini o‘rganish maqsadida unga mos umumlashgan Fridrixs modeli kiritilgan. Umumlashgan Fridrixs modeli ikkita xos qiymatga ega ekanligi isbotlangan. So‘ngra, $\mathcal{A}_\mu$ operatorli matritsaning muhim spektri ko‘pi bilan 3 ta kesmadan iborat ekanligi ko‘rsatilgan. Shuningdek, $\mathcal{A}_\mu$ operatorli matritsa muhim spektrining ikki zarrachali va uch zarrachali tarmoqlari ajratilgan hamda ularning chegaralari uchun tegishli baholashlar olingan.

1. Uchinchi tartibli operatorli matritsaning aniqlanishi.

$\mathbb{T}^1$- bir o‘lchamli tor, $\mathcal{H}_0 \coloneqq \mathbb{C}$ - kompleks sonlar fazosi, $\mathcal{H}_1 \coloneqq L_2(\mathbb{T}^1)$ - $\mathbb{T}^1$ da aniqlangan kvadrati bilan integrallanuvchi (kompleks o‘zgaruvchili) funksiyalar Hilbert fazosi va $\mathcal{H}_2 \coloneqq L_2^s(\mathbb{T}^2)$ $\mathbb{T}^2$ da aniqlangan kvadrati bilan integrallanuvchi (kompleks o‘zgaruvchili) simmetrik funksiyalar Hilbert fazosi bo‘lsin. $\mathcal{H}$ orqali $\mathcal{H}_0, \mathcal{H}_1$ va $\mathcal{H}_2$ fazolarning to‘g‘ri yig‘indisini qaraymiz, ya’ni $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

Zamonaviy matematik fizikada $\mathcal{H}$ Hilbert fazosiga Fok fazosining qirqilgan uch zarrachali qism fazosi deyiladi. Bu fazoning ixtiyoriy f elementi $f = (f_0, f_1, f_2), f_i \in \mathcal{H}_i, i = 0,1,2$ ko‘rinishga ega bo‘lib, uning normasi

$$\|f\| = \left[|f_0|^2 + \int_{\mathbb{T}^1} |f_1(x)|^2 dx + \int_{\mathbb{T}^2} |f_2(x,y)|^2 dx dy \right]^{1/2}$$

tenglik yordamida hisoblanadi.

$\mathcal{H}$ Hilbert fazosida

$$\mathcal{A}_\mu = \begin{pmatrix} A_{00} & \mu A_{01} & 0 \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ 0 & \mu A_{12}^* & A_{22} \end{pmatrix}$$

ko‘rinishdagi uchinchi tartibli operatorli matritsani qaraymiz. Bu yerda $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i, i \le j, i,j = 0,1,2$ matritsaviy elementlar quyidagi formulalar orqali aniqlangan:

$$A_{00} f_0 = \varepsilon f_0, \qquad A_{01} f_1 = \int_{\mathbb{T}^1} \sin(3t)\, f_1(t)\, dt$$

$$(A_{11} f_1)(x) = (\varepsilon + 1 - (\cos(3x))) f_1(x), \qquad A_{12} f_2(x) = \int_{\mathbb{T}^1} \sin(3t)\, f_2(x,t) dt$$

$$(A_{22} f_2)(x,y) = (\varepsilon + 2 - (\cos(3x)) - (\cos(3y))) f_2(x,y),$$
$$f_i \in \mathcal{H}_i, i = 0,1,2.$$

Ta’kidlash joizki, ushbu ko’rinishda berilgan $\mathcal{A}_\mu$ operatorli matritsa $\mathcal{H}$ Hilbert fazosida chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma operator bo‘ladi.

Matematik fizikada A_{01}, A_{12} operatorlarga yo‘qotish operatorlari, ularga qo‘shma bo‘lgan A_{01}^*, A_{12}^* operatorlarga esa paydo qilish operatorlari deyiladi.

Sodda hisoblashlarga ko‘ra

$$(A_{01}^*, f_0)(x) = \sin(3x)\, f_0, \qquad f_0 \in \mathcal{H}_0;$$
$$(A_{12}^* f_1)(x,y) = \sin(3y)\, f_1(x), \qquad f_1 \in \mathcal{H}_1.$$

Odatda $\mathcal{A}_\mu, \mu > 0$ operatorli matritsa bir o‘lchamli panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasiga mos Gamiltonian sifatida qaraladi.

