ABSTRACTS

of the international conference

MATHEMATICAL ANALYSIS AND ITS APPLICATIONS IN MODERN MATHEMATICAL PHYSICS

PART I

Samarkand
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3. *There are exactly five TISGMs*

   (a) *if* $\theta \in (0, \theta_4]$ *and* $\lambda \in (\lambda_2(\theta), \lambda_3(\theta))$

   (b) *if* $\theta \in (\theta_4, \theta_3)$ *and* $\lambda \in (\lambda_4(\theta), \lambda_3(\theta))$

   (c) *if* $\theta \in (0, \theta_4)$ *and* $\lambda \in (\lambda_1(\theta), \lambda_4(\theta)]$

   (d) *if* $\theta \in [\theta_4, \theta_2)$ *and* $\lambda \in (\lambda_1(\theta), \lambda_2(\theta))$

4. *There are exactly four such measures if* $\theta \in (0, \theta_2)$ *and* $\lambda = \lambda_1(\theta)$ *or if* $\theta \in [\theta_4, \theta_2)$ *and* $\lambda = \lambda_2(\theta)$

5. *There are exactly three such measures*

   (a) *if* $\theta \in (0, \theta_2)$ *and* $\lambda \in (0, \lambda_1(\theta))$

   (b) *if* $\theta \in (0, \theta_3)$ *and* $\lambda = \lambda_3(\theta)$

   (c) *if* $\theta \in (\theta_4, \theta_2)$ *and* $\lambda \in (\lambda_2(\theta), \lambda_4(\theta)]$

   (d) *if* $\theta \in [\theta_2, \theta_c')$ *and* $\lambda \in (0, \lambda_4(\theta))$

   (e) *if* $\theta \in [\theta_2, \theta_3)$ *and* $\lambda = \lambda_4(\theta)$

6. *Otherwise there exists a unique such measure.*

References

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2. *Kuelske, C., Rozikov, U. A.* Extremality of translation-invariant phases for a three-state SOS-model on the binary tree, J. Stat. Phys., 160, 2015, 659-680.

Usual, quadratic and cubic numerical ranges corresponding to a 3×3 operator matrices

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Operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces [1]. They arise in various areas of mathematics and its applications.

Let $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ be the direct sum of the complex Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. With respect to this decomposition, every bounded self-adjoint linear operator $\mathcal{A} \in L(\mathcal{H})$ has a 3×3 operator matrix representation

$$\mathcal{A} := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix}, \tag{1}$$

with bounded linear entries $A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i), i, j = 1, 2, 3$ such that $A_{ii} = A_{ii}^*, i = 1, 2, 3$.

For a Hilbert space X we denote by $\mathbb{S}_X$ the unite sphere in X :

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\}.$$

Definition 1. *The set*

$$W(\mathcal{A}) := \{(\mathcal{A}f, f) : f \in \mathbb{S}_{\mathcal{H}}\}$$

is called the numerical range of $\mathcal{A}$.

The numbers

$$m_{\mathcal{A}} := \inf_{f \in \mathbb{S}_{\mathcal{H}}} (\mathcal{A}f, f), \quad M_{\mathcal{A}} := \sup_{f \in \mathbb{S}_{\mathcal{H}}} (\mathcal{A}f, f)$$

are called lower and upper bounds of $\mathcal{A}$, respectively. Then $\overline{W(\mathcal{A})} = [m_{\mathcal{A}}, M_{\mathcal{A}}]$. In addition, if $m_{\mathcal{A}}, M_{\mathcal{A}} \in \sigma_{\rm pp}(\mathcal{A})$, then $W(\mathcal{A}) = [m_{\mathcal{A}}, M_{\mathcal{A}}]$.

