



# **ABSTRACTS**

**of the international conference**

**MATHEMATICAL ANALYSIS AND ITS  
APPLICATIONS IN MODERN  
MATHEMATICAL PHYSICS**

## **PART I**

**Samarkand  
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3. There are exactly five TISGMs

(a) if  $\theta \in (0, \theta_4]$  and  $\lambda \in (\lambda_2(\theta), \lambda_3(\theta))$

(b) if  $\theta \in (\theta_4, \theta_3)$  and  $\lambda \in (\lambda_4(\theta), \lambda_3(\theta))$

(c) if  $\theta \in (0, \theta_4)$  and  $\lambda \in (\lambda_1(\theta), \lambda_4(\theta)]$

(d) if  $\theta \in [\theta_4, \theta_2)$  and  $\lambda \in (\lambda_1(\theta), \lambda_2(\theta))$

4. There are exactly four such measures if  $\theta \in (0, \theta_2)$  and  $\lambda = \lambda_1(\theta)$  or if  $\theta \in [\theta_4, \theta_2)$  and  $\lambda = \lambda_2(\theta)$

5. There are exactly three such measures

(a) if  $\theta \in (0, \theta_2)$  and  $\lambda \in (0, \lambda_1(\theta))$

(b) if  $\theta \in (0, \theta_3)$  and  $\lambda = \lambda_3(\theta)$

(c) if  $\theta \in (\theta_4, \theta_2)$  and  $\lambda \in (\lambda_2(\theta), \lambda_4(\theta)]$

(d) if  $\theta \in [\theta_2, \theta'_c)$  and  $\lambda \in (0, \lambda_4(\theta))$

(e) if  $\theta \in [\theta_2, \theta_3)$  and  $\lambda = \lambda_4(\theta)$

6. Otherwise there exists a unique such measure.

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### Usual, quadratic and cubic numerical ranges corresponding to a $3 \times 3$ operator matrices

<sup>1</sup>Rasulov T., <sup>2</sup>Sharipova M.

<sup>1</sup> Uzbekistan, Bukhara State University and Bukhara branch of the Institute of Mathematics  
e-mail: rth@mail.ru

<sup>2</sup> Uzbekistan, Bukhara State University  
e-mail: sh.mubina.sh@gmail.com

Operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces [1]. They arise in various areas of mathematics and its applications.

Let  $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$  be the direct sum of the complex Hilbert spaces  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_3$ . With respect to this decomposition, every bounded self-adjoint linear operator  $\mathcal{A} \in L(\mathcal{H})$  has a  $3 \times 3$  operator matrix representation

$$\mathcal{A} := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix}, \quad (1)$$

with bounded linear entries  $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$ ,  $i, j = 1, 2, 3$  such that  $A_{ii} = A_{ii}^*$ ,  $i = 1, 2, 3$ .

For a Hilbert space  $X$  we denote by  $\mathbb{S}_X$  the unite sphere in  $X$ :

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\}.$$

**Definition 1.** The set

$$W(\mathcal{A}) := \{(\mathcal{A}f, f) : f \in \mathbb{S}_{\mathcal{H}}\}$$

is called the numerical range of  $\mathcal{A}$ .

The numbers

$$m_{\mathcal{A}} := \inf_{f \in \mathbb{S}_{\mathcal{H}}} (\mathcal{A}f, f), \quad M_{\mathcal{A}} := \sup_{f \in \mathbb{S}_{\mathcal{H}}} (\mathcal{A}f, f)$$

are called lower and upper bounds of  $\mathcal{A}$ , respectively. Then  $\overline{W(\mathcal{A})} = [m_{\mathcal{A}}, M_{\mathcal{A}}]$ . In addition, if  $m_{\mathcal{A}}, M_{\mathcal{A}} \in \sigma_{\text{pp}}(\mathcal{A})$ , then  $W(\mathcal{A}) = [m_{\mathcal{A}}, M_{\mathcal{A}}]$ .