Mazkur maqolada Hilbert fazosidagi chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma A operatorning muhim spektri, diskret spektri va spektri uchun mos ravishda $\sigma_{\text{ess}}(A), \sigma_{\text{disc}}(A)$ va $\sigma(A)$ belgilashlar ishlatiladi.

Eslatib o‘tamiz: A operatorning barcha chekli karrali, yakkalangan xos qiymatlari to‘plamiga A operatorning diskret spektri deyiladi. A operatorning $\sigma_{\text{ess}}(A)$ muhim spektri esa $\sigma_{\text{ess}}(A) \coloneqq \sigma(A) \setminus \sigma_{\text{disc}}(A)$

tenglik orqali aniqlanadi.

Ushbu maqola $\mathcal{A}_\mu, \mu > 0$ operatorli matritsaning muhim spektri va uning tarmoqlarini tadqiq qilishga bag‘ishlangan.

2. Umumlashgan Fridrixs modeli xos qiymatlarining mavjudligi.

$\mathcal{A}_\mu$ operatorli matritsaning muhim spektrini o‘rganish maqsadida $\mathcal{H}_0 \oplus \mathcal{H}_1$ fazoda ta’sir qiluvchi umumlashgan Fridrixs modeli deb ataluvchi $h_\mu, \mu > 0$ operatorli matritsani kiritamiz:

$$h_\mu = \begin{pmatrix} A_{00} & \mu A_{01} \\ \mu A_{01}^* & A_{11} \end{pmatrix}.$$

Funksional analizning mos ta’riflaridan foydalanib h_μ operatorli matritsaning chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma ekanligini ko‘rsatish mumkin.

Qulaylik uchun $\mathcal{H}_0 \oplus \mathcal{H}_1$ Hilbert fazosida

$$h_0 = \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}$$

tenglik yordamida aniqlanuvchi h_0 operatorli matritsani qaraymiz. Ko‘rinib turibdiki, $h - h_0$ qo‘zg‘alish operatori o‘z-o‘ziga qo‘shma ikki o‘lchamli operatordir. Aniqlanishiga ko‘ra, h_0 diagonal operatorning spektri quyidagiga teng:

$$\sigma(h_0) = \sigma(A_{00}) \cup \sigma(A_{11}),$$

bunda

$$\sigma(A_{00}) = \sigma_{\text{disc}}(A_{00}) = \{\varepsilon\}; \qquad \sigma(A_{11}) = \sigma_{\text{ess}}(A_{11}) = [\varepsilon; \varepsilon + 2].$$

Oxirgi mulohazalardan

$$\sigma_{\text{ess}}(h_0) = [\varepsilon; \varepsilon + 2], \sigma_{disc}(h_0) = \{\varepsilon\}$$

ekanligini hosil qilamiz.

Chekli o‘lchamli qo‘zg‘alishlarda muhim spektrning o‘zgarmasligi haqidagi mashhur Veyl teoremasiga ko‘ra, h_μ va h_0 operatorli matritsalarning muhim spektrlari ustma- ust tushadi. Ya’ni,

$$\sigma_{\text{ess}}(h_\mu) = \sigma_{\text{ess}}(h_0) = [\varepsilon, \varepsilon + 2].$$

Faraz qilaylik, $z \in \mathbb{C} \setminus [\varepsilon, \varepsilon + 2]$ soni h_μ operatorning xos qiymati bo‘lib, $f = (f_0, f_1)$ unga mos xos vektor-funksiya bo‘lsin. U holda f vektor $h_\mu f = zf$ tenglikni qanoatlantiradi. Ya’ni, f_0 va f_1 elementlar

$$\begin{cases} \varepsilon f_0 + \mu \int_{\mathbb{T}^1} \sin(3t) f_1(t) dt = z f_0 \\ \mu \sin(3x) f_0 + (\varepsilon + 1 + \cos(3x)) f_1(x) = z f_1(x) \end{cases} \tag{1}$$

tenglamalar sistemasini qanoatlantiradi. (1) tenglamalar sistemasining ikkinchi tengligini

$$\mu \sin(3x) f_0 = (z - \varepsilon - 1 + \cos(3x)) f_1(x) \tag{2}$$

kabi yozib olamiz.