Let $\widehat{\mathcal{H}}_1 := \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\widehat{\mathcal{H}}_2 := \mathcal{H}_3$. To define the quadratic numerical range we consider the operator matrix $\mathcal{A}$ with respect to the decomposition $\mathcal{H} = \widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2$:

$$\mathcal{A} = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{12}^* & \widehat{A}_{22} \end{pmatrix} \tag{2}$$

with the entries $\widehat{A}_{ij} : \widehat{\mathcal{H}}_j \to \widehat{\mathcal{H}}_i$, $i \le j$, $i, j = 1, 2$:

$$\widehat{A}_{11} := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \quad \widehat{A}_{12} := \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \quad \widehat{A}_{22} := A_{33}.$$

Definition 2. *For $\widehat{f} = (\widehat{f}_1, \widehat{f}_2) \in \mathbb{S}_{\widehat{\mathcal{H}}_1} \times \mathbb{S}_{\widehat{\mathcal{H}}_2}$ we define the 2×2 matrix*

$$\mathcal{A}_{\widehat{f}} := \begin{pmatrix} (\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1) & (\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1) \\ (\widehat{A}_{12}^*\widehat{f}_1, \widehat{f}_2) & (\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2) \end{pmatrix} \in M_2(\mathbb{C}).$$

Then the set

$$W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A}) := \bigcup_{\widehat{f} \in \mathbb{S}_{\widehat{\mathcal{H}}_1} \times \mathbb{S}_{\widehat{\mathcal{H}}_2}} \sigma_{\rm pp}(\mathcal{A}_{\widehat{f}})$$

is called the quadratic numerical range of $\mathcal{A}$ (with respect to the operator matrix representation (2)).

For $\widehat{f}_i \in \widehat{\mathcal{H}}_i$, $\widehat{f}_i \neq 0$, $i = 1, 2$, we define

$$\lambda_{\pm}\begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} := \frac{1}{2}\left(\frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{\|\widehat{f}_1\|^2} + \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{\|\widehat{f}_2\|^2} \pm \sqrt{\left(\frac{(\widehat{A}_{11}\widehat{f}_1, \widehat{f}_1)}{\|\widehat{f}_1\|^2} - \frac{(\widehat{A}_{22}\widehat{f}_2, \widehat{f}_2)}{\|\widehat{f}_2\|^2} \right)^2 + 4\frac{\left|(\widehat{A}_{12}\widehat{f}_2, \widehat{f}_1)\right|^2}{\|\widehat{f}_1\|\|\widehat{f}_2\|}} \right)$$

and we let

$$\Lambda_{\pm}(\mathcal{A}) := \left\{ \lambda_{\pm}\begin{pmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{pmatrix} : \widehat{f}_i \in \widehat{\mathcal{H}}_i,\ \widehat{f}_i \neq 0,\ i = 1, 2 \right\}.$$

Then for the set $W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A})$ we have the equality

$$W_{\widehat{\mathcal{H}}_1 \oplus \widehat{\mathcal{H}}_2}(\mathcal{A}) = \Lambda_+(\mathcal{A}) \cup \Lambda_-(\mathcal{A}).$$

Definition 3. *The set of all eigenvalues of the 3×3 matrices*

$$\mathcal{A}_f := \begin{pmatrix} (A_{11}f_1, f_1) & (A_{12}f_2, f_1) & (A_{13}f_3, f_1) \\ (A_{12}^*f_1, f_2) & (A_{22}f_2, f_2) & (A_{23}f_3, f_2) \\ (A_{13}^*f_1, f_3) & (A_{23}^*f_2, f_3) & (A_{33}f_3, f_3) \end{pmatrix}, \quad f = (f_1, f_2, f_3) \in \mathbb{S}_{\mathcal{H}_1} \times \mathbb{S}_{\mathcal{H}_2} \times \mathbb{S}_{\mathcal{H}_3}$$

is called the cubic numerical range of $\mathcal{A}$ (with respect to the operator matrix representation (1)) *and will be denoted by $W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A})$. For a fixed decomposition of $\mathcal{H}$, we also write $W^3(\mathcal{A}) = W_{\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}(\mathcal{A})$.*