Let  $\widehat{\mathcal{H}}_1 := \mathcal{H}_1 \oplus \mathcal{H}_2$  and  $\widehat{\mathcal{H}}_2 := \mathcal{H}_3$ . To define the quadratic numerical range we consider the operator matrix  $\mathcal{A}$  with respect to the decomposition  $\mathcal{H} = \widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2$ :

$$\mathcal{A} = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{12}^* & \widehat{A}_{22} \end{pmatrix} \quad (2)$$

with the entries  $\widehat{A}_{ij} : \widehat{\mathcal{H}}_j \rightarrow \widehat{\mathcal{H}}_i$ ,  $i \leq j$ ,  $i, j = 1, 2$ :

$$\widehat{A}_{11} := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \quad \widehat{A}_{12} := \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \quad \widehat{A}_{22} := A_{33}.$$

**Definition 2.** For  $\widehat{f} = (\widehat{f}_1, \widehat{f}_2) \in \mathbb{S}_{\widehat{\mathcal{H}}_1} \times \mathbb{S}_{\widehat{\mathcal{H}}_2}$  we define the  $2 \times 2$  matrix

$$\mathcal{A}_{\widehat{f}} := \begin{pmatrix} (\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1) & (\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1) \\ (\widehat{A}_{12}^*\widehat{f}_1, \widehat{f}_2) & (\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2) \end{pmatrix} \in M_2(\mathbb{C}).$$

Then the set

$$W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A}) := \bigcup_{\widehat{f} \in \mathbb{S}_{\widehat{\mathcal{H}}_1} \times \mathbb{S}_{\widehat{\mathcal{H}}_2}} \sigma_{\text{pp}}(\mathcal{A}_{\widehat{f}})$$

is called the quadratic numerical range of  $\mathcal{A}$  (with respect to the operator matrix representation (2)).

For  $\widehat{f}_i \in \widehat{\mathcal{H}}_i$ ,  $\widehat{f}_i \neq 0$ ,  $i = 1, 2$ , we define

$$\lambda_{\pm} \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} := \frac{1}{2} \left( \frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{\|\widehat{f}_1\|^2} + \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{\|\widehat{f}_2\|^2} \pm \sqrt{\left( \frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{\|\widehat{f}_1\|^2} - \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{\|\widehat{f}_2\|^2} \right)^2 + 4 \frac{|(\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1)|^2}{\|\widehat{f}_1\| \|\widehat{f}_2\|}} \right)$$

and we let

$$\Lambda_{\pm}(\mathcal{A}) := \left\{ \lambda_{\pm} \begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} : \widehat{f}_i \in \widehat{\mathcal{H}}_i, \widehat{f}_i \neq 0, i = 1, 2 \right\}.$$

Then for the set  $W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A})$  we have the equality

$$W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A}) = \Lambda_+(\mathcal{A}) \cup \Lambda_-(\mathcal{A}).$$

**Definition 3.** The set of all eigenvalues of the  $3 \times 3$  matrices

$$\mathcal{A}_f := \begin{pmatrix} (A_{11}f_1, f_1) & (A_{12}f_2, f_1) & (A_{13}f_3, f_1) \\ (A_{12}^*f_1, f_2) & (A_{22}f_2, f_2) & (A_{23}f_3, f_2) \\ (A_{13}^*f_1, f_3) & (A_{23}^*f_2, f_3) & (A_{33}f_3, f_3) \end{pmatrix}, \quad f = (f_1, f_2, f_3) \in \mathbb{S}_{\mathcal{H}_1} \times \mathbb{S}_{\mathcal{H}_2} \times \mathbb{S}_{\mathcal{H}_3}$$

is called the cubic numerical range of  $\mathcal{A}$  (with respect to the operator matrix representation (1)) and will be denoted by  $W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A})$ . For a fixed decomposition of  $\mathcal{H}$ , we also write  $W^3(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A})$ .