Ixtiyoriy $x \in \mathbb{T}^1$ uchun $z \notin [\varepsilon; \varepsilon + 2]$ bo‘lganida $\varepsilon + 1 - \cos(3x) - z \neq 0$ munosabat bajariladi. U holda (2) tenglikdan $f_1(x)$ uchun

$$f_1(x) = \frac{\mu \sin(3x)}{z - \varepsilon - 1 + \cos(3x)} f_0 \tag{3}$$

ifodani hosil qilamiz.

(3) tenglikdagi $f_1(x)$ ni (1) tenglamalar sistemasining birinchi tengligiga qo‘yib,

$$\varepsilon f_0 + \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{z - \varepsilon - 1 + \cos(3t)}\, f_0 - z f_0 = 0$$

yoki,

$$f_0 \left(\varepsilon - z - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t) dt}{\varepsilon + 1 - \cos(3t) - z} \right) = 0 \tag{4}$$

tenglikni hosil qilamiz.

Agar (4) ko‘paytmada $f_0 = 0$ bo‘lsa, u holda (3) tenglikga ko‘ra $f_1(x) = 0$ bo‘ladi. Bu esa $f = (f_0, f_1)$ ning xos vektor ekanligiga zid. Demak,

$$\varepsilon - z - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{\varepsilon + 1 - \cos(3t) - z} = 0$$

ekan.

Fiksirlangan $\mu > 0$ uchun $\mathbb{C}\backslash[\varepsilon; \varepsilon + 2]$ da analitik bo‘lgan

$$\Delta_\mu(z) := \varepsilon - z - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{\varepsilon + 1 - \cos(3t) - z}$$

funksiyani qaraymiz.

Odatda, $\Delta_\mu(\cdot)$ funksiya h_μ operatorli matritsaga mos Fredgolm determinanti deb ataladi.

Quyidagi lemma h_μ operatorli matritsaning xos qiymatlari va $\Delta_\mu(\cdot)$ funksiyaning nollari orasidagi bog‘lanishni ifodalaydi.

1-lemma. Har bir $\mu > 0$ da $z_\mu \in \mathbb{C}\backslash[\varepsilon; \varepsilon + 2]$ soni h_μ operatorli matritsaning xos qiymati bo’lishi uchun $\Delta_\mu(z_\mu) = 0$ bo‘lishi zarur va yetarlidir.

1-lemmadan h_μ operatorli matritsaning diskret spektri uchun

$$\sigma_{\text{disc}}(h_\mu) = \{z \in \mathbb{C}\backslash[\varepsilon; \varepsilon + 2]: \Delta_\mu(z) = 0\}$$

tenglikni hosil qilamiz.

Endi $\Delta_\mu(\cdot)$ funksiya nollarini o‘rganish orqali h_μ operatorli matritsaning xos qiymatlari soni va joylashuv o‘rnini tahlil qilamiz.

2-lemma. h_μ operatorli matritsa ikkita oddiy xos qiymatga ega bo‘lib, ulardan biri ε dan chapda, ikkinchisi esa $\varepsilon + 2$ dan o‘ngda joylashgan bo‘ladi.

Ma’lumki, $u(x) = \varepsilon + 1 - \cos(3x)$ funksiya $x = 0, x = \frac{2\pi}{3}, x = -\frac{2\pi}{3}$ nuqtalarda minimumga va $x = \frac{\pi}{3}, x = -\frac{\pi}{3}, x = \pi$ nuqtalarda maksimumga erishadi hamda $\sin 0 = \sin \pi = 0$.

Aniqlanishiga ko‘ra $\Delta_\mu(\cdot)$ funksiya $(-\infty; \varepsilon)$ va $(\varepsilon + 2; \infty)$ oraliqlarda monoton kamayuvchi funksiya bo‘ladi. Integral belgisi ostida limitga o‘tish haqidagi Lebeg teoremasiga ko‘ra chekli

$$\Delta_\mu(\varepsilon) = \lim_{z \to \varepsilon - 0} \Delta_\mu(z) = \lim_{z \to \varepsilon - 0} \left(\varepsilon - z - \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{\varepsilon + 1 - \cos(3t) - z} \right) =$$

$$= \varepsilon - \varepsilon - \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{\varepsilon + 1 - \cos(3t) - \varepsilon} = - \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{1 - \cos(3t)} < 0$$