Let $a_{ij}(f) := (A_{ij}f_j, f_i)$ for $i, j = 1, 2, 3$ and

$$E_k(f) := \frac{1}{3}(a_{11}(f) + a_{22}(f) + a_{33}(f)) + 2\sqrt{-\frac{P(f)}{3}} \cos\frac{\Phi(f) + 2\pi k}{3},\ k = 1, 2, 3,$$

where

$$\begin{aligned}
P(f) &:= -\frac{1}{6}(a_{11}(f)-a_{22}(f))^2+(a_{11}(f)-a_{33}(f))^2+(a_{22}(f)-a_{33}(f))^2 \\
&\quad -|a_{12}(f)|^2-|a_{23}(f)|^2-|a_{13}(f)|^2; \\
Q(f) &:= -\frac{2}{27}\left(a_{11}(f)+a_{22}(f)+a_{33}(f)\right)^3+\frac{1}{3}\left(a_{11}(f)+a_{22}(f)+a_{33}(f)\right) \\
&\quad \times\left(a_{11}(f)a_{22}(f)+a_{22}(f)a_{33}(f)+a_{11}(f)a_{33}(f)-|a_{12}(f)|^2-|a_{23}(f)|^2-|a_{13}(f)|^2\right) \\
&\quad +a_{11}(f)a_{22}(f)a_{33}(f)+2\mathrm{Re}(a_{12}(f)a_{23}(f)\overline{a_{13}(f)})+|a_{12}(f)|^2a_{33}(f) \\
&\quad +|a_{23}(f)|^2a_{11}(f)+|a_{13}(f)|^2a_{22}(f); \\
\Phi(f) &:= \arccos\left(-\frac{3Q(f)}{2P(f)}\sqrt{-\frac{3}{P(f)}}\right).
\end{aligned}$$

The main result of this note is the following theorem.

Theorem 1. *For the cubic numerical range of $\mathcal{A}$ we have*

$$W^3(\mathcal{A})=\bigcup_{k=1}^{3}\bigcup_{f\in\mathbb{S}_{\mathcal{H}_1}\times\mathbb{S}_{\mathcal{H}_2}\times\mathbb{S}_{\mathcal{H}_3}}\{E_k(f)\}.$$

This theorem plays a key role in the estimate of the bounds of $\mathcal{A}$ in terms of cubic numerical range.

If $A_{13}=0$, then for the lower bound of $\mathcal{A}$ we have [2]

$$\min\sigma(\mathcal{A})\geq\min\{\min\sigma(A_{11}),\min\sigma(A_{22}),\min\sigma(A_{33})\}-\left(\|A_{12}\|^2+\|A_{23}\|^2\right)^{1/2}, \tag{3}$$

and strict inequality prevails if at least two of the lower bounds $\min\sigma(A_{ii})$, $i=1,2,3$, of diagonal elements are different. As an example we consider the case where $\mathcal{H}_1=\mathbb{C}$, $\mathcal{H}_2=L_2[\pi;\pi]$, $\mathcal{H}_3=L_2([\pi;\pi]^2)$ and

$$A_{11}f_1=\varepsilon f_1,\ (A_{22}f_2)(x)=(\varepsilon+1-\cos x)f_2(x),\ (A_{33}f_3)(x,y)=(\varepsilon+2-\cos x-\cos y)f_2(x,y),$$

$$A_{12}f_2=\int_{-\pi}^{\pi}\sin t f_2(t)dt,\ A_{13}=0,\ (A_{23}f_3)(x)=\int_{-\pi}^{\pi}\sin t f_3(x,t)dt$$

with $\varepsilon>0$. In the latter case $W(A_{22})=(\varepsilon;\varepsilon{+}2)$, $W(A_{33})=(\varepsilon;\varepsilon{+}4)$ and $W(A_{ii})\subset W^3(\mathcal{A})$, $i=2,3$. Then according to (3) the lower bound of $\mathcal{A}$ can be estimated as follows: $\min\sigma(\mathcal{A})\geq\varepsilon-\sqrt{2}\pi$. An obtained inequality is an important in determining the location of the first eigenvalue of $\mathcal{A}$.

References

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CERTIFICATE

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