Let  $a_{ij}(f) := (A_{ij}f_j, f_i)$  for  $i, j = 1, 2, 3$  and

$$E_k(f) := \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)) + 2\sqrt{-\frac{P(f)}{3}} \cos \frac{\Phi(f) + 2\pi k}{3}, \quad k = 1, 2, 3,$$

where

$$\begin{aligned}
 P(f) &:= -\frac{1}{6}(a_{11}(f) - a_{22}(f))^2 + (a_{11}(f) - a_{33}(f))^2 + (a_{22}(f) - a_{33}(f))^2 \\
 &\quad - |a_{12}(f)|^2 - |a_{23}(f)|^2 - |a_{13}(f)|^2; \\
 Q(f) &:= -\frac{2}{27}(a_{11}(f) + a_{22}(f) + a_{33}(f))^3 + \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)) \\
 &\quad \times (a_{11}(f)a_{22}(f) + a_{22}(f)a_{33}(f) + a_{11}(f)a_{33}(f) - |a_{12}(f)|^2 - |a_{23}(f)|^2 - |a_{13}(f)|^2) \\
 &\quad + a_{11}(f)a_{22}(f)a_{33}(f) + 2\operatorname{Re}(a_{12}(f)a_{23}(f)\overline{a_{13}(f)}) + |a_{12}(f)|^2 a_{33}(f) \\
 &\quad + |a_{23}(f)|^2 a_{11}(f) + |a_{13}(f)|^2 a_{22}(f); \\
 \Phi(f) &:= \arccos\left(-\frac{3Q(f)}{2P(f)}\sqrt{-\frac{3}{P(f)}}\right).
 \end{aligned}$$

The main result of this note is the following theorem.

**Theorem 1.** *For the cubic numerical range of  $\mathcal{A}$  we have*

$$W^3(\mathcal{A}) = \bigcup_{k=1}^3 \bigcup_{f \in \mathcal{S}_{\mathcal{H}_1} \times \mathcal{S}_{\mathcal{H}_2} \times \mathcal{S}_{\mathcal{H}_3}} \{E_k(f)\}.$$

This theorem plays a key role in the estimate of the bounds of  $\mathcal{A}$  in terms of cubic numerical range.

If  $A_{13} = 0$ , then for the lower bound of  $\mathcal{A}$  we have [2]

$$\min \sigma(\mathcal{A}) \geq \min\{\min \sigma(A_{11}), \min \sigma(A_{22}), \min \sigma(A_{33})\} - (\|A_{12}\|^2 + \|A_{23}\|^2)^{1/2}, \quad (3)$$

and strict inequality prevails if at least two of the lower bounds  $\min \sigma(A_{ii})$ ,  $i = 1, 2, 3$ , of diagonal elements are different. As an example we consider the case where  $\mathcal{H}_1 = \mathbb{C}$ ,  $\mathcal{H}_2 = L_2[\pi; \pi]$ ,  $\mathcal{H}_3 = L_2([\pi; \pi]^2)$  and

$$A_{11}f_1 = \varepsilon f_1, (A_{22}f_2)(x) = (\varepsilon + 1 - \cos x)f_2(x), (A_{33}f_3)(x, y) = (\varepsilon + 2 - \cos x - \cos y)f_2(x, y),$$

$$A_{12}f_2 = \int_{-\pi}^{\pi} \sin t f_2(t) dt, A_{13} = 0, (A_{23}f_3)(x) = \int_{-\pi}^{\pi} \sin t f_3(x, t) dt$$

with  $\varepsilon > 0$ . In the latter case  $W(A_{22}) = (\varepsilon; \varepsilon+2)$ ,  $W(A_{33}) = (\varepsilon; \varepsilon+4)$  and  $W(A_{ii}) \subset W^3(\mathcal{A})$ ,  $i = 2, 3$ . Then according to (3) the lower bound of  $\mathcal{A}$  can be estimated as follows:  $\min \sigma(\mathcal{A}) \geq \varepsilon - \sqrt{2}\pi$ . An obtained inequality is an important in determining the location of the first eigenvalue of  $\mathcal{A}$ .

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# CERTIFICATE

This certificate is presented to



**Mubina. Sharipova**  
Bukhara State University,  
Bukhara branch of the Institute of Mathematics,  
Bukhara, Uzbekistan.

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**Prof. A.A.Soleev**  
Vice-chairman  
24/09/2022, Samarkand