$$\Delta_\mu(\varepsilon + 2) = \lim_{z \to \varepsilon + 2 + 0} \left(\varepsilon - z - \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{\varepsilon + 1 - \cos(3t) - z} \right) =$$

$$= \varepsilon - 2 - \varepsilon - \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{\varepsilon + 1 - \cos(3t) - 2 - \varepsilon} = -2 + \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{1 + \cos(3t)} =$$

$$= -2 + \int_{-\pi}^{\pi} \frac{1 - \cos^2(3t)dt}{1 + \cos(3t)} = -2 + \int_{-\pi}^{\pi} \frac{(1 - \cos(3t))(1 + \cos(3t))dt}{1 + \cos(3t)} =$$

$$= -2 + \int_{-\pi}^{\pi} (1 - \cos(3t))\, dt = -2 + 2\pi > 0$$

limitlar mavjud bo‘ladi.

Ikkinchi tomondan

$$\lim_{z \to -\infty} \Delta_\mu(z) = +\infty \quad \text{va} \quad \lim_{z \to +\infty} \Delta_\mu(z) = -\infty$$

tengliklar o‘rinlidir.

Shunday qilib, $\Delta_\mu(\varepsilon)<0$, $\Delta_\mu(2+\varepsilon)>0$ va $\Delta_\mu(\cdot)$ funksiyaning $(-\infty;\varepsilon)$ va $(\varepsilon+2;+\infty)$ oraliqda monoton kamayuvchi ekanligidan bu funksiya $(-\infty;\varepsilon)$ va $(\varepsilon+2;+\infty)$ oraliqda yotuvchi bittadan oddiy (1 karrali) nolga ega bo‘ladi. Bu nollarni mos ravishda E_μ^1 va E_μ^2 orqali belgilaymiz. Aniqlik uchun $E_\mu^1\in(-\infty;\varepsilon)$ va $E_\mu^2\in(\varepsilon+2;+\infty)$ deb olamiz. 1-lemmaga ko‘ra E_μ^1 va E_μ^2 sonlari h_μ operatorli matritsaning xos qiymatlari bo‘ladi.

Shundan qilib 2-lemma isbotlandi.

1-lemma va 2-lemmalardan h_μ operatorli matritsaning spektri uchun xulosa sifatida ushbu

$$\sigma(h_\mu)=\{E_\mu^1\}\cup[\varepsilon;\varepsilon+2]\cup\{E_\mu^2\}$$

tenglikni hosil qilamiz, bunda

$$E_\mu^1<\varepsilon, E_\mu^2>\varepsilon+2.$$

Quyidagi teorema $\mathcal{A}_\mu$ operatorli matritsaning muhim spektrini tavsiflaydi.

1-teorema. $\mathcal{A}_\mu$ operatorli matritsaning muhim spektri uchun quyidagi tenglik o‘rinli:

$$\sigma_{\text{ess}}(\mathcal{A}_\mu)=[E_\mu^1;E_\mu^1+2]\cup[\varepsilon;\varepsilon+4]\cup[E_\mu^2;E_\mu^2+2].$$

Isbot.

$$[E_\mu^1;E_\mu^1+2]\cup[\varepsilon;\varepsilon+4]\cup[E_\mu^2;E_\mu^2+2]\subset\sigma_{\text{ess}}(\mathcal{A}_\mu)$$

munosabatni isbotlashda Veyl kriteriyasidan foydalanish qulaydir. Avval, $[\varepsilon;\varepsilon+2]\subset\sigma_{\text{ess}}(\mathcal{A}_\mu)$ munosabat o‘rinli ekanligini ko‘rsatamiz. Ixtiyoriy $z_0\in[\varepsilon;\varepsilon+2]$ nuqta uchun shunday $f^{(n)}=\left(f_0^{(n)},f_1^{(n)},f_2^{(n)}\right)\in\mathcal{H}$ ketma-ketlik topilib, $\|f^{(n)}\|=1$ va $\|(\mathcal{A}_\mu-z_0)f^{(n)}\|\xrightarrow[n\to\infty]{}0$ ekanligini isbotlaymiz.

$(\varepsilon+2-(\cos(3x))-(\cos(3y)))$ funksiya $\mathbb{T}^2$ kompakt to‘plamda uzluksiz bo‘lganligi bois, shunday $(x_0,y_0)\in\mathbb{T}^2$ nuqta topilib, $z_0=\varepsilon+2-(\cos(3x_0))-(\cos(3y_0))$ tenglik o‘rinli bo‘ladi.

Ixtiyoriy $n\in\mathbb{N}$ natural soni uchun $x_0\in\mathbb{T}$ nuqtaning quyidagi atrofini qaraymiz:

$$V_n(x_0):=\left\{x\in[-\pi;\pi]:\frac{1}{n+1}<|x-x_0|<\frac{1}{n}\right\}$$

Aniqlanishiga ko‘ra, $V_n(x_0)$ quyidagi kabi topiladi:

$$\frac{1}{n+1}<|x-x_0|<\frac{1}{n}$$

a) $\frac{1}{n+1}<x-x_0<\frac{1}{n}\Rightarrow x_0+\frac{1}{n+1}<x<\frac{1}{n}+x_0;$

b) $-\frac{1}{n}<x-x_0<-\frac{1}{n+1}\Rightarrow x_0-\frac{1}{n}<x<x_0-\frac{1}{n+1}.$

Demak,

$$V_n(x_0)=[-\pi;\pi]\cap\left\{\left(x_0+\frac{1}{n+1},\frac{1}{n}+x_0\right)\cup\left(x_0-\frac{1}{n},x_0-\frac{1}{n+1}\right)\right\}.$$

$$V_1(x_0)=[-\pi;\pi]\cap\left\{\left(x_0+\frac{1}{2},1+x_0\right)\cup\left(x_0-1,x_0-\frac{1}{2}\right)\right\}$$

$$V_2(x_0)=[-\pi;\pi]\cap\left\{\left(x_0+\frac{1}{3},\frac{1}{2}+x_0\right)\cup\left(x_0-\frac{1}{2},x_0-\frac{1}{3}\right)\right\}.$$

Ko‘rinib turibdiki, $V_n(x_0)$ o‘zaro kesishmaydigan atroflar ketma-ketligini aniqlaydi, ya’ni $n\neq m$ natural sonlari uchun

$$V_n(x_0)\cap V_m(x_0)=\emptyset$$

tenglik o‘rinli bo‘ladi.

$\mu(V_n(x_0))$ orqali $V_n(x_0)$ to‘plamning Lebeg o‘lchovini belgilaymiz. $f^{(n)}\in\mathcal{H}$ funksiyalar ketma-ketligini quyidagicha tanlaymiz:

$$f^{(n)}=\begin{pmatrix}0\\0\\f_2^{(n)}(x,y)\end{pmatrix},$$

bu yerda

$$f_2^{(n)}(x,y)=\begin{cases}\dfrac{1}{\sqrt{\mu\big(V_n(x_0)\big)}\sqrt{\mu\big(V_n(y_0)\big)}}, (x,y)\in V_n(x_0)\times V_n(y_0)\\ 0, (x,y)\notin V_n(x_0)\times V_n(y_0)\end{cases}.$$

Yuqoridagi mulohazalarga ko‘ra barcha $n\neq m$ natural sonlari uchun

$$\big(V_n(x_0)\times V_n(y_0)\big)\cap\big(V_m(x_0)\times V_m(y_0)\big)=\emptyset$$

munosabat o‘rinli. Bundan ko‘rinib turibdiki, $\{f^{(n)}\}$ ortogonal sistemani tashkil qiladi.

Endi $\|f^{(n)}\|=1$ ekanligini ko‘rsatamiz:

$$\|f^{(n)}\|^2=\left\|f_2^{(n)}\right\|^2=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|f_2^{(n)}(x,y)\right|^2dxdy=\frac{1}{\mu(V_n(x_0))\cdot\mu(V_n(y_0))}\int_{V_n(x_0)}\int_{V_n(y_0)}dxdy=$$

$$=\frac{1}{\mu(V_n(x_0))\cdot\mu(V_n(y_0))}\cdot\mu\big(V_n(x_0)\big)\cdot\mu\big(V_n(y_0)\big)=1.$$

Bundan, $\{f^{(n)}\}$ nafaqat ortogonal, balki ortonormal vektor-funksiyalar ketma-ketligi ekanligi kelib chiqadi.

Istalgan $n\in\mathbb{N}$ soni uchun $(\mathcal{A}_\mu-z_0I)f^{(n)}$ elementni qaraymiz va uning normasini baholaymiz:

$$\left\|\big(\mathcal{A}_\mu-z_0I\big)f^{(n)}\right\|^2\leq\mu\big(V_n(y_0)\big)+\sup_{(x,y)\in V_n(x_0)\times V_n(y_0)}|\,\varepsilon+2-(\cos(3x))-(\cos(3y))-z_0|^2.$$

$V_n(x_0)$ to‘plamning aniqlanishidan va $\varepsilon+2-(\cos(3x))-(\cos(3y))-z_0$ funksiyaning uzluksizligidan $n\to\infty$ bo‘lganda $\left\|(\mathcal{A}_\mu-z_0I)f^{(n)}\right\|\to 0$ bajarilishi kelib chiqadi, ya’ni $z_0\in\sigma_{\text{ess}}\big(\mathcal{A}_\mu\big)$ munosabat o‘rinli. z_0 ixtiyoriy nuqta bo‘lganligi bois $[\varepsilon;\varepsilon+4]\subset\sigma_{\text{ess}}\big(\mathcal{A}_\mu\big)$ munosabatga ega bo‘lamiz.

Endi $\big[E_\mu^1;E_\mu^1+2\big]\cup\big[E_\mu^2;E_\mu^2+2\big]\subset\sigma_{\text{ess}}\big(\mathcal{A}_\mu\big)$ ekanligini ko‘rsatamiz. Ixtiyoriy $z_0\in\big[E_\mu^1;E_\mu^1+2\big]\cup\big[E_\mu^2;E_\mu^2+2\big]$ nuqtani olib, $z_0\in\sigma_{\text{ess}}\big(\mathcal{A}_\mu\big)$ munosabat bajarilishini isbotlaymiz. Shunday $x_0\in[-\pi;\pi]$ nuqta topilib, $z_0-1+\cos(3x_0)\in\sigma_{\text{disc}}(h_\mu)$ tasdiq o‘rinli bo‘ladi. Shu sababli, $\psi=(\psi_0,\psi_1)\in\mathcal{H}_1\oplus\mathcal{H}_2$ nolmas vektor topilib,

$$\Big(h_\mu-(z_0+1-\cos(3x))\Big)\psi=0$$

tenglik o‘rinli bo‘ladi.

$\phi^{(n)}$ ortogonal vektor-funksiyalar ketma-ketligini quyidagicha tanlaymiz:

$$\phi^{(n)}:=\begin{pmatrix}0\\ \phi_1^{(n)}(x)\\ \phi_2^{(n)}(x,y)\end{pmatrix},$$

bu yerda

$$\phi_1^{(n)}(x)=\frac{\chi_{V_n(x_0)}(x)}{\sqrt{\mu(V_n(x_0))}}\cdot\frac{\psi_0}{\|\psi\|};$$

$$\phi_2^{(n)}(x,y)=\frac{\chi_{V_n(x_0)}(x)}{\sqrt{\mu(V_n(x_0))}}\cdot\frac{\psi_1(y)}{\|\psi\|}.$$

Bunda $\chi_{V_n(x_0)}(x)$ orqali $V_n(x_0)$ to‘plamning xarakteristik funksiyasi belgilangan. Tanlanishiga ko‘ra $\{\phi^{(n)}\}$ ortogonal sistemani tashkil qiladi. Haqiqatan ham, $n\neq m$ sonlari uchun $(\phi_1^{(n)},\phi_1^{(m)})=0$, $\left(\phi_2^{(n)},\phi_2^{(m)}\right)=0$ tengliklar o‘rinli. Bundan esa

$$\big(\phi^{(n)},\phi^{(m)}\big)=(\phi_1^{(n)},\phi_1^{(m)})+(\phi_2^{(n)},\phi_2^{(m)})=0+0=0$$

ekanligi kelib chiqadi. $\phi^{(n)}$ vektor-funksiyaning normasini baholaymiz:

$$\|\phi^{(n)}\|^2 = \|\phi_1^{(n)}\|^2 + \|\phi_2^{(n)}\|^2 = \int_{-\pi}^{\pi} |\phi_1^{(n)}(x)|^2 dx + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \|\phi_2^{(n)}(x,y)\|^2 dxdy =$$
$$= \frac{\|\psi_0\|^2}{\|\psi\|^2}\int_{V_n(x_0)} \frac{dx}{\mu(V_n(x_0))} + \frac{\|\psi_1\|^2}{\|\psi\|^2}\int_{V_n(x_0)} \frac{dx}{\mu(V_n(x_0))} =$$
$$= \frac{\|\psi_0\|^2}{\|\psi\|^2}\cdot\frac{1}{\mu(V_n(x_0))}\cdot\mu(V_n(x_0)) + \frac{\|\psi_1\|^2}{\|\psi\|^2}\cdot\frac{1}{\mu(V_n(x_0))}\cdot\mu(V_n(x_0)) == \frac{\|\psi_0\|^2+\|\psi_1\|^2}{\|\psi\|^2}$$
$$= \frac{\|\psi\|^2}{\|\psi\|^2} = 1.$$

Shunday qilib, $\{\phi^{(n)}\}$ ortonormal vektor-funksiyalar ketma-ketligi ekan.

Endi, istalgan $n \in \mathbb{N}$ natural soni uchun $(\mathcal{A}_\mu - z_0)\phi^{(n)}$ vektor funksiyani qaraymiz va uning normasini baholaymiz:

$$\|(\mathcal{A}_\mu - z_0)\phi^{(n)}\|^2 \le C\mu(V_n(x_0)) + 2\sup_{x\in V_n(x_0)} |\cos(3x) - \cos(3x_0)|^2, \qquad (4)$$

Bunda $C > 0$ biror son.

$n \to \infty$ bo‘lganda $\mu(V_n(x_1)) \to 0$ va $\sup_{x\in V_n(x_0)}|\cos(3x) - \cos(3x_0)|^2 \to 0$ munosabatlar bajarilgani bois, (4) tengsizlikdan $n \to \infty$ bo‘lganda $\|(\mathcal{A}_\mu - z_0)\phi^{(n)}\| \to 0$ munosabat bajarilishi kelib chiqadi. Bundan esa $z_0 \in \sigma_{\text{ess}}(\mathcal{A}_\mu)$ ekanligini hosil qilamiz. Shunday qilib, $[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2] \subset \sigma_{\text{ess}}(\mathcal{A}_\mu)$ munosabat isbotlandi.

Teskari tasdiqni isbotlash uchun har bir $z \in \mathbb{C}\backslash\{[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]\}$ soni uchun $\mathcal{H}_0 \oplus \mathcal{H}_1$ Hilbert fazosida

$$\mathcal{T}_\mu(z) = \begin{pmatrix} T_{00}(z) & T_{01}(z) \\ T_{10}(z) & T_{11}(z) \end{pmatrix}$$

kabi aniqlangan ikkinchi tartibli operatorli matritsalar sinfini qaraymiz. Bu yerda $T_{ij}(z): \mathcal{H}_j \to \mathcal{H}_i, i \le j, i,j = 0,1$ matritsaviy elementlar

$$T_{00}(z)g_0 = (1 + \varepsilon - z)g_0 \qquad T_{01}(z)g_1 = \int_{\mathbb{T}^1} \sin(3t)\, g_1(t)dt$$
$$(T_{10}(z)g_0)(x) = -\frac{\sin(3x)\, g_0}{\Delta_\mu(z + 1 - \cos(3x))}$$
$$(T_{11}(z)g_1)(x) = \frac{\sin(3x)}{2\Delta_\mu(z + 1 - \cos(3x))}\int_{\mathbb{T}^1} \frac{\sin(3t)\, g_1(t)dt}{\varepsilon + 2 - \cos(3x) - \cos(3y) - z}$$

tengliklar yordamida aniqlangan va $g_i \in \mathcal{H}_i, i = 0,1$. Ta’kidlash joizki, ixtiyoriy $z \in \mathbb{C}\backslash\{[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]\}$ soni uchun $T_{00}(z), T_{01}(z), T_{10}(z)$ matritsaviy elementlar 1 o‘lchamli operatorlar bo‘lib, $T_{11}(z)$ operator esa Hilbert-Shmidt sinfiga tegishlidir. Shu sababli $\mathcal{T}_\mu(z)$ kompakt operator bo‘ladi.

Quyidagi lemma $\mathcal{A}_\mu$ operatorli matritsa uchun mashhur Faddeyev natijasining analogi bo‘lib, $\mathcal{A}_\mu$ va $\mathcal{T}_\mu(z)$ operatorli matritsalarning xos qiymatlari orasidagi bog‘lanishni ifodalaydi.

3-lemma. Har bir tayinlangan $z \in \mathbb{C}\backslash\{[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]\}$ soni $\mathcal{A}_\mu$ operatorli matritsaning xos qiymati bo‘lishi uchun $\lambda = 1$ soni $\mathcal{T}_\mu(z)$ operatorli matritsaning xos qiymati bo‘lishi zarur va yetarlidir. Bundan tashqari, z va 1 xos qiymatlar bir xil karralikga egadir.

Ushbu lemma [8] ishdagi kabi isbotlanadi.

Yuqoridagi mulohazalardan $(I - \mathcal{T}_\mu(z))^{-1}$ operatorning mavjudligi kelib chiqadi. Fredgolmning analitik teoremasidan $\sigma(\mathcal{A}_\mu)\backslash\{[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]\}$ to‘plamning yakkalangan nuqtalardan iboratligi va bu to‘plam nuqtalarining quyuqlashish nuqtalari faqatgina shu to‘plamning chegarasi bo‘lishi mumkinligi kelib chiqadi. Shunday qilib,

$$\sigma(\mathcal{A}_\mu)\backslash\{[E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]\} \subset \sigma_{\text{disc}}(\mathcal{A}_\mu) = \sigma(\mathcal{A}_\mu)\backslash\sigma_{\text{ess}}(\mathcal{A}_\mu).$$

Demak, $\sigma_{\text{ess}}(\mathcal{A}_\mu) \subset [E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]$ munosabat o'rinli. Bundan, $\sigma_{\text{ess}}(\mathcal{A}_\mu) = [E_\mu^1; E_\mu^1 + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^2; E_\mu^2 + 2]$ tenglikni hosil qilamiz. Shunday qilib, 1-teorema isbotlandi.

1-ta'rif: Ushbu

$$[E_\mu^1; E_\mu^1 + 2] \cup [E_\mu^2; E_\mu^2 + 2] \text{ va } [\varepsilon; \varepsilon + 4]$$

to'plamlarga mos ravishda $\mathcal{A}_\mu$ operatorli matritsa muhim spektrining ikki zarrachali va uch zarrachali tarmoqlari deyiladi. Bu tarmoqlarni $\sigma_{\text{two}}(\mathcal{A}_\mu)$ va $\sigma_{\text{three}}(\mathcal{A}_\mu)$ kabi belgilaymiz, ya'ni

$$\sigma_{\text{two}}(\mathcal{A}_\mu) = [E_\mu^1; E_\mu^1 + 2] \cup [E_\mu^2; E_\mu^2 + 2],$$

$$\sigma_{\text{three}}(\mathcal{A}_\mu) = [\varepsilon; \varepsilon + 4].$$

Ta'kidlash joizki, [9,10] maqolalar dispersiya funksiyasi bir nechta nuqtalarda aynimagan minimumga ega bo'ladigan hol tahlil qilingan. Bu holda operatorli matritsa xos qiymatlari sonining chekli yoki cheksiz bo'lishi shu nuqtalarga bog'liq ravishda o'rganilgan.

Xulosa. Ushbu maqolada Fok fazosining qirqilgan uch zarrachali qism fazosida ta'sir qiluvchi $\mathcal{A}_\mu, \mu > 0$ uchinchi tartibli operatorli matritsa o'rganilgan. $\mathcal{A}_\mu, \mu > 0$ operatorli matritsaning muhim spektrini aniqlash maqsadida umumlashgan Fridrixs modeli kiritilgan va uning xos qiymatlari soni hamda joylashuv o'rni o'rganilgan. Buning yordamida $\mathcal{A}_\mu$ operatorli matritsa muhim spektrining tarmoqlari aniqlangan. Bunda mashhur Veyl mezoni va $\mathcal{A}_\mu, \mu > 0$ operatorli matritsa xos vektorlariga mos Faddeyev tenglamasi xossalaridan foydalanilgan. Muhim spektrning ikki va uch zarrachali tarmoqlari quyi chegaralari taqqoslangan.

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